Bicomplex extensions of zero mean curvature surfaces in $\mathbf{R}^{2,1}$ and $\mathbf{R}^{2,2}$

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Abstract. In this paper, we construct zero mean curvature complex surfaces in $\mathbf{C}^N$ with a various type of standard metric, each of which changes its type from positive definite to neutral, by means of bicomplex numbers. By applying them as bicomplex extensions, we describe the correspondence between fold singularities and type-changing of zero mean curvature real surfaces in $\mathbf{R}^{2,1}$ and $\mathbf{R}^{2,2}$. In particular, we show that any fold singularity consists of branch points of the bicomplex extension. We also show that type-changing across a lightlike line segment occurs on an incomplete end on a fold singularity.

1 Introduction

Spacelike maximal surfaces in the Lorentzian 3-space $\mathbf{R}^{2,1}$ are correspondents to minimal surfaces in the Euclidean 3-space $\mathbf{R}^3$ as zero mean curvature surfaces. They have quite similar natures to each others. One of the essential differences is the fact that spacelike maximal surfaces have singularities in general. Among these, we are interested in double surfaces with *fold singularities*. Here we say a surface has a fold singularity if the surface can be expressed locally by an analytic map $F : \mathbf{R}^2 \supset U \to \mathbf{R}^{N_+,N_-}$ satisfying $F(x, y) = F(x, -y)$ for a suitable coordinate system. In particular, we mainly consider the case that the fold singularity is nondegenerate in the sense that $F_x(x, 0) \neq {}^t(0, \ldots, 0)$ holds on each, or at least generic, point on the singularity. Although fold singularities do not appear on spacelike maximal surfaces in usual, they are not so rare and many examples are known. Recently they are studied actively by focusing on analytic extensions of spacelike maximal surfaces to timelike minimal surfaces across their fold singularities since such type-changing occurs only across a null curve of fold singularities or a lightlike line segment ([13], [12], [8], [4], [5], [6], [7], [9], [20], [1], etc.).

One of the strong tools to analyze these surfaces is the Weierstrass type of representation formula. By this formula, each zero mean curvature surface is presented locally as a real part of the integral of a $\mathbf{C}^3$-valued (resp. $\mathbf{D}^3$-valued) holomorphic 1-form on a suitable domain in $\mathbf{C}$ in the case of spacelike maximal surfaces (resp. in $\mathbf{D}$, the algebra of *paracomplex numbers*, in the case of timelike minimal surfaces). In this paper, we unify the formulas on these surfaces by considering the extensions to $\mathbf{B}$, the algebra of *bicomplex numbers*. $\mathbf{B}$ is the Clifford algebra isomorphic to $\mathbf{C}l_1$ in [15], which includes both $\mathbf{C}$ and $\mathbf{D}$ as subalgebras. $\mathbf{B}$ was first introduced essentially by Cockle [3] as *tessarines*, and later, defined formally by Segre [18] as bicomplex numbers. $\mathbf{B}$ has been investigated itself, and used for analyzing various subjects by many authors. However it seems that there are not so many studies on geometry of submanifolds by using $\mathbf{B}$, for instance, as Baird-Wood [2] which treats representation formulas of harmonic morphisms, etc..

In §2, we summarize basic facts on $\mathbf{B}$ and bicomplex holomorphic functions in a form suitable for our purpose, and in §3, consider zero mean curvature complex surfaces in $\mathbf{C}^N$ which are given by projections of bicomplex holomorphic maps, and give a generic results for such maps to have fold singularities. In §§4-5, we observe bicomplex extensions of zero mean curvature surfaces in $\mathbf{R}^{2,1}$, and as an application, give a transformation formula of the Weierstrass data for spacelike maximal and timelike minimal surfaces in $\mathbf{R}^{2,1}$ extended analytically to each other across their fold singularities. In §§6-7, we observe bicomplex extensions of zero mean curvature surfaces in $\mathbf{R}^{2,2}$, and as an application, give a degenerate result of negative definite domains which appear in deformations of "spacelike" maximal surfaces in $\mathbf{R}^{2,2}$. In §8, we observe cross sections including the degenerate directions of spacelike maximal and/or timelike minimal surfaces around fold singularities, incomplete ends and cuspidal edges. In §9, we discuss the flux around zero-devisors and a global meaning of timelike minimal surfaces.

2 Basic facts on bicomplex numbers

The algebra of *bicomplex numbers* is a 4-dimensional real vector space

$$\mathbf{B} := \{\tilde{z} = x_1 + i_1 x_2 + i_2 x_3 + j x_4 \mid x_1, x_2, x_3, x_4 \in \mathbf{R}\}$$

equipped with the multiplication defined by

$${i_1}^2 = {i_2}^2 = -1, \; j = i_1 i_2 = i_2 i_1, \; \text{and hence } j^2 = 1.$$

Set

$$
\begin{aligned}
\mathbf{C}(i_1) &:= \{\zeta = x_1 + i_1 x_2 \mid x_1, x_2 \in \mathbf{R}\},\\
\mathbf{C}(i_2) &:= \{z = x_1 + i_2 x_3 \mid x_1, x_3 \in \mathbf{R}\},\\
\mathbf{D} &:= \{\check{z} = x_1 + j x_4 \mid x_1, x_4 \in \mathbf{R}\}.
\end{aligned}
$$

Both $\mathbf{C}(i_1)$ and $\mathbf{C}(i_2)$ are the fields of complex numbers, and $\mathbf{D}$ is the algebra of *para-complex numbers*. We can regard $\mathbf{B}$ as a complexification of $\mathbf{C}(i_2)$ (resp. $\mathbf{D}$) in the sense that $\mathbf{B} = \mathbf{C}(i_2) \otimes_{\mathbf{R}} \mathbf{C}(i_1)$ (resp. $\mathbf{D} \otimes_{\mathbf{R}} \mathbf{C}(i_1)$). Three *conjugations* are defined on $\mathbf{B}$ as follows:

$$
\begin{aligned}
\widetilde{z} &= x_1 + i_1 x_2 + i_2 x_3 + j x_4 = \zeta_1 + i_2 \zeta_2 = \zeta_1 + j\zeta_3,\\
\widetilde{z}^{\dagger_1} &:= x_1 - i_1 x_2 + i_2 x_3 - j x_4 = \overline{\zeta_1} + i_2 \overline{\zeta_2} = \overline{\zeta_1} - j\overline{\zeta_3},\\
\widetilde{z}^{\dagger_2} &:= x_1 + i_1 x_2 - i_2 x_3 - j x_4 = \zeta_1 - i_2 \zeta_2 = \zeta_1 - j\zeta_3,\\
\widetilde{z}^{\dagger_3} &:= \widetilde{z}^{\dagger_1 \dagger_2}\\
&= x_1 - i_1 x_2 - i_2 x_3 + j x_4 = \overline{\zeta_1} - i_2 \overline{\zeta_2} = \overline{\zeta_1} + j\overline{\zeta_3},
\end{aligned}
$$

where $\zeta_1, \zeta_2, \zeta_3 \in \mathbf{C}(i_1)$ and $\overline{\cdot}$ is the usual conjugation of $\mathbf{C}(i_1)$. It holds that

$$
\begin{aligned}
\widetilde{z} + \widetilde{z}^{\dagger_1} &= 2(x_1 + i_2 x_3) \in \mathbf{C}(i_2),\\
\widetilde{z} + \widetilde{z}^{\dagger_2} &= 2(x_1 + i_1 x_2) \in \mathbf{C}(i_1),\\
\widetilde{z} + \widetilde{z}^{\dagger_3} &= 2(x_1 + j x_4) \in \mathbf{D},\\
\widetilde{z}\widetilde{z}^{\dagger_1} &= (x_1 + i_2 x_3)^2 + (x_2 + i_2 x_4)^2\\
&= ({x_1}^2 + {x_2}^2 - {x_3}^2 - {x_4}^2) + 2i_2(x_1 x_3 + x_2 x_4) \in \mathbf{C}(i_2),\\
\widetilde{z}\widetilde{z}^{\dagger_2} &= (x_1 + i_1 x_2)^2 + (x_3 + i_1 x_4)^2\\
&= ({x_1}^2 - {x_2}^2 + {x_3}^2 - {x_4}^2) + 2i_1(x_1 x_2 + x_3 x_4) \in \mathbf{C}(i_1),\\
\widetilde{z}\widetilde{z}^{\dagger_3} &= (x_1 + j x_4)^2 + (x_2 - j x_3)^2\\
&= ({x_1}^2 + {x_2}^2 + {x_3}^2 + {x_4}^2) + 2j(x_1 x_4 - x_2 x_3) \in \mathbf{D}.
\end{aligned}
$$

The *real part* and the three *imaginary parts* of a bicomplex number $\widetilde{z} = x_1 + i_1 x_2 + i_2 x_3 + j x_4$ are calculated by

$$
\begin{aligned}
x_1 &= \frac{1}{4}(\widetilde{z} + \widetilde{z}^{\dagger_1} + \widetilde{z}^{\dagger_2} + \widetilde{z}^{\dagger_3}),\\
x_2 &= \frac{1}{4i_1}(\widetilde{z} - \widetilde{z}^{\dagger_1} + \widetilde{z}^{\dagger_2} - \widetilde{z}^{\dagger_3}),
\end{aligned}
$$

$$x_3 = \frac{1}{4i_2}(\tilde{z} + \tilde{z}^{\dagger_1} - \tilde{z}^{\dagger_2} - \tilde{z}^{\dagger_3}),$$
$$x_4 = \frac{1}{4j}(\tilde{z} - \tilde{z}^{\dagger_1} - \tilde{z}^{\dagger_2} + \tilde{z}^{\dagger_3}).$$

We denote these parts also by $(\tilde{z})_1, (\tilde{z})_2, (\tilde{z})_3, (\tilde{z})_4$ or $\{\tilde{z}\}_1, \{\tilde{z}\}_2, \{\tilde{z}\}_3, \{\tilde{z}\}_4$.

In usual, the absolute values and arguments of bicomplex numbers are defined for each conjugation, and they take complex or paracomplex values in general. In this paper, we use the nonnegative real *absolute value* $|\tilde{z}|$ and the *Euclidean norm* $||\tilde{z}||$ given by

$$|\tilde{z}|^4 = \tilde{z}\tilde{z}^{\dagger_1}\tilde{z}^{\dagger_2}\tilde{z}^{\dagger_3} = \{(x_1 - x_4)^2 + (x_2 + x_3)^2\}\{(x_1 + x_4)^2 + (x_2 - x_3)^2\},$$
$$||\tilde{z}||^2 = \mathrm{Re}(\tilde{z}\tilde{z}^{\dagger_3}) = {x_1}^2 + {x_2}^2 + {x_3}^2 + {x_4}^2.$$

For $z = x_1 + i_2 x_3 \in \mathbf{C}(i_2)$ and $\check{z} = x_1 + j x_4 \in \mathbf{D}$, we also use the notation $|z|^2_{\mathbf{C}(i_2)} := {x_1}^2 + {x_3}^2$ and $|\check{z}|^2_{\mathbf{D}} := {x_1}^2 - {x_4}^2$ respectively. It holds that $|\tilde{z}|^4 = |\tilde{z}\tilde{z}^{\dagger_1}|^2_{\mathbf{C}(i_2)} = |\tilde{z}\tilde{z}^{\dagger_3}|^2_{\mathbf{D}}$.

$\mathbf{B}$ has *zero-divisors*, and we denote the set of zero-divisors by $\mathcal{S}$. It is easy to see that

$$\begin{aligned}
\mathcal{S} &= \{\tilde{z} \mid |\tilde{z}| = 0\} \\
&= \{\tilde{z} \mid x_1 - x_4 = x_2 + x_3 = 0 \text{ or } x_1 + x_4 = x_2 - x_3 = 0\} \\
&= \{(1+j)(x_1 + i_1 x_2) \mid x_1, x_2 \in \mathbf{R}\} \cup \{(1-j)(x_1 + i_1 x_2) \mid x_1, x_2 \in \mathbf{R}\} \\
&= (1+j)\mathbf{C}(i_1) \cup (1-j)\mathbf{C}(i_1) = (1+j)\mathbf{B} \cup (1-j)\mathbf{B}.
\end{aligned}$$

Let $|\tilde{z}|_\pm$ be a nonnegative number given by

$$|\tilde{z}|^2_\pm = \mathrm{Re}(\tilde{z}\tilde{z}^{\dagger_3}) \mp \mathrm{Im}_{\mathbf{D}}(\tilde{z}\tilde{z}^{\dagger_3}) = (x_1 \mp x_4)^2 + (x_2 \pm x_3)^2.$$

Then it holds that $|\tilde{z}|^2 = |\tilde{z}|_+|\tilde{z}|_-$ and $(1 \pm j)\mathbf{B} = \{\tilde{z} \mid |\tilde{z}|_\pm = 0\}$. For any $\tilde{z} \notin \mathcal{S}$, its *inverse* is given by

$$\frac{1}{\tilde{z}} = \frac{\tilde{z}^{\dagger_1}\tilde{z}^{\dagger_2}\tilde{z}^{\dagger_3}}{|\tilde{z}|^4}.$$

Now, it is natural to consider three *arguments* θ_1, θ_2, θ_3 satisfying

$$\tilde{z} = r e^{i_1\theta_1 + i_2\theta_2 + j\theta_3}$$

for $\tilde{z} \notin \mathcal{S}$, where $r = |\tilde{z}|$ and

$$e^{i_1\theta_1} = \cos\theta_1 + i_1 \sin\theta_1, \quad e^{i_2\theta_2} = \cos\theta_2 + i_2 \sin\theta_2, \quad e^{j\theta_3} = \cosh\theta_3 + j \sinh\theta_3.$$

These arguments are calculated by

$$r^2 e^{2i_1\theta_1} \;=\; \tilde{z}\tilde{z}^{\dagger_2}, \quad r^2 e^{2i_2\theta_2} \;=\; \tilde{z}\tilde{z}^{\dagger_1}, \quad r^2 e^{2j\theta_3} \;=\; \tilde{z}\tilde{z}^{\dagger_3}.$$

In particular (θ_1, θ_2) is well-defined modulo $(\pi,\pi)\mathbf{Z} + (\pi, -\pi)\mathbf{Z}$. Note here that $|j| = 1$ and $j \;=\; i_1 i_2 \;=\; e^{i_1\pi(N_1+1/2)+i_2\pi(N_2+1/2)}$ $(N_1, N_2 \in \mathbf{Z}, N_1+N_2$: even) under our definition.

Now, let

$$\tilde{w}(\tilde{z}) \;=\; u_1(x_1,x_2,x_3,x_4) + i_1 u_2(x_1,x_2,x_3,x_4) + i_2 u_3(x_1,x_2,x_3,x_4) + j u_4(x_1,x_2,x_3,x_4)$$

be a bicomplex function defined on an open subset Ω of $\mathbf{B}$. Then the bicomplex derivative $\tilde{w}_{\tilde{z}}$ of $\tilde{w}(\tilde{z})$ is defined as a natural extension of the complex derivative of a complex function so that $\tilde{w}_{\tilde{z}} \;=\; \tilde{w}_{x_1} \;=\; \tilde{w}_{x_2}/i_1 \;=\; \tilde{w}_{x_3}/i_2 \;=\; \tilde{w}_{x_4}/j$, and $\tilde{w}(\tilde{z})$ is *bicomplex differentiable* or *bicomplex holomorphic* if and only if it satisfies the following *bicomplex Cauchy-Riemann equations*:

$$\text{(2.1)} \qquad \begin{cases} (u_1)_{x_1} &=& (u_2)_{x_2} &=& (u_3)_{x_3} &=& (u_4)_{x_4}, \\ (u_2)_{x_1} &=& -(u_1)_{x_2} &=& (u_4)_{x_3} &=& -(u_3)_{x_4}, \\ (u_3)_{x_1} &=& (u_4)_{x_2} &=& -(u_1)_{x_3} &=& -(u_2)_{x_4}, \\ (u_4)_{x_1} &=& -(u_3)_{x_2} &=& -(u_2)_{x_3} &=& (u_1)_{x_4}. \end{cases}$$

Note here that any bicomplex holomorphic function satisfies

$$\text{(2.2)} \qquad \begin{aligned} (u_\ell)_{x_1x_1} + (u_\ell)_{x_2x_2} \;&=\; (u_\ell)_{x_3x_3} + (u_\ell)_{x_4x_4} \\ \;=\; (u_\ell)_{x_1x_3} + (u_\ell)_{x_2x_4} \;&=\; (u_\ell)_{x_1x_4} - (u_\ell)_{x_2x_3} \;=\; 0 \end{aligned} \qquad (\ell = 1,2,3,4)$$

and

$$\text{(2.3)} \qquad (u_\ell)_{x_1x_1} + (u_\ell)_{x_3x_3} \;=\; (u_\ell)_{x_1x_1} - (u_\ell)_{x_4x_4} \;=\; 0 \qquad (\ell = 1,2,3,4).$$

3 Zero mean curvature complex surfaces and $\dagger_2$ equivariance

Let $\varphi^n(\tilde{z})$ be a bicomplex holomorphic functions defined on an open subset Ω of $\mathbf{B}$ for $n = 1, 2, \ldots, N$. Set

$$\Phi \;:=\; {}^t(\varphi^1, \varphi^2, \ldots, \varphi^N),$$

and denote the projection of Φ to $\mathbf{C}(i_1)^N$ by F, namely

$$\begin{aligned} F \;\;:\;\; & \mathbf{B} \supset \Omega \to \mathbf{C}(i_1)^N \\ & \tilde{z} \mapsto \Phi_1(\tilde{z}) + i_1\Phi_2(\tilde{z}) \;=\; {}^t(\varphi^1_1 + i_1\varphi^1_2, \varphi^2_1 + i_1\varphi^2_2, \ldots, \varphi^N_1 + i_1\varphi^N_2)(\tilde{z}), \end{aligned}$$

where we denote $(\Phi)_\ell$ and $(\varphi^n)_\ell$ by Φ_ℓ and φ^n_ℓ respectively ($\ell = 1,2,3,4$). Set $\zeta_1 := x_1 + i_1 x_2$ and $\zeta_2 := x_3 + i_1 x_4$ as in §2. Then $\tilde{z} = (x_1 + i_1 x_2) + i_2(x_3 + i_1 x_4) = \zeta_1 + i_2 \zeta_2$, and F can be regarded as the map from $\Omega \subset \mathbf{C}(i_1)^2$ into $\mathbf{C}(i_1)^N$. By (2.1), we see that F is holomorphic with respect to the variables (ζ_1, ζ_2).

Now, regard $\mathbf{C}(i_1)^N$ as $\mathbf{R}^{2N_+, 2N_-}$ ($N_+ + N_- = N$), and denote the standard metric by ds^2, namely

$$ds^2 := \sum_{n=1}^{N_+} \{(du_1^n)^2 + (du_2^n)^2\} - \sum_{n=N_+ +1}^{N_+ + N_-} \{(du_1^n)^2 + (du_2^n)^2\}.$$

Set $h := F^*(ds^2)$, and

$$h_{\ell m} := h\left(\frac{\partial}{\partial x_\ell}, \frac{\partial}{\partial x_m}\right) = ds^2\left(\frac{\partial F}{\partial x_\ell}, \frac{\partial F}{\partial x_m}\right) \qquad (\ell, m = 1,2,3,4).$$

Then it holds that

$$(h_{\ell m})_{\ell, m = 1,2,3,4} = \begin{pmatrix} a & 0 & c & d \\ 0 & a & -d & c \\ c & -d & b & 0 \\ d & c & 0 & b \end{pmatrix}$$

with

$$(3.1) \qquad \begin{cases} a = \langle (\Phi_1)_{x_1}, (\Phi_1)_{x_1} \rangle + \langle (\Phi_2)_{x_1}, (\Phi_2)_{x_1} \rangle, \\ b = \langle (\Phi_3)_{x_1}, (\Phi_3)_{x_1} \rangle + \langle (\Phi_4)_{x_1}, (\Phi_4)_{x_1} \rangle, \\ c = -\langle (\Phi_1)_{x_1}, (\Phi_3)_{x_1} \rangle - \langle (\Phi_2)_{x_1}, (\Phi_4)_{x_1} \rangle, \\ d = \langle (\Phi_1)_{x_1}, (\Phi_4)_{x_1} \rangle - \langle (\Phi_2)_{x_1}, (\Phi_3)_{x_1} \rangle, \end{cases}$$

where we use (2.1), and denote the standard inner product on $\mathbf{R}^{N_+, N_-}$ by $\langle \cdot, \cdot \rangle$, namely

$$\langle {}^t(u^1, \ldots, u^N), {}^t(v^1, \ldots, v^N) \rangle := \sum_{n=1}^{N_+} u^n v^n - \sum_{n=N_+ +1}^{N_+ + N_-} u^n v^n.$$

In particular, it holds that

$$\det(h_{\ell m}) = (ab - c^2 - d^2)^2.$$

Since

$$(h^{m\ell}) = (h_{\ell m})^{-1} = \frac{1}{ab - c^2 - d^2} \begin{pmatrix} b & 0 & -c & -d \\ 0 & b & d & -c \\ -c & d & a & 0 \\ -d & -c & 0 & a \end{pmatrix},$$

by direct computation, we can show that the Laplacian with respect to h is of the following form:

$$\begin{aligned}\triangle_h &= \frac{1}{ab-c^2-d^2}\left[b\left(\frac{\partial^2}{\partial {x_1}^2}+\frac{\partial^2}{\partial {x_2}^2}\right)+a\left(\frac{\partial^2}{\partial {x_3}^2}+\frac{\partial^2}{\partial {x_4}^2}\right)\right. \\ &\quad -2c\left(\frac{\partial^2}{\partial x_1\partial x_3}+\frac{\partial^2}{\partial x_2\partial x_4}\right)-2d\left(\frac{\partial^2}{\partial x_1\partial x_4}-\frac{\partial^2}{\partial x_2\partial x_3}\right) \\ &\quad +\left\{(b_{x_1}-c_{x_3}-d_{x_4})\frac{\partial}{\partial x_1}+(b_{x_2}-c_{x_4}+d_{x_3})\frac{\partial}{\partial x_2}\right. \\ &\quad \left.\left.+(a_{x_3}-c_{x_1}+d_{x_2})\frac{\partial}{\partial x_3}+(a_{x_4}-c_{x_2}-d_{x_1})\frac{\partial}{\partial x_4}\right\}\right].\end{aligned}$$

Applying (2.1), we see that each coefficient of the term of the differential operator of the first order vanishes. By using this fact and (2.2), we get the following

Theorem 3.1. *The mean curvature vector field of F vanishes on the regular point set of F, namely it holds that $\triangle_h F \equiv \mathbf{0}$.*

Denote the restriction of F to $\Omega\cap\mathbf{C}(i_2)$ (resp. $\Omega\cap\mathbf{D}$) by $F|_{\mathbf{C}(i_2)}$ (resp. $F|_{\mathbf{D}}$). By (2.3), $F|_{\mathbf{C}(i_2)}$ (resp. $F|_{\mathbf{D}}$) is a harmonic map with respect to the metric $dx_1^2+dx_3^2$ on $\Omega\cap\mathbf{C}(i_2)$ (resp. $dx_1^2-dx_4^2$ on $\Omega\cap\mathbf{D}$).

Theorem 3.2. *Suppose that Φ satisfies the condition*

$$\Phi(\tilde{z}^{\dagger_2}) = \Phi(\tilde{z})^{\dagger_2}+C \tag{3.2}$$

for some constant vector $C\in\mathbf{B}^N$. Then $F|_{\mathbf{C}(i_2)}$ and $F|_{\mathbf{D}}$ have a common fold singularity on $F(\Omega\cap\mathbf{R})\cap\{a\neq 0\}$, and $F|_{\mathbf{D}}$ is an analytic extension of $F|_{\mathbf{C}(i_2)}$ across the fold singularity. The image of the extension is nondegenerate on any point $F(x_1)$ on the fold singularity such that $\Phi_{\tilde{z}}(x_1)$ and $\Phi_{\tilde{z}\tilde{z}}(x_1)$ are linear independent.

Proof. Under the assumption, we see that

$$\Phi(\zeta_1+i_2(-\zeta_2)) = \Phi(\zeta_1-i_2\zeta_2) = \Phi((\zeta_1+i_2\zeta_2)^{\dagger_2}) = \Phi(\zeta_1+i_2\zeta_2)^{\dagger_2}+C$$

for $\zeta_1,\zeta_2\in\mathbf{C}(i_1)$ such that $\zeta_1+i_2\zeta_2\in\Omega\cap\Omega^{\dagger_2}$, and hence

$$\Phi(\zeta_1+i_2(-\zeta_2))-\Phi(\zeta_1) = (\Phi(\zeta_1+i_2\zeta_2)-\Phi(\zeta_1))^{\dagger_2}.$$

Therefore $F = \Phi_1 + i_1\Phi_2$ satisfies

$$F(\zeta_1 + i_2(-\zeta_2)) - F(\zeta_1) \ = \ F(\zeta_1 + i_2\zeta_2) - F(\zeta_1),$$

namely $F(\zeta_1 + i_2\zeta_2)$ is even with respect to ζ_2. On the other hand, by (3.2), we also see that $(\Phi_3)_{x_1} = (\Phi_4)_{x_1} = 0$ on $\Omega \cap \mathbf{C}(i_1)$ and hence $b = c = d = 0$ on $\Omega \cap \mathbf{C}(i_1)$. Therefore both $F|_{\mathbf{C}(i_2)}$ and $F|_{\mathbf{D}}$ are double surfaces with fold singularities on $F(\Omega \cap \mathbf{R}) \cap \{a \neq 0\}$, and $F(x_1 + jx_4) = F(x_1 + i_2 \cdot i_1 x_4)$ is an analytic extension of $F(x_1 + i_2 x_3)$ in the sense that

$$F_{\text{fld}}(s,t) \ := \ \begin{cases} F(s + i_2\sqrt{t}) & (t \geq 0,\ s + i_2\sqrt{t} \in \Omega), \\ F(s + j\sqrt{-t}) & (t \leq 0,\ s + j\sqrt{-t} \in \Omega) \end{cases} \tag{3.3}$$

is analytic with respect to (s,t). Note here that $i_2\Phi_3(\zeta_1) + j\Phi_4(\zeta_1)$ is independent of $\zeta_1 \in \Omega \cap \mathbf{C}(i_1)$ under the assumption (3.2). Since

$$F_{\text{fld}}(s,t) \ = \ -(i_2\Phi_3(s) + j\Phi_4(s)) + \Phi(s) - \frac{1}{2}\Phi_{\widetilde{z}\widetilde{z}}(s)t + O(t^2)$$

holds around $(s,0)$ for any $s \in \Omega \cap \mathbf{R}$, the image of F_{fld} is nondegenerate on $(s,0)$ for any s such that $\dfrac{\partial F_{\text{fld}}}{\partial s}(s,0) = \Phi_{\widetilde{z}}(s)$ and $\dfrac{\partial F_{\text{fld}}}{\partial t}(s,0) = -\dfrac{1}{2}\Phi_{\widetilde{z}\widetilde{z}}(s)$ are linear independent. □

Theorem 3.3. *Suppose that Φ satisfies* (3.2)*. In addition, assume that $\Omega = \Omega_\infty \setminus (\widetilde{q} + \mathcal{S})$ for some domain $\Omega_\infty \subset \mathbf{B}$ and $\widetilde{q} \in \Omega_\infty \cap \mathbf{R}$, and that Φ has a pole of order K with only odd ordered terms at $\widetilde{q}$, namely, Φ can be written as*

$$\Phi(\widetilde{z}) \ = \ \sum_{k=1;k:\text{odd}}^{K} \frac{1}{(\widetilde{z} - \widetilde{q})^k} C_k + \Phi_{\text{hol}}(\widetilde{z}),$$

where $C_k \in \mathbf{C}(i_1)^N$ $(k = 1, \ldots, K; k : \text{odd})$, $C_K \neq {}^t(0, \ldots, 0)$ and Φ_{hol} is a bicomplex holomorphic map from Ω_∞ to $\mathbf{B}^N$ satisfying (3.2)*. Then $F|_{\mathbf{D}}$ is an analytic extension of $F|_{\mathbf{C}(i_2)}$ across a subset of a line parallel to C_K. The image of the extension is nondegenerate on an open subset of the line if C_K and either $(\Phi_{\text{hol}})_{\widetilde{z}\widetilde{z}}(\widetilde{q})$ or*

$$\begin{cases} (\Phi_{\text{hol}})_{\widetilde{z}}(\widetilde{q}) & (K = 1), \\ C_{K-2} & (K \geq 3; K : \text{odd}) \end{cases}$$

are linear independent.

Proof. We may assume $\widetilde{q}=0$ without loss of generality. In this case, F is of the following form:

$$F(\zeta_1+i_2\zeta_2) \;=\; \sum_{k=1;k:\mathrm{odd}}^{K}\left\{\sum_{\tau=0;\tau:\mathrm{even}}^{k-1}\frac{(-1)^{\tau/2}\binom{k}{\tau}{\zeta_1}^{k-\tau}{\zeta_2}^{\tau}}{({\zeta_1}^2+{\zeta_2}^2)^k}\right\}C_k+F_{\mathrm{hol}}(\zeta_1+i_2\zeta_2),$$

where $F_{\mathrm{hol}}:=(\Phi_{\mathrm{hol}})_1+i_1(\Phi_{\mathrm{hol}})_2$ is holomorphic with respect to (ζ_1,ζ_2), and even with respect to ζ_2. Now, for any $s\in\mathbf{R}$ such that $s{\zeta_2}^{K+1}+i_2\zeta_2\in\Omega_\infty$ holds for $\zeta_2\in\mathbf{C}(i_1)$ near to 0, it holds that

$$\begin{aligned}
F(s{\zeta_2}^{K+1}+i_2\zeta_2) \;&=\; \sum_{k=1;k:\mathrm{odd}}^{K}\left\{\sum_{\tau=0;\tau:\mathrm{even}}^{k-1}\frac{(-1)^{\tau/2}\binom{k}{\tau}s^{k-\tau}{\zeta_2}^{(K-1)k-K\tau}}{(s^2{\zeta_2}^{2K}+1)^k}\right\}C_k \\
&\qquad +F_{\mathrm{hol}}(s{\zeta_2}^{K+1}+i_2\zeta_2) \\
&= F_{\mathrm{hol}}(0)-\frac{1}{2}(\Phi_{\mathrm{hol}})_{\widetilde{z}\widetilde{z}}(0){\zeta_2}^2+O({\zeta_2}^4) \\
&\qquad +\begin{cases}(s-s^3{\zeta_2}^2)C_1+s{\zeta_2}^2(\Phi_{\mathrm{hol}})_{\widetilde{z}}(0) & (K=1),\\ (-1)^{(K-1)/2}KsC_K+(-1)^{(K-3)/2}(K-2)s{\zeta_2}^2C_{K-2} & (K\geq 3; K:\mathrm{odd}),\end{cases}
\end{aligned}$$

and hence $F((-1)^{(K+1)/2}s{x_4}^{K+1}+jx_4)=F(s(i_1x_4)^{K+1}+i_2\cdot i_1x_4)$ is an analytic extension of $F(s{x_3}^{K+1}+i_2x_3)$ in the sense that

$$(3.4)\qquad F_{\mathrm{end}}(s,t) \;:=\; \begin{cases} F(st^{(K+1)/2}+i_2\sqrt{t}) & (t>0,\ st^{(K+1)/2}+i_2\sqrt{t}\in\Omega_\infty),\\ (-1)^{(K-1)/2}KsC_K+F_{\mathrm{hol}}(0) & (t=0),\\ F(st^{(K+1)/2}+j\sqrt{-t}) & \\ \qquad (-s^{-2/K}<t<0,\ st^{(K+1)/2}+j\sqrt{-t}\in\Omega_\infty) & \end{cases}$$

is analytic with respect to (s,t). The image of F_{end} is nondegenerate at least on the following subset of the borderline:

$$\begin{aligned}\{(-1)^{(K-1)/2}KsC_K+F_{\mathrm{hol}}(0)\mid s\in\mathbf{R},\\
-\frac{1}{2}(\Phi_{\mathrm{hol}})_{\widetilde{z}\widetilde{z}}(0)+\begin{cases}-s^3C_1+s(\Phi_{\mathrm{hol}})_{\widetilde{z}}(0) & (K=1),\\ (-1)^{(K-3)/2}(K-2)sC_{K-2} & (K\geq 3; K:\mathrm{odd})\end{cases}\\
\text{and } C_K \text{ are linear independent.}\}.\end{aligned}$$

□

For any real analytic map from an open interval into $\mathbf{R}^N$, its bicomplex extension satisfies (3.2) with $C\in\mathbf{R}^N$, and hence we can apply Theorems 3.2 and 3.3.

We note here that the first examples of analytic extensions across lightlike line segments were given in [5, Theorem 1.1]. In Example 9.3, we give another family of such extensions by applying Theorem 3.3.

4 Bicomplex extensions of zero mean curvature surfaces in $\mathbf{R}^{2,1}$

In this paper, we are interested in the case that the holomorphic map $F = \Phi_1 + i_1\Phi_2$ we defined in §3 is a bicomplex extension of some spacelike minimal or timelike maximal immersion into $\mathbf{R}^{2,1}$ or $\mathbf{R}^{2,2}$. In §§4-5, we restrict our attention to the case that $(N_+, N_-) = (2, 1)$ and $\Phi = {}^t(\varphi^1, \varphi^2, \varphi^3)$ satisfies

$$(\varphi^1_{\widetilde{z}})^2 + (\varphi^2_{\widetilde{z}})^2 - (\varphi^3_{\widetilde{z}})^2 \;=\; 0.$$

Set

$$\widetilde{f} \;:=\; \frac{\varphi^1_{\widetilde{z}} + \varphi^3_{\widetilde{z}}}{2},$$
$$\widetilde{g} \;:=\; \frac{\varphi^2_{\widetilde{z}}}{\varphi^1_{\widetilde{z}} + \varphi^3_{\widetilde{z}}} \;=\; \frac{-\varphi^1_{\widetilde{z}} + \varphi^3_{\widetilde{z}}}{\varphi^2_{\widetilde{z}}}.$$

Then $\widetilde{f}$ (resp. $\widetilde{g}$) is a bicomplex holomorphic function on Ω (resp. $\Omega \setminus \{|\widetilde{f}| = 0\}$), and $\Phi \;=\; \Phi_1 + i_1\Phi_2 + i_2\Phi_3 + j\Phi_4$ is rewritten as follows:

$$\Phi(\widetilde{z}) \;=\; \int^{\widetilde{z}} {}^t((1-\widetilde{g}^2)\widetilde{f}, 2\widetilde{g}\widetilde{f}, (1+\widetilde{g}^2)\widetilde{f})d\widetilde{z}. \tag{4.1}$$

In this case, we use the notation $\widetilde{X} \;=\; \Phi_1 + i_1\Phi_2$ instead of F. Conversely, for any pair $(\widetilde{g}, \widetilde{f})$ of bicomplex holomorphic functions on a domain $\Omega \subset \mathbf{B}$, the map $\widetilde{X} \;=\; \Phi_1 + i_1\Phi_2$ with (4.1) is locally well-defined. We note here that this form of representation formula can be found in [14, Theorem 3.2] for timelike minimal surfaces, and in [17, Theorem 1] for null curves.

By (2.1), it holds that

$$\widetilde{X}_{x_1} \;=\; (\Phi_1)_{x_1} + i_1(\Phi_2)_{x_1}, \qquad \widetilde{X}_{x_3} \;=\; -(\Phi_3)_{x_1} - i_1(\Phi_4)_{x_1}.$$

Since

$$\widetilde{X}_{\zeta_1} \times \widetilde{X}_{\zeta_2} \;=\; \widetilde{X}_{x_1} \times \widetilde{X}_{x_3} \;=\; i_2(\widetilde{g} - \widetilde{g}^{\dagger_2})\widetilde{f}\widetilde{f}^{\dagger_2} \begin{pmatrix} \widetilde{g}\widetilde{g}^{\dagger_2} - 1 \\ -(\widetilde{g} + \widetilde{g}^{\dagger_2}) \\ -(\widetilde{g}\widetilde{g}^{\dagger_2} + 1) \end{pmatrix},$$

where $\times$ is the outer product of type $(2,1)$, the Gauss map of $\widetilde{X}$ with respect to the complex inner product of $\mathbf{C}(i_1)^3$ regarded as $\mathbf{C}(i_1)^{2,1} = \mathbf{R}^{2,1} \otimes_{\mathbf{R}} \mathbf{C}(i_1)$ is given by

$$\widetilde{G} \;=\; \frac{1}{-i_2(\widetilde{g} - \widetilde{g}^{\dagger_2})} \begin{pmatrix} \widetilde{g}\widetilde{g}^{\dagger_2} - 1 \\ -(\widetilde{g} + \widetilde{g}^{\dagger_2}) \\ -(\widetilde{g}\widetilde{g}^{\dagger_2} + 1) \end{pmatrix},$$

and the Gauss map of $\widetilde{X}$ with respect to the real inner product of $\mathbf{C}(i_1)^3$ regarded as $\mathbf{R}^{4,2}$ is given by

$$\widetilde{G}^{\dagger_1} \;=\; \frac{1}{-i_2(\widetilde{g}^{\dagger_1} - \widetilde{g}^{\dagger_3})} \begin{pmatrix} \widetilde{g}^{\dagger_1}\widetilde{g}^{\dagger_3} - 1 \\ -(\widetilde{g}^{\dagger_1} + \widetilde{g}^{\dagger_3}) \\ -(\widetilde{g}^{\dagger_1}\widetilde{g}^{\dagger_3} + 1) \end{pmatrix}$$

in the sense that the normal vector space at any point is $\widetilde{G}^{\dagger_1}\mathbf{C}(i_1)$.

Denote the metric h induced by $\widetilde{X}$ from the standard metric on $\mathbf{C}(i_1)^3$ regarded as $\mathbf{R}^{4,2}$ (resp. $\mathbf{R}^6$) by h_- (resp. h_+), and $(\widetilde{g})_\ell$ and $(\widetilde{f})_\ell$ by g_ℓ and f_ℓ respectively ($\ell = 1,2,3,4$). Then we get the following

Proposition 4.1. *For any point in* $\Omega \setminus \{\widetilde{f} = 0\}$, *the metric* h_- *induced by* $\mathbf{R}^{4,2}$ *satisfies the following*:

(1) h_- *is positive definite if* $|\widetilde{f}| \neq 0$, $\widetilde{g} \notin \mathbf{C}(i_1)$ *and* $2{g_2}^2 + {g_4}^2 < {g_3}^2$.

(2) h_- *is rank-*2 *positive semidefinite if* (i) *or* (ii) *holds*:

(i) $|\widetilde{f}| \neq 0$, $\widetilde{g} \in \mathbf{C}(i_1) \setminus \mathbf{R}$ *or* $2{g_2}^2 + {g_4}^2 = {g_3}^2$; (ii) $|\widetilde{f}|_\pm = 0$ *and* $g_2 \mp g_3 \neq 0$.

(3) h_- *is* 0 *if* (i) *or* (ii) *holds*:

(i) $|\widetilde{f}| \neq 0$ *and* $\widetilde{g} \in \mathbf{R}$; (ii) $|\widetilde{f}|_\pm = 0$ *and* $g_2 \mp g_3 = 0$.

(4) h_- *is neutral if* $|\widetilde{f}| \neq 0$, $\widetilde{g} \notin \mathbf{C}(i_1)$ *and* $2{g_2}^2 + {g_4}^2 > {g_3}^2$.

Proof. Note here that the characteristic polynomial of $(h_{\ell m})$ is given by

$$\{\lambda^2 - (a+b)\lambda + (ab - c^2 - d^2)\}^2 \tag{4.2}$$

in general. The coefficients a, b, c, d of the metric $(h_{-,\ell m})_{\ell,m=1,2,3,4}$ induced by $\mathbf{R}^{4,2}$ is given by (3.1) with

$$\langle(\Phi_\ell)_{x_1}, (\Phi_m)_{x_1}\rangle \;=\; 2\{2(\widetilde{g}\widetilde{f})_\ell(\widetilde{g}\widetilde{f})_m - (\widetilde{f})_\ell(\widetilde{g}^2\widetilde{f})_m - (\widetilde{g}^2\widetilde{f})_\ell(\widetilde{f})_m\},$$

and hence

$$(4.3) \qquad \begin{cases} a &= -\{(\widetilde{g}-\widetilde{g}^{\dagger_1})^2\widetilde{f}\widetilde{f}^{\dagger_1}\}_1 - \{(\widetilde{g}-\widetilde{g}^{\dagger_3})^2\widetilde{f}\widetilde{f}^{\dagger_3}\}_1, \\ b &= \{(\widetilde{g}-\widetilde{g}^{\dagger_1})^2\widetilde{f}\widetilde{f}^{\dagger_1}\}_1 - \{(\widetilde{g}-\widetilde{g}^{\dagger_3})^2\widetilde{f}\widetilde{f}^{\dagger_3}\}_1, \\ c &= \{(\widetilde{g}-\widetilde{g}^{\dagger_1})^2\widetilde{f}\widetilde{f}^{\dagger_1}\}_3, \\ d &= -\{(\widetilde{g}-\widetilde{g}^{\dagger_3})^2\widetilde{f}\widetilde{f}^{\dagger_3}\}_4. \end{cases}$$

Since

$$\begin{aligned} ab - c^2 - d^2 &= \{|\widetilde{g}-\widetilde{g}^{\dagger_3}|^4_{\mathbf{D}} - |\widetilde{g}-\widetilde{g}^{\dagger_1}|^4_{\mathbf{C}(i_2)}\}|\widetilde{f}|^4 \\ &= -16({g_3}^2+{g_4}^2)(2{g_2}^2-{g_3}^2+{g_4}^2)|\widetilde{f}|^4, \end{aligned}$$

$ab-c^2-d^2>0$ (resp. $=, <$) holds if and only if $|\widetilde{f}|\neq 0$ and ${g_3}^2+{g_4}^2>0$ and $2{g_2}^2+{g_4}^2<{g_3}^2$ (resp. $|\widetilde{f}|=0$ or $g_3=g_4=0$ or $2{g_2}^2+{g_4}^2={g_3}^2$, $|\widetilde{f}|\neq 0$ and ${g_3}^2+{g_4}^2>0$ and $2{g_2}^2+{g_4}^2>{g_3}^2$). On the other hand, since

$$\begin{aligned} a+b &= -2\{(\widetilde{g}-\widetilde{g}^{\dagger_3})^2\widetilde{f}\widetilde{f}^{\dagger_3}\}_1 \\ &= 8\{(g_2-g_3)^2||\widetilde{f}||^2 + 2g_2g_3|\widetilde{f}|^2_+\} \\ &= 8\{(g_2+g_3)^2||\widetilde{f}||^2 - 2g_2g_3|\widetilde{f}|^2_-\} \\ &\geq 0, \end{aligned}$$

if $|\widetilde{f}|\neq 0$ and $a+b=0$ holds, then $g_2=g_3=0$. Combining these conditions, we get our conclusion. □

Proposition 4.2. *For any point in $\Omega\setminus\{\widetilde{f}=0\}$, the metric h_+ induced by $\mathbf{R}^6$ is rank-2 positive semidefinite if $\widetilde{g}\in\mathbf{C}(i_1)$ or $|\widetilde{f}|=0$. Otherwise h_+ is positive definite.*

Proof. Recall (4.2). The coefficients a,b,c,d of the metric $(h_{+,\ell m})_{\ell,m=1,2,3,4}$ induced by $\mathbf{R}^6$ is given by (3.1) with

$$\langle(\Phi_\ell)_{x_1},(\Phi_m)_{x_1}\rangle = 2\{2(\widetilde{g}\widetilde{f})_\ell(\widetilde{g}\widetilde{f})_m + (\widetilde{f})_\ell(\widetilde{f})_m + (\widetilde{g}^2\widetilde{f})_\ell(\widetilde{g}^2\widetilde{f})_m\},$$

and hence

$$(4.4) \qquad \begin{cases} a &= \{(1+\widetilde{g}\widetilde{g}^{\dagger_1})^2\widetilde{f}\widetilde{f}^{\dagger_1}\}_1 + \{(1+\widetilde{g}\widetilde{g}^{\dagger_3})^2\widetilde{f}\widetilde{f}^{\dagger_3}\}_1, \\ b &= -\{(1+\widetilde{g}\widetilde{g}^{\dagger_1})^2\widetilde{f}\widetilde{f}^{\dagger_1}\}_1 + \{(1+\widetilde{g}\widetilde{g}^{\dagger_3})^2\widetilde{f}\widetilde{f}^{\dagger_3}\}_1, \\ c &= -\{(1+\widetilde{g}\widetilde{g}^{\dagger_1})^2\widetilde{f}\widetilde{f}^{\dagger_1}\}_3, \\ d &= \{(1+\widetilde{g}\widetilde{g}^{\dagger_3})^2\widetilde{f}\widetilde{f}^{\dagger_3}\}_4. \end{cases}$$

Since

$$
\begin{aligned}
ab - c^2 - d^2 &= \{|1 + \tilde{g}\tilde{g}^{\dagger_3}|^4_{\mathbf{D}} - |1 + \tilde{g}\tilde{g}^{\dagger_1}|^4_{\mathbf{C}(i_2)}\}|\tilde{f}|^4 \\
&= 8({g_3}^2 + {g_4}^2)\{({g_1}^2 - {g_4}^2)^2 + ({g_2}^2 - {g_3}^2)^2 \\
&\qquad + 2(g_1 g_2 + g_3 g_4)^2 + 2(g_1 g_3 + g_2 g_4)^2 + 2({g_1}^2 + {g_2}^2) + 1\}|\tilde{f}|^4 \\
&\geq 0,
\end{aligned}
$$

$ab - c^2 - d^2 = 0$ holds if and only if $g_3 = g_4 = 0$ or $|\tilde{f}| = 0$. On the other hand, since

$$
\begin{aligned}
a + b &= 2\{(1 + \tilde{g}\tilde{g}^{\dagger_3})^2 \tilde{f}\tilde{f}^{\dagger_3}\}_1 \\
&= 2\{(1 + |\tilde{g}|^2_-)^2 ||\tilde{f}||^2 - 4(g_1 g_4 - g_2 g_3)(1 + ||\tilde{g}||^2)|\tilde{f}|^2_+\} \\
&= 2\{(1 + |\tilde{g}|^2_+)^2 ||\tilde{f}||^2 + 4(g_1 g_4 - g_2 g_3)(1 + ||\tilde{g}||^2)|\tilde{f}|^2_-\} \\
&> 0,
\end{aligned}
$$

we get our conclusion. □

Combining Propositions 4.1 and 4.2, we see that the map $\widetilde{X}$ is degenerate only on $\{\tilde{z} \mid \tilde{g}(\tilde{z}) \in \mathbf{C}(i_1) \text{ or } |\tilde{f}(\tilde{z})| = 0\}$ and changes its type on $\{\tilde{z} \mid 2g_2(\tilde{z})^2 + g_4(\tilde{z})^2 = g_3(\tilde{z})^2, |\tilde{f}(\tilde{z})| \neq 0\}$ without singularities.

5 Transformation of Weierstrass data

Let $\widetilde{X}$, $\tilde{g}$ and $\tilde{f}$ be as in §4. Denote the restriction of $\widetilde{X}$ to $\Omega \cap \mathbf{C}(i_2)$ (resp. $\Omega \cap \mathbf{D}$, $\Omega \cap \mathbf{R}$) by $\widetilde{X}|_{\mathbf{C}(i_2)}$ (resp. $\widetilde{X}|_{\mathbf{D}}$, $\widetilde{X}|_{\mathbf{R}}$) etc. as before. Suppose that $(\tilde{g}, \tilde{f})$ satisfies the condition

(5.1) $$\tilde{g}(\tilde{z}^{\dagger_1}) = \tilde{g}(\tilde{z})^{\dagger_1}, \qquad \tilde{f}(\tilde{z}^{\dagger_1}) = \tilde{f}(\tilde{z})^{\dagger_1} \qquad (\forall \tilde{z} \in \Omega \cap \Omega^{\dagger_1} \subset \mathbf{B}).$$

Then it holds that

(5.2) $$\tilde{g}(z),\ \tilde{f}(z) \in \mathbf{C}(i_2) \qquad (\forall z = x_1 + i_2 x_3 \in \Omega \cap \mathbf{C}(i_2)),$$

and (the projection of) $\widetilde{X}|_{\mathbf{C}(i_2)} = \Phi_1|_{\mathbf{C}(i_2)} = \mathrm{Re}\ \Phi|_{\mathbf{C}(i_2)}$ (to $\mathbf{R}^{2,1}$) is a spacelike maximal immersion from $\Omega \cap \mathbf{C}(i_2)$ into $\mathbf{R}^{2,1} \subset \mathbf{C}(i_1)^3$. By (4.3) (resp. (4.4)) and (5.2), the induced metric is given by

$$
\begin{aligned}
h_-|_{\mathbf{C}(i_2)} &= 4(\mathrm{Im}_{\mathbf{C}(i_2)}\widetilde{g})^2|\widetilde{f}|^2_{\mathbf{C}(i_2)}|dz|^2_{\mathbf{C}(i_2)} \;=\; 4{g_3}^2({f_1}^2+{f_3}^2)({dx_1}^2+{dx_3}^2),\\
(\text{resp. } h_+|_{\mathbf{C}(i_2)} &= (1+|\widetilde{g}|^2_{\mathbf{C}(i_2)})^2|\widetilde{f}|^2_{\mathbf{C}(i_2)}|dz|^2_{\mathbf{C}(i_2)} + \mathrm{Re}\,\{(1+\widetilde{g}^2)^2\widetilde{f}^2dz^2\}\\
&= 2[2\{\mathrm{Re}(\widetilde{g}\widetilde{f}dz)\}^2+\{\mathrm{Re}(\widetilde{f}dz)\}^2+\{\mathrm{Re}(\widetilde{g}^2\widetilde{f}dz)\}^2]\;).
\end{aligned}
$$

By (5.2), it holds that $\widetilde{g}=\widetilde{g}^{\dagger_1}$ and $\widetilde{g}^{\dagger_2}=\widetilde{g}^{\dagger_3}$ on $\Omega\cap\mathbf{C}(i_2)$, and hence the Gauss map of $\widetilde{X}|_{\mathbf{C}(i_2)}$ is given by

$$
\widetilde{G}^{\dagger_1} \;=\; \frac{1}{-i_2(\widetilde{g}-\widetilde{g}^{\dagger_2})}\begin{pmatrix} |\widetilde{g}|^2_{\mathbf{C}(i_2)}-1\\ -(\widetilde{g}+\widetilde{g}^{\dagger_2})\\ -(|\widetilde{g}|^2_{\mathbf{C}(i_2)}+1)\end{pmatrix} \;=\; \frac{1}{2\mathrm{Im}_{\mathbf{C}(i_2)}\,\widetilde{g}}\begin{pmatrix} |\widetilde{g}|^2_{\mathbf{C}(i_2)}-1\\ -2\mathrm{Re}\,\widetilde{g}\\ -(|\widetilde{g}|^2_{\mathbf{C}(i_2)}+1)\end{pmatrix}.
$$

By using the Möbius transformation of $\widetilde{g}|_{\mathbf{C}(i_2)}$, we can describe the correspondence between this representation and the usual one (cf. [13, Theorem 1.1])

$$
(5.3)\qquad X_{\max}(z) \;=\; \mathrm{Re}\int^z {}^t((1+{g_{\max}}^2)f_{\max},\, i_2(1-{g_{\max}}^2)f_{\max},\, -2g_{\max}f_{\max})dz
$$

as follows:

Lemma 5.1. *Let $(\widetilde{g},\widetilde{f})$ be as above. Set*

$$
g_{\max} \;:=\; \left.\frac{\widetilde{g}-i_2}{\widetilde{g}+i_2}\right|_{\mathbf{C}(i_2)}, \qquad f_{\max} \;:=\; \left.\frac{-(\widetilde{g}+i_2)^2}{2}\widetilde{f}\,\right|_{\mathbf{C}(i_2)}.
$$

Then $(g_{\max}, f_{\max})$ gives a representation of the same immersion by the usual Enneper-Weierstrass representation formula (5.3)*. Conversely, for any Weierstrass data $(g_{\max}, f_{\max})$ for* (5.3)*, set*

$$
g \;:=\; -i_2\frac{g_{\max}+1}{g_{\max}-1}, \qquad f \;:=\; \frac{(g_{\max}-1)^2}{2}f_{\max},
$$

and denote the bicomplex extensions of g and f by $\widetilde{g}$ and $\widetilde{f}$. Then $(g,f)=(\widetilde{g}|_{\mathbf{C}(i_2)},\widetilde{f}|_{\mathbf{C}(i_2)})$ gives a representation $\widetilde{X}|_{\mathbf{C}(i_2)}=\mathrm{Re}\,\Phi|_{\mathbf{C}(i_2)}$ with (4.1) *for the same immersion.*

Suppose that $(\widetilde{g},\widetilde{f})$ satisfies the condition

$$
(5.4)\qquad \widetilde{g}(\widetilde{z}^{\dagger_3}) \;=\; \widetilde{g}(\widetilde{z})^{\dagger_3}, \qquad \widetilde{f}(\widetilde{z}^{\dagger_3}) \;=\; \widetilde{f}(\widetilde{z})^{\dagger_3} \qquad (\forall\widetilde{z}\in\Omega\cap\Omega^{\dagger_3}\subset\mathbf{B}).
$$

Then it holds that

$$(5.5)\qquad \widetilde{g}(\check{z}),\ \widetilde{f}(\check{z}) \in \mathbf{D} \qquad (\forall \check{z} = x_1 + jx_4 \in \Omega \cap \mathbf{D}),$$

and (the projection of) $\widetilde{X}|_{\mathbf{D}} = \Phi_1|_{\mathbf{D}} = \mathrm{Re}\ \Phi|_{\mathbf{D}}$ (to $\mathbf{R}^{2,1}$) is a timelike minimal immersion from $\Omega \cap \mathbf{D}$ into $\mathbf{R}^{2,1} \subset \mathbf{C}(i_1)^3$. By (4.3) (resp. (4.4)) and (5.5), the induced metric is given by

$$\begin{aligned} h_-|_{\mathbf{D}} &= -4(\mathrm{Im}_{\mathbf{D}}\widetilde{g})^2|\widetilde{f}|^2_{\mathbf{D}}|d\check{z}|^2_{\mathbf{D}} = -4{g_4}^2({f_1}^2 - {f_4}^2)({dx_1}^2 - {dx_4}^2), \\ (\text{resp. } h_+|_{\mathbf{D}} &= (1 + |\widetilde{g}|^2_{\mathbf{D}})^2|\widetilde{f}|^2_{\mathbf{D}}|d\check{z}|^2_{\mathbf{D}} + \mathrm{Re}\ \{(1+\widetilde{g}^2)^2\widetilde{f}^2 d\check{z}^2\} \\ &= 2[2\{\mathrm{Re}(\widetilde{g}\widetilde{f}d\check{z})\}^2 + \{\mathrm{Re}(\widetilde{f}d\check{z})\}^2 + \{\mathrm{Re}(\widetilde{g}^2\widetilde{f}d\check{z})\}^2]\ ). \end{aligned}$$

By (5.5), it holds that $\widetilde{g} = \widetilde{g}^{\dagger_3}$ and $\widetilde{g}^{\dagger_1} = \widetilde{g}^{\dagger_2}$ on $\Omega \cap \mathbf{D}$, and hence the Gauss map of $\widetilde{X}|_{\mathbf{D}}$ is given by

$$(i_1\widetilde{G})^{\dagger_1} = \frac{1}{j(\widetilde{g} - \widetilde{g}^{\dagger_2})}\begin{pmatrix} |\widetilde{g}|^2_{\mathbf{D}} - 1 \\ -(\widetilde{g} + \widetilde{g}^{\dagger_2}) \\ -(|\widetilde{g}|^2_{\mathbf{D}} + 1) \end{pmatrix} = \frac{1}{2\mathrm{Im}_{\mathbf{D}}\ \widetilde{g}}\begin{pmatrix} |\widetilde{g}|^2_{\mathbf{D}} - 1 \\ -2\mathrm{Re}\ \widetilde{g} \\ -(|\widetilde{g}|^2_{\mathbf{D}} + 1) \end{pmatrix}.$$

In this case also, by using the Möbius transformation of $\widetilde{g}|_{\mathbf{D}}$, we can describe the correspondence between this representation and another type of ones (cf. [19], [1, Fact A.7])

$$(5.6_{\pm})\qquad \check{X}_{\min}(z) = \mathrm{Re}\int^{\check{z}} {}^t(2\check{g}_{\min}\check{f}_{\min}, \pm j(1 - \check{g}^2_{\min})\check{f}_{\min}, -(1 + \check{g}^2_{\min})\check{f}_{\min})d\check{z}$$

as follows:

Lemma 5.2. *Let* $(\widetilde{g}, \widetilde{f})$ *be as above. Set*

$$\check{g}_{\min} := \left.\frac{\widetilde{g} \pm j}{\widetilde{g} \mp j}\right|_{\mathbf{D}}, \qquad \check{f}_{\min} := \left.\frac{-(\widetilde{g} \mp j)^2}{2}\widetilde{f}\,\right|_{\mathbf{D}}.$$

Then $(\check{g}_{\min}, \check{f}_{\min})$ *gives a representation of the same immersion by the usual Enneper-Weierstrass type representation formula* $(5.6_{\pm})$*. Conversely, for any Weierstrass data* $(\check{g}_{\min}, \check{f}_{\min})$ *for* $(5.6_{\pm})$*, set*

$$\check{g} := \pm j\frac{\check{g}_{\min} + 1}{\check{g}_{\min} - 1}, \qquad \check{f} := \frac{-(\check{g}_{\min} - 1)^2}{2}\check{f}_{\min},$$

and denote the bicomplex extensions of $\check{g}$ *and* $\check{f}$ *by* $\widetilde{g}$ *and* $\widetilde{f}$. *Then* $(\check{g}, \check{f}) = (\widetilde{g}|_{\mathbf{D}}, \widetilde{f}|_{\mathbf{D}})$ *gives a representation* $\widetilde{X}|_{\mathbf{D}} = \mathrm{Re}\,\Phi|_{\mathbf{D}}$ *with* (4.1) *for the same immersion.*

Suppose that $(\widetilde{g}, \widetilde{f})$ satisfies the condition

$$\widetilde{g}(\widetilde{z}^{\dagger_2}) = \widetilde{g}(\widetilde{z})^{\dagger_2}, \qquad \widetilde{f}(\widetilde{z}^{\dagger_2}) = \widetilde{f}(\widetilde{z})^{\dagger_2} \qquad (\forall \widetilde{z} \in \Omega \cap \Omega^{\dagger_2} \subset \mathbf{B}). \tag{5.7}$$

Then, since $\Phi(\widetilde{z}^{\dagger_2}) = \Phi(\widetilde{z})^{\dagger_2} + C$ holds for some constant vector $C \in \mathbf{B}^3$, as we have already observed in the proof of Theorem 3.2, $\widetilde{X}(\zeta_1 + i_2\zeta_2)$ is even with respect to ζ_2, both $\widetilde{X}|_{\mathbf{C}(i_2)}$ and $\widetilde{X}|_{\mathbf{D}}$ are double surfaces with fold singularities on $\Omega \cap \mathbf{R}$, and $\widetilde{X}|_{\mathbf{D}}$ is an analytic extension of $\widetilde{X}|_{\mathbf{C}(i_2)}$ even if we do not assume that $\widetilde{X}|_{\mathbf{C}(i_2)}$ (resp. $\widetilde{X}|_{\mathbf{D}}$) is maximal (resp. minimal). The image of this extension is nondegenerate on any point on the fold singularities such that $\widetilde{f} \neq 0$ and $\widetilde{g}_{\widetilde{z}} \neq 0$.

The map $X_{\max} : (\mathbf{C}(i_2) \supset)\Omega_0 \to \mathbf{R}^{2,1}$ defined by (5.3) is a double surface with fold singularities if its Weierstrass data $(g_{\max}, f_{\max})$ satisfies

$$g_{\max} \circ I(z) = \frac{1}{g_{\max}(z)^{\dagger_2}}, \qquad f_{\max} \circ I(z) = (g_{\max}(z)^2 f_{\max}(z))^{\dagger_2} \qquad (\forall z \in \Omega_0).$$

for some antiholomorphic involution $I(z)$ of $\mathbf{C}(i_2)$. This condition with $I(z) = z^{\dagger_2}$ is equivalent with the condition (5.7) for the corresponding $(\widetilde{g}, \widetilde{f})$. On the other hand, the map $\check{X}_{\min} : (\mathbf{D} \supset)\check{\Omega}_0 \to \mathbf{R}^{2,1}$ defined by (5.6) is a double surface with fold singularities if its Weierstrass data $(\check{g}_{\min}, \check{f}_{\min})$ satisfies

$$\check{g}_{\min} \circ \check{I}(\check{z}) = \frac{1}{\check{g}_{\min}(\check{z})^{\dagger_2}}, \qquad \check{f}_{\min} \circ \check{I}(\check{z}) = (\check{g}_{\min}(\check{z})^2 \check{f}_{\min}(\check{z}))^{\dagger_2} \qquad (\forall \check{z} \in \check{\Omega}_0)$$

for some antiparaholomorphic involution $\check{I}(\check{z})$ of $\mathbf{D}$. This condition with $\check{I}(\check{z}) = \check{z}^{\dagger_2}$ is also equivalent with the condition (5.7) for the corresponding $(\widetilde{g}, \widetilde{f})$. Conversely, if $(\widetilde{g}, \widetilde{f})$ satisfies (5.1), (5.4) and hence (5.7) also, then it does not only satisfy both (5.2) and (5.5) but also satisfies

$$\widetilde{g}(x),\ \widetilde{f}(x) \in \mathbf{R} \qquad (\forall x = x_1 \in \Omega \cap \mathbf{R}), \tag{5.8}$$

and hence both corresponding $X_{\max}$ and $\check{X}_{\min}$ are double surfaces with fold singularities on $\Omega_0 \cap \mathbf{R}$ and $\check{\Omega}_0 \cap \mathbf{R}$ respectively. Now, by combining Lemmas 5.1 and 5.2, we get the following formula of the transformation of Weierstrass data:

Theorem 5.3. *Let $X_{\max} : \Omega_0 \to \mathbf{R}^{2,1}$ be a spacelike maximal immersion defined by* (5.3) *with Weierstrass data $(g_{\max}, f_{\max})$. Suppose that $X_{\max}$ is a double surface with fold singularities on $\Omega_0 \cap \mathbf{R} \subset \mathbf{C}(i_2)$ and satisfies $|g_{\max}(x)|_{\mathbf{C}(i_2)} = 1$ $(\forall x \in \Omega_0 \cap \mathbf{R})$. Denote the bicomplex extension of $(g_{\max}, f_{\max})$ by $(\widetilde{g}_{\max}, \widetilde{f}_{\max})$. Then the Weierstrass data $(\check{g}_{\min}, \check{f}_{\min})$ of a timelike minimal immersion $\check{X}_{\min}$ defined by* (5.6) *with the common fold singularities is given by*

$$\check{g}_{\min} = \left.\frac{\pm i_2 j \widetilde{g}_{\max} + 1}{\widetilde{g}_{\max} \pm i_2 j}\right|_{\mathbf{D}}, \qquad \check{f}_{\min} = \left.\frac{(\widetilde{g}_{\max} \pm i_2 j)^2}{\pm 2 i_2 j} \widetilde{f}_{\max}\right|_{\mathbf{D}}.$$

It is well known that $\check{X}_{\min}$ satisfies

$$\check{X}_{\min}(x_1 + j x_4) = \frac{\check{X}_{\min}(x_1 + x_4) + \check{X}_{\min}(x_1 - x_4)}{2}, \tag{5.9}$$

and we can derive the same assertion as above also by using this formula with the condition $\check{X}_{\min} = X_{\max}$ on $\check{\Omega}_0 \cap \mathbf{R}$. As we shall see later, we can regard $\check{X}_{\min}$ as a global extension of $X_{\max}$ by considering the bicomplex extension $\widetilde{X}$, and hence $(\check{g}_{\min}, \check{f}_{\min})$ also has a global meaning.

Example 5.4. For any polynomial $\psi(z) = \sum_{k=0}^{n} (\alpha_k + i_2 \beta_k) z^k$ on $\mathbf{C}(i_2)$ such that $\alpha_k, \beta_k \in \mathbf{R}$ $(k = 0, \ldots, n)$, set $\overline{\psi}(z) := \psi(z^{\dagger_2})^{\dagger_2} = \sum_{k=0}^{n} (\alpha_k - i_2 \beta_k) z^k$, and denote $\widetilde{\psi}|_{\mathbf{D}}$ by $\check{\psi}$, namely $\check{\psi}(\check{z}) := \sum_{k=0}^{n} (\alpha_k + i_2 \beta_k) \check{z}^k$. Moreover, set $\hat{\psi}(\check{z}) := \sum_{k=0}^{n} (\alpha_k \mp j \beta_k) \check{z}^k$. Then $\hat{\overline{\psi}}(\check{z}) := \sum_{k=0}^{n} (\alpha_k \pm j \beta_k) \check{z}^k$. Note here that

$$\begin{aligned}
\pm i_2 j \check{\overline{\psi}}(\check{z}) + \check{\psi}(\check{z}) &= \sum_{k=0}^{n} \{\pm i_2 j (\alpha_k + i_2 \beta_k)^{\dagger_2} + (\alpha_k + i_2 \beta_k)\} \check{z}^k \\
&= (1 \pm i_2 j) \sum_{k=0}^{n} (\alpha_k \pm j \beta_k) \check{z}^k = (1 \pm i_2 j) \hat{\overline{\psi}}(\check{z}), \\
\check{\overline{\psi}}(\check{z}) \pm i_2 j \check{\psi}(\check{z}) &= \sum_{k=0}^{n} \{(\alpha_k + i_2 \beta_k)^{\dagger_2} \pm i_2 j (\alpha_k + i_2 \beta_k)\} \check{z}^k \\
&= (1 \pm i_2 j) \sum_{k=0}^{n} (\alpha_k \mp j \beta_k) \check{z}^k = (1 \pm i_2 j) \hat{\psi}(\check{z}).
\end{aligned}$$

In particular, if $\overline{\psi} = \psi$, then $\psi(z) = \sum_{k=0}^{n} \alpha_k z^k$ and hence $\hat{\psi}(\check{z}) = \sum_{k=0}^{n} \alpha_k \check{z}^k = \check{\psi}(\check{z})$.

Let $X_{\max}$ be as in Theorem 5.3. Consider the case that both $g_{\max}$ and $f_{\max}$ are rational functions on $\mathbf{C}(i_2)$. Let g_a, g_b, f_a, f_b be polynomials on $\mathbf{C}(i_2)$ satisfying $g_{\max} = g_a/g_b$ and $f_{\max} = f_a/f_b$. By $g_{\max}(z^{\dagger_2}) = 1/g_{\max}(z)^{\dagger_2}$ and $f_{\max}(z^{\dagger_2}) = (g_{\max}(z)^2 f_{\max}(z))^{\dagger_2}$, we have

$$\frac{g_a(z^{\dagger_2})}{g_b(z^{\dagger_2})} = \frac{g_b(z)^{\dagger_2}}{g_a(z)^{\dagger_2}}, \qquad \frac{f_a(z^{\dagger_2})}{f_b(z^{\dagger_2})} = \frac{(g_a(z)^2 f_a(z))^{\dagger_2}}{(g_b(z)^2 f_b(z))^{\dagger_2}}.$$

Hence we can choose g_a and g_b so that $g_a(z) = g_b(z^{\dagger_2})^{\dagger_2} = \overline{g_b}(z)$ and hence $g_{\max} = \overline{g_b}/g_b$. Now we have

$$\frac{f_a(z)}{g_b(z)^2 f_b(z)} = \frac{f_a(z^{\dagger_2})^{\dagger_2}}{(g_b(z^{\dagger_2})^2 f_b(z^{\dagger_2}))^{\dagger_2}} = \frac{\overline{f_a}(z)}{\overline{g_b}(z)^2 \overline{f_b}(z)},$$

and hence we can also choose f_a and f_b so that $f_a(z) = \overline{f_a}(z)$ and $g_b(z)^2 f_b(z) = \overline{g_b}(z)^2 \overline{f_b}(z)$. By applying Theorem 5.3, we see that the Weierstrass data of $\check{X}_{\min}$ is given by

$$\begin{aligned}
\check{g}_{\min} &= \frac{\pm i_2 j \check{\overline{g_b}} + \check{g}_b}{\check{\overline{g_b}} \pm i_2 j \check{g}_b} = \frac{(1 \pm i_2 j)\hat{\overline{g_b}}}{(1 \pm i_2 j)\hat{g}_b} = \frac{\hat{g}_a}{\hat{g}_b}, \\
\check{f}_{\min} &= \frac{(\check{\overline{g_b}} \pm i_2 j \check{g}_b)^2}{\pm 2 i_2 j} \frac{\check{f}_a}{\check{g}_b^2 \check{f}_b} = \frac{(1 \pm i_2 j)^2 \hat{g}_b^2}{\pm 2 i_2 j} \frac{\hat{f}_a}{\hat{g}_b^2 \hat{f}_b} = \frac{\hat{f}_a}{\hat{f}_b}.
\end{aligned}$$

This formula is valid also for the case that both $g_{\max}$ and $f_{\max}$ are meromorphic functions on a domain in $\mathbf{C}(i_2)$.

However, if we employ the representation $\widetilde{X} = \Phi_1 + i_1 \Phi_2$ with (4.1), then we do not need such a transformation, and both $X_{\max}$ and $\check{X}_{\min}$ are expressed as $X_{\max} = \widetilde{X}|_{\mathbf{C}(i_2)}$ and $\check{X}_{\min} = \widetilde{X}|_{\mathbf{D}}$ by using the common data.

6 Bicomplex extensions of zero mean curvature surfaces in $\mathbf{R}^{2,2}$

In §§6-7, we restrict our attention to the case that $(N_+, N_-) = (2, 2)$ and $\Phi = {}^t(\varphi^1, \varphi^2, \varphi^3, \varphi^4)$ satisfies

$$(\varphi^1_{\tilde{z}})^2 + (\varphi^2_{\tilde{z}})^2 - (\varphi^3_{\tilde{z}})^2 - (\varphi^4_{\tilde{z}})^2 = 0.$$

By the quite similar calculation and consideration as in §§4-5, we can show the corresponding results for this case. Set

$$\widetilde{f} := \frac{\varphi^1_{\tilde{z}} + \varphi^3_{\tilde{z}}}{2},$$

$$\widetilde{g}_I := \frac{\varphi^2_{\widetilde{z}} + \varphi^4_{\widetilde{z}}}{\varphi^1_{\widetilde{z}} + \varphi^3_{\widetilde{z}}} = \frac{-\varphi^1_{\widetilde{z}} + \varphi^3_{\widetilde{z}}}{\varphi^2_{\widetilde{z}} - \varphi^4_{\widetilde{z}}},$$
$$\widetilde{g}_{II} := \frac{\varphi^2_{\widetilde{z}} - \varphi^4_{\widetilde{z}}}{\varphi^1_{\widetilde{z}} + \varphi^3_{\widetilde{z}}} = \frac{-\varphi^1_{\widetilde{z}} + \varphi^3_{\widetilde{z}}}{\varphi^2_{\widetilde{z}} + \varphi^4_{\widetilde{z}}}.$$

Then $\widetilde{f}$ (resp. $\widetilde{g}_I$, $\widetilde{g}_{II}$) is a bicomplex holomorphic function on Ω (resp. $\Omega \setminus \{|\widetilde{f}| = 0\}$), and $\Phi = \Phi_1 + i_1\Phi_2 + i_2\Phi_3 + j\Phi_4$ is rewritten as follows:

$$\text{(6.1)} \qquad \Phi(\widetilde{z}) = \int^{\widetilde{z}} {}^t((1 - \widetilde{g}_I\widetilde{g}_{II})\widetilde{f}, (\widetilde{g}_I + \widetilde{g}_{II})\widetilde{f}, (1 + \widetilde{g}_I\widetilde{g}_{II})\widetilde{f}, (\widetilde{g}_I - \widetilde{g}_{II})\widetilde{f})d\widetilde{z}.$$

In this case also, we use the notation $\widetilde{X} = \Phi_1 + i_1\Phi_2$ instead of F. Conversely, for any triplet $(\widetilde{g}_I, \widetilde{g}_{II}, \widetilde{f})$ of bicomplex holomorphic functions on a domain $\Omega \subset \mathbf{B}$, the map $\widetilde{X} = \Phi_1 + i_1\Phi_2$ with (6.1) is locally well-defined.

Denote the metric h induced by $\widetilde{X}$ from the standard metric on $\mathbf{C}(i_1)^4$ regarded as $\mathbf{R}^{4,4}$ (resp. $\mathbf{R}^8$) by $h_\flat$ (resp. $h_\sharp$), and $(\widetilde{g}_I)_\ell$, $(\widetilde{g}_{II})_\ell$ and $(\widetilde{f})_\ell$ by $g_{I,\ell}$, $g_{II,\ell}$ and f_ℓ respectively ($\ell = 1, 2, 3, 4$). Then we get the following

Proposition 6.1. *Set* $\Delta_1 := ({g_{I,2}}^2 - {g_{I,3}}^2)({g_{II,2}}^2 - {g_{II,3}}^2) - ({g_{I,2}}^2 + {g_{I,4}}^2)({g_{II,2}}^2 + {g_{II,4}}^2)$, $\Delta_2 := g_{I,2}g_{II,2} + g_{I,3}g_{II,3}$ *and* $\Delta_3 := g_{I,2}g_{II,3} + g_{I,3}g_{II,2}$. *For any point in* $\Omega \setminus \{\widetilde{f} = 0\}$*, the metric* $h_\flat$ *induced by* $\mathbf{R}^{4,4}$ *satisfies the following*:

(1) $h_\flat$ *is positive* (*resp. negative*) *definite if* $|\widetilde{f}| \neq 0$, $\Delta_1 > 0$ *and* $\Delta_2 > 0$ (*resp.* < 0).

(2) $h_\flat$ *is rank-*2 *positive* (*resp. negative*) *semidefinite if* (i) *or* (ii) *holds*:

(i) $|\widetilde{f}| \neq 0$, $\Delta_1 = 0$ *and* $\Delta_2 > 0$ (*resp.* < 0) ; (ii) $|\widetilde{f}|_\pm = 0$ *and* $\Delta_2 \mp \Delta_3 \neq 0$.

(3) $h_\flat$ *is* 0 *if* (i) *or* (ii) *holds*:

(i) $|\widetilde{f}| \neq 0$, $\Delta_1 = 0$ *and* $\Delta_2 = 0$, *in another word, satisfying one of the following conditions*: $\widetilde{g}_I \in \mathbf{R}$; $\widetilde{g}_{II} \in \mathbf{R}$; $\widetilde{g}_I \in \mathbf{C}(i_2)$ *and* $\widetilde{g}_{II} \in \mathbf{D}$; $\widetilde{g}_I \in \mathbf{D}$ *and* $\widetilde{g}_{II} \in \mathbf{C}(i_2)$;

(ii) $|\widetilde{f}|_\pm = 0$ *and* $\Delta_2 \mp \Delta_3 = 0$.

(4) $h_\flat$ *is neutral if* $|\widetilde{f}| \neq 0$ *and* $\Delta_1 < 0$.

Proof. Recall (4.2). The coefficients a, b, c, d of the metric $(h_{\flat,\ell m})_{\ell,m=1,2,3,4}$ induced by $\mathbf{R}^{4,4}$ is given by

$$\langle(\Phi_\ell)_{x_1}, (\Phi_m)_{x_1}\rangle = 2\{(\widetilde{g}_I\widetilde{f})_\ell(\widetilde{g}_{II}\widetilde{f})_m + (\widetilde{g}_{II}\widetilde{f})_\ell(\widetilde{g}_I\widetilde{f})_m - (\widetilde{f})_\ell(\widetilde{g}_I\widetilde{g}_{II}\widetilde{f})_m - (\widetilde{g}_I\widetilde{g}_{II}\widetilde{f})_\ell(\widetilde{f})_m\},$$

and hence

$$
(6.2)\qquad \begin{cases} a &= -\{(\widetilde{g}_I-\widetilde{g}_I^{\dagger_1})(\widetilde{g}_{II}-\widetilde{g}_{II}^{\dagger_1})\widetilde{f}\widetilde{f}^{\dagger_1}\}_1-\{(\widetilde{g}_I-\widetilde{g}_I^{\dagger_3})(\widetilde{g}_{II}-\widetilde{g}_{II}^{\dagger_3})\widetilde{f}\widetilde{f}^{\dagger_3}\}_1,\\ b &= \{(\widetilde{g}_I-\widetilde{g}_I^{\dagger_1})(\widetilde{g}_{II}-\widetilde{g}_{II}^{\dagger_1})\widetilde{f}\widetilde{f}^{\dagger_1}\}_1-\{(\widetilde{g}_I-\widetilde{g}_I^{\dagger_3})(\widetilde{g}_{II}-\widetilde{g}_{II}^{\dagger_3})\widetilde{f}\widetilde{f}^{\dagger_3}\}_1,\\ c &= \{(\widetilde{g}_I-\widetilde{g}_I^{\dagger_1})(\widetilde{g}_{II}-\widetilde{g}_{II}^{\dagger_1})\widetilde{f}\widetilde{f}^{\dagger_1}\}_3,\\ d &= -\{(\widetilde{g}_I-\widetilde{g}_I^{\dagger_3})(\widetilde{g}_{II}-\widetilde{g}_{II}^{\dagger_3})\widetilde{f}\widetilde{f}^{\dagger_3}\}_4. \end{cases}
$$

Since

$$
\begin{aligned}
ab-c^2-d^2 &= \{|(\widetilde{g}_I-\widetilde{g}_I^{\dagger_3})(\widetilde{g}_{II}-\widetilde{g}_{II}^{\dagger_3})|^2_{\mathbf{D}}-|(\widetilde{g}_I-\widetilde{g}_I^{\dagger_1})(\widetilde{g}_{II}-\widetilde{g}_{II}^{\dagger_1})|^2_{\mathbf{C}(i_2)}\}|\widetilde{f}|^4\\
&= 8\Delta_1\cdot|\widetilde{f}|^4,
\end{aligned}
$$

$ab-c^2-d^2>0$ (resp. $=,<$) holds if and only if $|\widetilde{f}|\neq 0$ and $\Delta_1>0$ (resp. $|\widetilde{f}|=0$ or $\Delta_1=0$, $|\widetilde{f}|\neq 0$ and $\Delta_1<0$). In particular, if $ab-c^2-d^2\geq 0$, then it holds that

$$
\begin{aligned}
{\Delta_2}^2-{\Delta_3}^2 &= ({g_{I,2}}^2-{g_{I,3}}^2)({g_{II,2}}^2-{g_{II,3}}^2)\\
&\geq ({g_{I,2}}^2+{g_{I,4}}^2)({g_{II,2}}^2+{g_{II,4}}^2)\ \geq\ 0.
\end{aligned}
$$

On the other hand, since

$$
\begin{aligned}
a+b &= -2\{(\widetilde{g}_I-\widetilde{g}_I^{\dagger_3})(\widetilde{g}_{II}-\widetilde{g}_{II}^{\dagger_3})\widetilde{f}\widetilde{f}^{\dagger_3}\}_1\\
&= 8\{\Delta_2||\widetilde{f}||^2-2\Delta_3(f_1f_4-f_2f_3)\}\\
&= 8\{(\Delta_2-\Delta_3)||\widetilde{f}||^2+\Delta_3|\widetilde{f}|^2_+\}\\
&= 8\{(\Delta_2+\Delta_3)||\widetilde{f}||^2-\Delta_3|\widetilde{f}|^2_-\},
\end{aligned}
$$

if $|\widetilde{f}|\neq 0$ and $a+b=0$ holds, then $\Delta_2=\Delta_3=0$. If $\Delta_2>0$ (resp. <0), then $\Delta_2>|\Delta_3|\geq 0$ or $\Delta_2=|\Delta_3|>0$ (resp. $\Delta_2<-|\Delta_3|\leq 0$ or $\Delta_2=-|\Delta_3|<0$) and hence $a+b>0$ (resp. $a+b<0$). Combining these conditions, we get our conclusion. □

Proposition 6.2. *For any point in $\Omega\setminus\{\widetilde{f}=0\}$, the metric $h_\sharp$ induced by $\mathbf{R}^8$ is rank-2 positive semidefinite if $\widetilde{g}_I,\widetilde{g}_{II}\in\mathbf{C}(i_1)$ or $|\widetilde{f}|=0$. Otherwise $h_\sharp$ is positive definite.*

Proof. Recall (4.2). The coefficients a,b,c,d of the metric $(h_{\sharp,\ell m})_{\ell,m=1,2,3,4}$ induced by $\mathbf{R}^8$ is given by

$$
\langle(\Phi_\ell)_{x_1},(\Phi_m)_{x_1}\rangle\ =\ 2\{(\widetilde{g}_I\widetilde{f})_\ell(\widetilde{g}_I\widetilde{f})_m+(\widetilde{g}_{II}\widetilde{f})_\ell(\widetilde{g}_{II}\widetilde{f})_m+(\widetilde{f})_\ell(\widetilde{f})_m+(\widetilde{g}_I\widetilde{g}_{II}\widetilde{f})_\ell(\widetilde{g}_I\widetilde{g}_{II}\widetilde{f})_m\},
$$

and hence

$$(6.3)\qquad \begin{cases} a &= \{(1+\tilde g_I\tilde g_I^{\dagger_1})(1+\tilde g_{II}\tilde g_{II}^{\dagger_1})\tilde f\tilde f^{\dagger_1}\}_1 + \{(1+\tilde g_I\tilde g_I^{\dagger_3})(1+\tilde g_{II}\tilde g_{II}^{\dagger_3})\tilde f\tilde f^{\dagger_3}\}_1,\\ b &= -\{(1+\tilde g_I\tilde g_I^{\dagger_1})(1+\tilde g_{II}\tilde g_{II}^{\dagger_1})\tilde f\tilde f^{\dagger_1}\}_1 + \{(1+\tilde g_I\tilde g_I^{\dagger_3})(1+\tilde g_{II}\tilde g_{II}^{\dagger_3})\tilde f\tilde f^{\dagger_3}\}_1,\\ c &= -\{(1+\tilde g_I\tilde g_I^{\dagger_1})(1+\tilde g_{II}\tilde g_{II}^{\dagger_1})\tilde f\tilde f^{\dagger_1}\}_3,\\ d &= \{(1+\tilde g_I\tilde g_I^{\dagger_3})(1+\tilde g_{II}\tilde g_{II}^{\dagger_3})\tilde f\tilde f^{\dagger_3}\}_4. \end{cases}$$

Since

$$\begin{aligned} ab-c^2-d^2 &= \{|(1+\tilde g_I\tilde g_I^{\dagger_3})(1+\tilde g_{II}\tilde g_{II}^{\dagger_3})|^2_{\mathbf{D}} - |(1+\tilde g_I\tilde g_I^{\dagger_1})(1+\tilde g_{II}\tilde g_{II}^{\dagger_1})|^2_{\mathbf{C}(i_2)}\}|\tilde f|^4\\ &= 4[({g_{I,3}}^2+{g_{I,4}}^2)\{({g_{II,1}}^2-{g_{II,4}}^2)^2+({g_{II,2}}^2-{g_{II,3}}^2)^2\\ &\qquad +2(g_{II,1}g_{II,2}+g_{II,3}g_{II,4})^2+2(g_{II,1}g_{II,3}+g_{II,2}g_{II,4})^2+2({g_{II,1}}^2+{g_{II,2}}^2)+1\}\\ &\quad +({g_{II,3}}^2+{g_{II,4}}^2)\{({g_{I,1}}^2-{g_{I,4}}^2)^2+({g_{I,2}}^2-{g_{I,3}}^2)^2\\ &\qquad +2(g_{I,1}g_{I,2}+g_{I,3}g_{I,4})^2+2(g_{I,1}g_{I,3}+g_{I,2}g_{I,4})^2+2({g_{I,1}}^2+{g_{I,2}}^2)+1\}]|\tilde f|^4\\ &\geq 0, \end{aligned}$$

$ab-c^2-d^2=0$ holds if and only if $g_{I,3}=g_{I,4}=g_{II,3}=g_{II,4}=0$ or $|\tilde f|=0$. On the other hand, since

$$\begin{aligned} a+b &= 2\{(1+\tilde g_I\tilde g_I^{\dagger_3})(1+\tilde g_{II}\tilde g_{II}^{\dagger_3})\tilde f\tilde f^{\dagger_3}\}_1\\ &= 2[(1+|\tilde g_I|^2_-)(1+|\tilde g_{II}|^2_-)||\tilde f||^2\\ &\qquad -2\{(g_{I,1}g_{I,4}-g_{I,2}g_{I,3})(1+||\tilde g_{II}||^2)+(g_{II,1}g_{II,4}-g_{II,2}g_{II,3})(1+||\tilde g_I||^2)\}|\tilde f|^2_+]\\ &= 2[(1+|\tilde g_I|^2_+)(1+|\tilde g_{II}|^2_+)||\tilde f||^2\\ &\qquad +2\{(g_{I,1}g_{I,4}-g_{I,2}g_{I,3})(1+||\tilde g_{II}||^2)+(g_{II,1}g_{II,4}-g_{II,2}g_{II,3})(1+||\tilde g_I||^2)\}|\tilde f|^2_-]\\ &> 0, \end{aligned}$$

we get our conclusion. □

Combining Propositions 6.1 and 6.2, we see that the map $\widetilde{X}$ is degenerate only on $\{\tilde z \mid \tilde g_I(\tilde z),\ \tilde g_{II}(\tilde z) \in \mathbf{C}(i_1) \text{ or } |\tilde f(\tilde z)|=0\}$ and changes its type on $\{\tilde z \mid \Delta_1=0, \Delta_2\neq 0, |\tilde f(\tilde z)|\neq 0\}$ without singularities.

7 Degeneration of negative definite domains and singularities

Let $\widetilde{X}$, $\widetilde{g}_I$, $\widetilde{g}_{II}$ and $\widetilde{f}$ be as in §6. Denote the restriction of $\widetilde{X}$ to $\Omega \cap \mathbf{C}(i_2)$ (resp. $\Omega \cap \mathbf{D}$, $\Omega \cap \mathbf{R}$) by $\widetilde{X}|_{\mathbf{C}(i_2)}$ (resp. $\widetilde{X}|_{\mathbf{D}}$, $\widetilde{X}|_{\mathbf{R}}$) etc. as before. If $(\widetilde{g}_I, \widetilde{g}_{II}, \widetilde{f})$ satisfies the condition (5.1) with $\widetilde{g} = \widetilde{g}_I, \widetilde{g}_{II}$, then (5.2) holds with $\widetilde{g} = \widetilde{g}_I, \widetilde{g}_{II}$, and (the projection of) $\widetilde{X}|_{\mathbf{C}(i_2)} = \Phi_1|_{\mathbf{C}(i_2)} = \text{Re } \Phi|_{\mathbf{C}(i_2)}$ (to $\mathbf{R}^{2,2}$) is a spacelike maximal immersion from $\Omega \cap \mathbf{C}(i_2)$ into $\mathbf{R}^{2,2} \subset \mathbf{C}(i_1)^4$. By (6.2) (resp. (6.3)) and (5.2), the induced metric is given by

$$
\begin{aligned}
h_\flat|_{\mathbf{C}(i_2)} &= 4(\text{Im}_{\mathbf{C}(i_2)}\widetilde{g}_I)(\text{Im}_{\mathbf{C}(i_2)}\widetilde{g}_{II})|\widetilde{f}|^2_{\mathbf{C}(i_2)}|dz|^2_{\mathbf{C}(i_2)} \\
&= 4g_{I,3}g_{II,3}({f_1}^2 + {f_3}^2)({dx_1}^2 + {dx_3}^2), \\
(\text{resp. } h_\sharp|_{\mathbf{C}(i_2)} &= (1 + |\widetilde{g}_I|^2_{\mathbf{C}(i_2)})(1 + |\widetilde{g}_{II}|^2_{\mathbf{C}(i_2)})|\widetilde{f}|^2_{\mathbf{C}(i_2)}|dz|^2_{\mathbf{C}(i_2)} + \text{Re }\{(1 + \widetilde{g}_I^2)(1 + \widetilde{g}_{II}^2)\widetilde{f}^2 dz^2\} \\
&= 2[\{\text{Re}(\widetilde{g}_I \widetilde{f} dz)\}^2 + \{\text{Re}(\widetilde{g}_{II} \widetilde{f} dz)\}^2 + \{\text{Re}(\widetilde{f} dz)\}^2 + \{\text{Re}(\widetilde{g}_I \widetilde{g}_{II} \widetilde{f} dz)\}^2]\ ).
\end{aligned}
$$

On the other hand, if $(\widetilde{g}_I, \widetilde{g}_{II}, \widetilde{f})$ satisfies the condition (5.4) with $\widetilde{g} = \widetilde{g}_I, \widetilde{g}_{II}$, then (5.5) holds with $\widetilde{g} = \widetilde{g}_I, \widetilde{g}_{II}$, and (the projection of) $\widetilde{X}|_{\mathbf{D}} = \Phi_1|_{\mathbf{D}} = \text{Re } \Phi|_{\mathbf{D}}$ (to $\mathbf{R}^{2,2}$) is a timelike minimal immersion from $\Omega \cap \mathbf{D}$ into $\mathbf{R}^{2,2} \subset \mathbf{C}(i_1)^4$. By (6.2) (resp. (6.3)) and (5.5), the induced metric is given by

$$
\begin{aligned}
h_\flat|_{\mathbf{D}} &= -4(\text{Im}_{\mathbf{D}}\widetilde{g}_I)(\text{Im}_{\mathbf{D}}\widetilde{g}_{II})|\widetilde{f}|^2_{\mathbf{D}}|d\check{z}|^2_{\mathbf{D}} \\
&= -4g_{I,4}g_{II,4}({f_1}^2 - {f_4}^2)({dx_1}^2 - {dx_4}^2), \\
(\text{resp. } h_\sharp|_{\mathbf{D}} &= (1 + |\widetilde{g}_I|^2_{\mathbf{D}})(1 + |\widetilde{g}_{II}|^2_{\mathbf{D}})|\widetilde{f}|^2_{\mathbf{D}}|d\check{z}|^2_{\mathbf{D}} + \text{Re }\{(1 + \widetilde{g}_I^2)(1 + \widetilde{g}_{II}^2)\widetilde{f}^2 d\check{z}^2\} \\
&= 2[\{\text{Re}(\widetilde{g}_I \widetilde{f} d\check{z})\}^2 + \{\text{Re}(\widetilde{g}_{II} \widetilde{f} d\check{z})\}^2 + \{\text{Re}(\widetilde{f} d\check{z})\}^2 + \{\text{Re}(\widetilde{g}_I \widetilde{g}_{II} \widetilde{f} d\check{z})\}^2]\ ).
\end{aligned}
$$

Now, if $(\widetilde{g}_I, \widetilde{g}_{II}, \widetilde{f})$ satisfies (5.1), (5.4) and hence (5.7) also with $\widetilde{g} = \widetilde{g}_I, \widetilde{g}_{II}$, then it does not only satisfy both (5.2) and (5.5) but also satisfies (5.8) with $\widetilde{g} = \widetilde{g}_I, \widetilde{g}_{II}$. Since $\Phi(\widetilde{z}^{\dagger_2}) = \Phi(\widetilde{z})^{\dagger_2} + C$ holds for some constant vector $C \in \mathbf{B}^4$, as we have already observed in the proof of Theorem 3.2, $\widetilde{X}(\zeta_1 + i_2\zeta_2)$ is even with respect to ζ_2, both $\widetilde{X}|_{\mathbf{C}(i_2)}$ and $\widetilde{X}|_{\mathbf{D}}$ are double surfaces with fold singularities on $\Omega \cap \mathbf{R}$, and $\widetilde{X}|_{\mathbf{D}}$ is an analytic extension of $\widetilde{X}|_{\mathbf{C}(i_2)}$. The image of this extension is nondegenerate on any point on the fold singularities such that $\widetilde{f} \neq 0$ and either $(\widetilde{g}_I)_{\widetilde{z}} \neq 0$ or $(\widetilde{g}_{II})_{\widetilde{z}} \neq 0$.

In general, if there exists a nonzero constant vector $V \in \mathbf{R}^4 \setminus \{0\}$ satisfying

$$V \perp {}^t((1 - \widetilde{g}_I\widetilde{g}_{II})\widetilde{f}, (\widetilde{g}_I + \widetilde{g}_{II})\widetilde{f}, (1 + \widetilde{g}_I\widetilde{g}_{II})\widetilde{f}, (\widetilde{g}_I - \widetilde{g}_{II})\widetilde{f}) \qquad \text{on} \quad \Omega,$$

then both $\widetilde{X}|_{\mathbf{C}(i_2)}$ and $\widetilde{X}|_{\mathbf{D}}$ are not full in $\mathbf{R}^{2,2}$. In particular, if $\widetilde{g}_I = \widetilde{g}_{II}$ holds on Ω, then their 4-th components are constant functions, in another word, they coincide with $\widetilde{X}|_{\mathbf{C}(i_2)}$ and $\widetilde{X}|_{\mathbf{D}}$ given by (4.1). By this observation, we get the following

Theorem 7.1. *Let $\widetilde{X}_\epsilon$ be a 1-parameter family of deformation of maps on Ω each of which is given by* (6.1)*. Suppose that $\widetilde{X}_0$ satisfies $\widetilde{g}_I = \widetilde{g}_{II}$ on Ω. Then, if ϵ goes near to* 0*, then the image of the set $g_{I,3}g_{II,3} \leq 0$ (resp. $g_{I,4}g_{II,4} \leq 0$) degenerates to the set $g_{I,3} = g_{II,3} = 0$ (resp. $g_{I,4} = g_{II,4} = 0$) that is the set of singularities of $\widetilde{X}|_{\mathbf{C}(i_2)}$ (resp. $\widetilde{X}|_{\mathbf{D}}$).*

Example 7.2. Define $\widetilde{X}_\epsilon$ by the following Weierstrass data:

$$\widetilde{g}_I = -(\widetilde{z} - i_2\epsilon), \qquad \widetilde{g}_{II} = -(\widetilde{z} + i_2\epsilon), \qquad \widetilde{f} = -\frac{1}{\widetilde{z}^2}.$$

By direct computation, we have

$$\begin{aligned}
\Phi_\epsilon(\widetilde{z}) &= \int^{\widetilde{z}} {}^t(1 - \widetilde{g}_I\widetilde{g}_{II}, \widetilde{g}_I + \widetilde{g}_{II}, 1 + \widetilde{g}_I\widetilde{g}_{II}, \widetilde{g}_I - \widetilde{g}_{II})\widetilde{f}d\widetilde{z} \\
&= \int^{\widetilde{z}} {}^t\left(-\frac{1-\epsilon^2}{\widetilde{z}^2} + 1, \frac{2}{\widetilde{z}}, -\frac{1+\epsilon^2}{\widetilde{z}^2} - 1, -\frac{2i_2\epsilon}{\widetilde{z}^2}\right) d\widetilde{z} \\
&= {}^t\left(\frac{1-\epsilon^2}{\widetilde{z}} + \widetilde{z}, 2\log\widetilde{z}, \frac{1+\epsilon^2}{\widetilde{z}} - \widetilde{z}, \frac{2i_2\epsilon}{\widetilde{z}}\right) + C
\end{aligned}$$

for some $C \in \mathbf{B}^4$. Here we set $C := {}^t(0,0,0,0)$. Since

$$\frac{1}{x_1 + i_2x_3} = \frac{x_1 - i_2x_3}{{x_1}^2 + {x_3}^2}, \qquad \frac{1}{x_1 + jx_4} = \frac{x_1 - jx_4}{{x_1}^2 - {x_4}^2},$$

its spacelike maximal part is given by the following:

$$\widetilde{X}_\epsilon(x_1 + i_2x_3) = {}^t\left(\frac{(1-\epsilon^2)x_1}{{x_1}^2 + {x_3}^2} + x_1,\ \log({x_1}^2 + {x_3}^2),\ \frac{(1+\epsilon^2)x_1}{{x_1}^2 + {x_3}^2} - x_1,\ \frac{2i_2\epsilon x_3}{{x_1}^2 + {x_3}^2}\right) \quad ((x_1, x_3) \neq (0,0)),$$

$\widetilde{X}_0(x_1 + i_2x_3)$ is called the helicoid of the 2-nd kind in $\mathbf{R}^{2,1}(= \mathbf{R}^{2,1} \times \{0\} \subset \mathbf{R}^{2,2})$ that is the correspondent to the catenoid in $\mathbf{R}^3$, and $\widetilde{X}_\epsilon(x_1 + i_2x_3)$ is a 1-parameter deformation in $\mathbf{R}^{2,2}$. By using some other parametrizations, it can be expressed also by the following:

$$\widetilde{X}_\epsilon(r(\cos\theta_2 + i_2\sin\theta_2)) = {}^t\left((\frac{1-\epsilon^2}{r} + r)\cos\theta_2,\ 2\log r,\ (\frac{1+\epsilon^2}{r} - r)\cos\theta_2,\ \frac{2\epsilon}{r}\sin\theta_2\right)$$

$$= {}^t\Big(\{e^R + (1-\epsilon^2)e^{-R}\}\ \cos\theta_2,\ 2R,\ \{-e^R + (1+\epsilon^2)e^{-R}\}\ \cos\theta_2,\ 2\epsilon e^{-R}\ \sin\theta_2\Big)$$

$$(R \in \mathbf{R},\ 0 \le \theta_2 < 2\pi).$$

Since

$$|(\widetilde{X}_\epsilon)_R|^2 \;=\; |(\widetilde{X}_\epsilon)_{\theta_2}|^2 \;=\; 4(\sin^2\theta_2 - \epsilon^2 e^{-2R}) \;=\; 4e^{-2R}({x_3}^2 - \epsilon^2),$$

$\widetilde{X}_\epsilon|_{\mathbf{C}(i_2)}$ is positive (resp. negative) indefinite on the set $|x_3| > \epsilon$ (resp. $|x_3| < \epsilon$). However, by Theorem 6.2, it does not degenerate with respect to the metric $h_\sharp$ on the set $|x_3| = \epsilon$. If ϵ goes near to 0, then the image of the set $|x_3| \le \epsilon$ converges to the fold singularities of $\widetilde{X}_0 = \widetilde{X}$.

Figure 7.1 shows some samples of the projections ${}^t(\widetilde{X}^1_\epsilon, \widetilde{X}^2_\epsilon, 0.8\widetilde{X}^3_\epsilon + 0.6\widetilde{X}^4_\epsilon)|_{\mathbf{C}(i_2)}$ of $\widetilde{X}_\epsilon|_{\mathbf{C}(i_2)}$. Each thick line means a null curve of type-changing.

8 Cross sections including degenerate directions

In this section, we observe the behaviour of the bicomplex extension $\widetilde{X}$ around the singularities.

First, suppose that $(\widetilde{g}, \widetilde{f})$ satisfies (5.1), (5.4) and hence (5.7) also. Then, by (4.4) and (5.8), the restriction of the metric h_+ to the singular set $\Omega \cap \mathbf{R}$ is given by

$$(h_{+,\ell m})|_{\mathbf{R}} \;=\; 2(1+{g_1}^2)^2 \begin{pmatrix} {f_1}^2 & 0 & 0 & 0 \\ 0 & {f_1}^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Since

$$\mathrm{Ker}\ (h_{+,\ell m})|_{\mathbf{R}} \;=\; {}^t(0,0,1,0)\mathbf{R} + {}^t(0,0,0,1)\mathbf{R} \;=\; i_2\mathbf{R} + j\mathbf{R} \;=\; i_2\mathbf{C}(i_1),$$

$x_1 + i_2\mathbf{C}(i_1)$ is a cross section around $x_1 \in \Omega \cap \mathbf{R}$ including the degenerate directions. Set

$$Y(\zeta) \;:=\; \widetilde{X}(x_1 + i_2\zeta) \qquad (\forall\zeta \in \mathbf{C}(i_1) \text{ s.t. } x_1 + i_2\zeta \in \Omega).$$

Then Y is an even holomorphic map of ζ as we have already seen in §3.

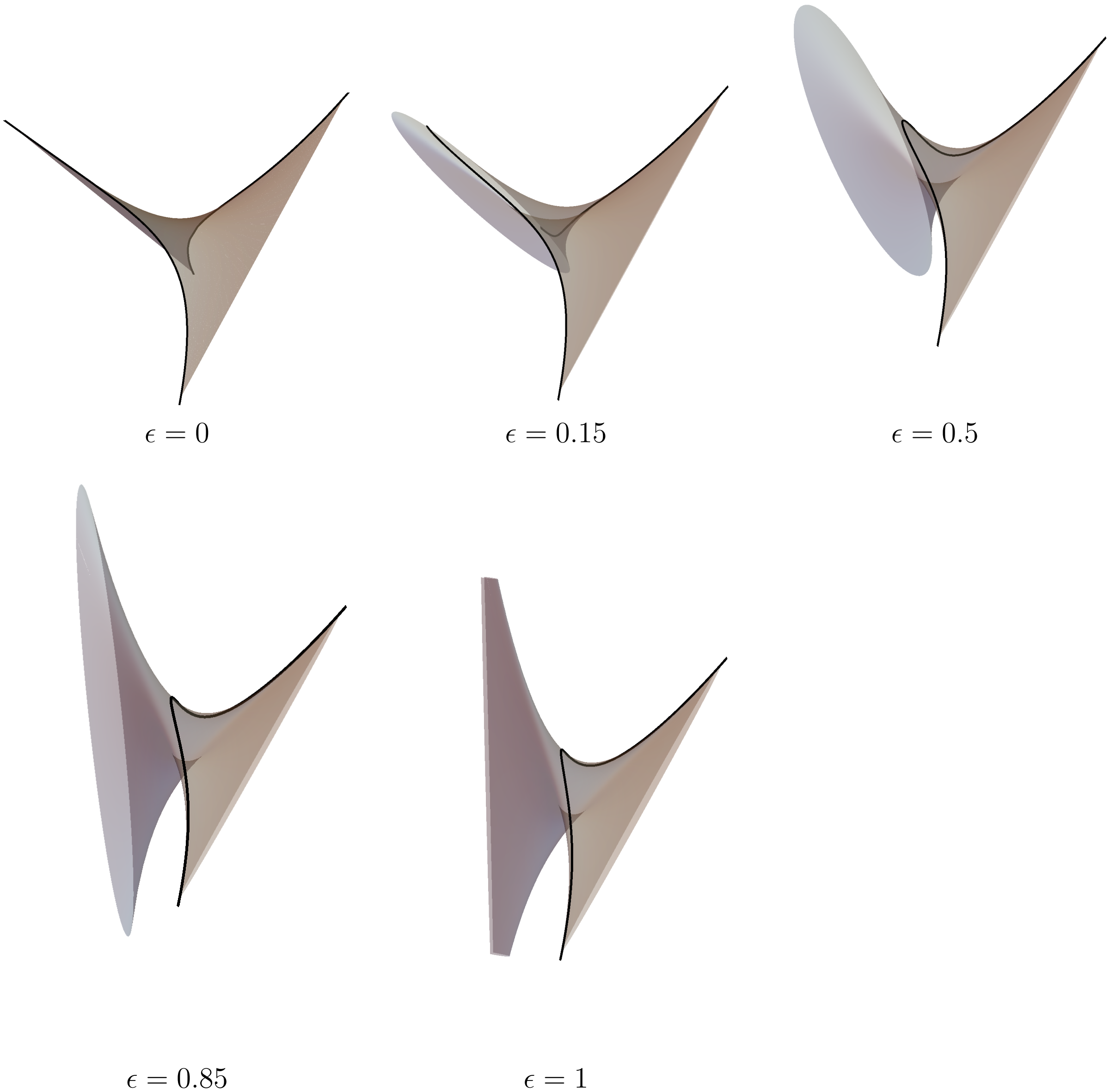

FIGURE 7.1.

If $(\widetilde{f}, \widetilde{g}\widetilde{f}, \widetilde{g}^2\widetilde{f})$ has a pole of order $K+1$ with even ordered terms only at $0 \in \Omega_\infty$ additionally, then

$$Y_{\text{end}}(\zeta) \ := \ \widetilde{X}(s\zeta^{K+1} + i_2\zeta) \qquad (\forall \zeta \in \mathbf{C}(i_1) \text{ s.t. } s\zeta^{K+1} + i_2\zeta \in \Omega_\infty)$$

is also an even holomorphic map of ζ.

Also in the case that $(\widetilde{g}_I, \widetilde{g}_{II}, \widetilde{f})$ satisfies (5.1), (5.4) and (5.7) with $\widetilde{g} = \widetilde{g}_I, \widetilde{g}_{II}$, by (6.3) and (5.8), the restriction of the metric $h_\sharp$ to the singular set $\Omega \cap \mathbf{R}$ is given by

$$(h_{\sharp,\ell m})|_{\mathbf{R}} \;=\; 2(1+{g_{I,1}}^2)(1+{g_{II,1}}^2)\begin{pmatrix} {f_1}^2 & 0 & 0 & 0 \\ 0 & {f_1}^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and hence we get the same conclusion as for $(\widetilde{g}, \widetilde{f})$ above.

Now, consider the case that $(\widetilde{g}, \widetilde{f})$ satisfies (5.1) only. We may assume that $\widetilde{g}(x) \in \mathbf{R}$ $(\forall x \in \Omega \cap \mathbf{R})$ without loss of generality by changing coordinate if necessary. Then, by (4.4) and (5.2), the restriction of the metric h_+ to the singular set $\Omega \cap \mathbf{R}$ is given by

$$(h_{+,\ell m})|_{\mathbf{R}} \;=\; 2(1+{g_1}^2)^2\begin{pmatrix} {f_1}^2 & 0 & -f_1f_3 & 0 \\ 0 & {f_1}^2 & 0 & -f_1f_3 \\ -f_1f_3 & 0 & {f_3}^2 & 0 \\ 0 & -f_1f_3 & 0 & {f_3}^2 \end{pmatrix}.$$

Since

$$\begin{aligned} \mathrm{Ker}\,(h_{+,\ell m})|_{\mathbf{R}} &= {}^t(f_3, 0, f_1, 0)\mathbf{R} + {}^t(0, f_3, 0, f_1)\mathbf{R} \\ &= (f_3 + i_2 f_1)\mathbf{R} + (i_1 f_3 + j f_1)\mathbf{R} \;=\; (f_3 + i_2 f_1)\mathbf{C}(i_1) \end{aligned}$$

on $\Omega \cap \mathbf{R}$ and

$$f_3(x_1) + i_2 f_1(x_1) \;=\; i_2(f_1(x_1) + i_1 \cdot 0 - i_2 f_3(x_1) - j \cdot 0) \;=\; i_2 \widetilde{f}(x_1)^{\dagger_2},$$

$x_1 + i_2\widetilde{f}(x_1)^{\dagger_2}\mathbf{C}(i_1)$ is a cross section around $x_1 \in \Omega \cap \mathbf{R}$ including the degenerate directions. Set

$$Y(\zeta) \;:=\; \widetilde{X}(x_1 + i_2\widetilde{f}(x_1)^{\dagger_2}\zeta) \qquad (\forall \zeta \in \mathbf{C}(i_1) \text{ s.t. } x_1 + i_2\widetilde{f}(x_1)^{\dagger_2}\zeta \in \Omega),$$

and denote $\{\cdot\}_1 + i_1\{\cdot\}_2$ by $\{\cdot\}_{12}$. Then Y is a holomorphic map of ζ and satisfies

$$\begin{aligned} Y'(\zeta) &= \frac{d}{d\zeta}\widetilde{X}(x_1 + i_2\widetilde{f}(x_1)^{\dagger_2}\zeta) \\ &= \{i_2\widetilde{f}(x_1)^{\dagger_2} \cdot \Phi'(x_1 + i_2\widetilde{f}(x_1)^{\dagger_2}\zeta)\}_{12} \\ &= \{i_2\widetilde{f}(x_1)^{\dagger_2} \cdot {}^t(1-\widetilde{g}^2, 2\widetilde{g}, 1+\widetilde{g}^2)\widetilde{f}|_{\widetilde{z}=x_1+i_2\widetilde{f}(x_1)^{\dagger_2}\zeta}\}_{12}, \end{aligned}$$

$$\begin{aligned} Y'(0) &= \left.\frac{d}{d\zeta}\right|_{\zeta=0} \widetilde{X}(x_1 + i_2\widetilde{f}(x_1)^{\dagger_2}\zeta) \\ &= \{i_2\widetilde{f}(x_1)^{\dagger_2} \cdot {}^t(1-\widetilde{g}(x_1)^2, 2\widetilde{g}(x_1), 1+\widetilde{g}(x_1)^2)\widetilde{f}(x_1)\}_{12} \\ &= \{i_2|\widetilde{f}(x_1)|^2_{\mathbf{C}(i_2)} \cdot {}^t(1-g_1(x_1)^2, 2g_1(x_1), 1+g_1(x_1)^2)\}_{12} \\ &= {}^t(0,0,0). \end{aligned}$$

Since $\Phi(\tilde{z}^{\dagger_1}) = \Phi(\tilde{z})^{\dagger_1} + C$ holds for some constant vector $C \in \mathbf{B}^3$, we see that

$$\Phi(x_1 + i_2\widetilde{f}(x_1)^{\dagger_2}\zeta^{\dagger_1}) = \Phi((x_1 + i_2\widetilde{f}(x_1)^{\dagger_2}\zeta)^{\dagger_1}) = \Phi(x_1 + i_2\widetilde{f}(x_1)^{\dagger_2}\zeta)^{\dagger_1} + C$$

for $\zeta \in \mathbf{C}(i_1)$ such that $x_1 + i_2\widetilde{f}(x_1)^{\dagger_2}\zeta \in \Omega \cap \Omega^{\dagger_1}$, and hence

$$\Phi(x_1 + i_2\widetilde{f}(x_1)^{\dagger_2}\zeta^{\dagger_1}) - \Phi(x_1) = (\Phi(x_1 + i_2\widetilde{f}(x_1)^{\dagger_2}\zeta) - \Phi(x_1))^{\dagger_1}.$$

Therefore $\widetilde{X} = \Phi_1 + i_1\Phi_2$ satisfies

$$\begin{aligned}\Phi_1(x_1 + i_2\widetilde{f}(x_1)^{\dagger_2}\zeta^{\dagger_1}) - \Phi_1(x_1) &= \Phi_1(x_1 + i_2\widetilde{f}(x_1)^{\dagger_2}\zeta) - \Phi_1(x_1), \\ \Phi_2(x_1 + i_2\widetilde{f}(x_1)^{\dagger_2}\zeta^{\dagger_1}) - \Phi_2(x_1) &= -(\Phi_2(x_1 + i_2\widetilde{f}(x_1)^{\dagger_2}\zeta) - \Phi_2(x_1)),\end{aligned}$$

namely $(Y(x_3') - Y(0))_2 = 0$, $(Y(i_1x_4') - Y(0))_1$ is even, and $(Y(i_1x_4') - Y(0))_2$ is odd for $x_3', x_4' \in \mathbf{R}$ such that $x_1 + i_2\widetilde{f}(x_1)^{\dagger_2}x_3' \in \Omega$ or $x_1 + i_2\widetilde{f}(x_1)^{\dagger_2}i_1x_4' = x_1 + j\widetilde{f}(x_1)^{\dagger_2}x_4' \in \Omega$. In another word, $Y(x_3') - Y(0) \in \mathbf{R}$ and "the even (resp. odd) part of $Y(i_1x_4') - Y(0)$"$\in \mathbf{R}$ (resp. $i_1\mathbf{R}$) for $x_3', x_4' \in \mathbf{R}$ as above. Hence $\widetilde{X}(x_1 + j\widetilde{f}(x_1)^{\dagger_2}\sqrt{-t})$ $(t \leq 0)$ is only a nonanalytic $C^{1,\alpha}$ extension of $\widetilde{X}(x_1 + i_2\widetilde{f}(x_1)^{\dagger_2}\sqrt{t})(= \widetilde{X}|_{\mathbf{C}(i_2)} = \text{Re }\Phi|_{\mathbf{C}(i_2)})$ $(t \geq 0)$ with respect to the parameter t, but "the even part of $\widetilde{X}(x_1 + j\widetilde{f}(x_1)^{\dagger_2}\sqrt{-t})$ "$(= \text{Re }\Phi(x_1 + j\widetilde{f}(x_1)^{\dagger_2}\sqrt{-t}))$ $(t \leq 0)$ is an analytic extension of "that of $\widetilde{X}(x_1 + i_2\widetilde{f}(x_1)^{\dagger_2}\sqrt{t})$ "$(t \geq 0)$.

Also in the case that $(\tilde{g}, \widetilde{f})$ satisfies (5.4) only, we may assume that $\tilde{g}(x) \in \mathbf{R}$ $(\forall x \in \Omega \cap \mathbf{R})$ as above. Then, by (4.4) and (5.5), the restriction of the metric h_+ to the singular set $\Omega \cap \mathbf{R}$ is given by

$$(h_{+,\ell m})|_{\mathbf{R}} = 2(1 + {g_1}^2)^2 \begin{pmatrix} {f_1}^2 & 0 & 0 & f_1f_4 \\ 0 & {f_1}^2 & -f_1f_4 & 0 \\ 0 & -f_1f_4 & {f_4}^2 & 0 \\ f_1f_4 & 0 & 0 & {f_4}^2 \end{pmatrix}.$$

Since

$$\begin{aligned}\text{Ker }(h_{+,\ell m})|_{\mathbf{R}} &= {}^t(f_4, 0, 0, -f_1)\mathbf{R} + {}^t(0, f_4, f_1, 0)\mathbf{R} \\ &= (f_4 - jf_1)\mathbf{R} + (i_1f_4 + i_2f_1)\mathbf{R} = (f_4 - jf_1)\mathbf{C}(i_1)\end{aligned}$$

on $\Omega \cap \mathbf{R}$ and

$$f_4(x_1) - jf_1(x_1) = -j(f_1(x_1) + i_1 \cdot 0 - i_2 \cdot 0 - jf_4(x_1)) = -j\widetilde{f}(x_1)^{\dagger_2},$$

$x_1 - j\widetilde{f}(x_1)^{\dagger_2}\mathbf{C}(i_1)$ is a cross section around $x_1 \in \Omega \cap \mathbf{R}$ including the degenerate directions. Set

$$Y(\zeta) \ := \ \widetilde{X}(x_1 - j\widetilde{f}(x_1)^{\dagger_2}\zeta) \qquad (\forall \zeta \in \mathbf{C}(i_1) \text{ s.t. } x_1 - j\widetilde{f}(x_1)^{\dagger_2}\zeta \in \Omega).$$

Then Y is a holomorphic map of ζ, and we get the same conclusion as above.

9 Flux around zero-devisors

Let $\check{X} : \check{\Omega}_0 \to \mathbf{R}^{2,1}$ (resp. $\mathbf{R}^{2,2}$) be a timelike minimal immersion. Even if the Weierstrass data $(\check{g}, \check{f})$ (resp. $(\check{g}_I, \check{g}_{II}, \check{f})$) of $\check{X}$ is given by rational functions on $\mathbf{D}$, the domain of $(\check{g}, \check{f})$ (resp. $(\check{g}_I, \check{g}_{II}, \check{f})$) is not connected, and $\check{X}$ cannot be defined globally on $\mathbf{D} \setminus (\mathcal{P} + \mathcal{S})$ only by the integral of Weierstrass type of representation formula, where $\mathcal{P}$ is the set of poles of $\check{f}$ and/or $\check{g}^2\check{f}$ (resp. $(\check{g}_I\check{f}, \check{g}_{II}\check{f}, \check{g}_I\check{g}_{II}\check{f})$), and $\mathcal{S}$ is the set of zero-devisors. Suppose that $(\widetilde{g}, \widetilde{f})$ (resp. $(\widetilde{g}_I, \widetilde{g}_{II}, \widetilde{f})$) be the bicomplex extension of $(\check{g}, \check{f})$ (resp. $(\check{g}_I, \check{g}_{II}, \check{f})$) defined on $\Omega \subset \mathbf{B}$. It satisfies (5.4), and the principal part of the Laurent expansion of each of $\widetilde{f}$, $\widetilde{g}\widetilde{f}$ and $\widetilde{g}^2\widetilde{f}$ (resp. $\widetilde{g}_I\widetilde{f}$, $\widetilde{g}_{II}\widetilde{f}$ and $\widetilde{g}_I\widetilde{g}_{II}\widetilde{f}$) around each pole has the term of the form $(a_1 + ja_4)/\widetilde{z}$ with $a_1, a_4 \in \mathbf{R}$ in general. Since

$$\begin{aligned}
\int^{\widetilde{z}} \frac{a_1 + ja_4}{\widetilde{z}} d\widetilde{z} \ &= \ (a_1 + ja_4)\log \widetilde{z} \\
&= \ (a_1 + ja_4)(\log r + i_1\theta_1 + i_2\theta_2 + j\theta_3) \\
&= \ (a_1 \log r + a_4\theta_3) + i_1(a_1\theta_1 - a_4\theta_2) + i_2(a_1\theta_2 - a_4\theta_1) \\
&\qquad + j(a_1\theta_3 + a_4 \log r),
\end{aligned}$$

it holds for any loop γ in $\Omega \setminus (\mathcal{P} + \mathcal{S})$ that

$$\int_\gamma \frac{a_1 + ja_4}{\widetilde{z}} d\widetilde{z} \ = \ i_1(a_1N_1\pi - a_4N_2\pi) + i_2(a_1N_2\pi - a_4N_1\pi) \ \in \ i_1\mathbf{R} + i_2\mathbf{R}$$

for some $N_1, N_2 \in \mathbf{Z}$ such that $N_1 + N_2$ is even, and, in particular,

$$\mathrm{Re} \int_\gamma \frac{a_1 + ja_4}{\widetilde{z}} d\widetilde{z} \ = \ 0.$$

Hence $\Phi_1 = \mathrm{Re}\ \Phi = \mathrm{Re}\ \widetilde{X}$ is well-defined on $\Omega \setminus (\mathcal{P} + \mathcal{S})$ though the bicomplex extension $\widetilde{X}$ itself is not well-defined. Now, it is clear that $\mathrm{Re}\ (\widetilde{X}(\check{z}) - \widetilde{X}(\check{z}_0))$ is independent of the choice of the path for any $\check{z}, \check{z}_0 \in (\Omega \cap \mathbf{D}) \setminus (\mathcal{P} + \mathcal{S})$. At the same time, this observation justifies (5.9) in a global meaning, and we get the following

Theorem 9.1. *$\check{X}$ extends "analytically" beyond $\mathcal{P}+\mathcal{S}$ of the Weierstrass data $(\check{g}, \check{f})$ or $(\check{g}_I, \check{g}_{II}, \check{f})$.*

Example 9.2. Let us observe the bicomplex extension of the helicoid of the 2nd kind in $\mathbf{R}^{2,1}$ again. Its Weierstrass data is given by the following:

$$\widetilde{g} = -\widetilde{z}, \qquad \widetilde{f} = -\frac{1}{\widetilde{z}^2}.$$

By direct computation, we have

$$\begin{aligned}\Phi(\widetilde{z}) &= \int^{\widetilde{z}} {}^t(1-\widetilde{g}^2, 2\widetilde{g}, 1+\widetilde{g}^2)\widetilde{f}d\widetilde{z} \\ &= \int^{\widetilde{z}} {}^t\left(-\frac{1}{\widetilde{z}^2}+1,\ \frac{2}{\widetilde{z}},\ -\frac{1}{\widetilde{z}^2}-1\right)d\widetilde{z} \\ &= {}^t\left(\frac{1}{\widetilde{z}}+\widetilde{z},\ 2\log\widetilde{z},\ \frac{1}{\widetilde{z}}-\widetilde{z}\right)+C\end{aligned}$$

for some $C \in \mathbf{B}^3$. Set $C := {}^t(0,0,0)$. Then its spacelike maximal part and timelike minimal parts are given by the following:

$$\begin{aligned}\widetilde{X}(x_1+i_2x_3) &= {}^t\left(\frac{x_1}{{x_1}^2+{x_3}^2}+x_1,\ \log({x_1}^2+{x_3}^2),\ \frac{x_1}{{x_1}^2+{x_3}^2}-x_1\right) \quad ((x_1,x_3)\neq(0,0)), \\ \widetilde{X}(x_1+jx_4) &= {}^t\left(\frac{x_1}{{x_1}^2-{x_4}^2}+x_1,\ \log|{x_1}^2-{x_4}^2|,\ \frac{x_1}{{x_1}^2-{x_4}^2}-x_1\right) \quad (|x_1|\neq|x_4|).\end{aligned}$$

It is clear that the image of $\widetilde{X}(x_1+jx_4)|_{\{|x_1|>|x_4|\}}$ is an analytic extension of that of $\widetilde{X}(x_1+i_2x_3)|_{\{(x_1,x_3)\neq(0,0)\}}$ in the sense that

$$\widetilde{X}_{\text{fld}}(s,t) = {}^t\left(\frac{s}{s^2+t}+s,\ \log(s^2+t),\ \frac{s}{s^2+t}-s\right) \qquad (s^2+t>0)$$

is analytic with respect to (s,t). On the other hand, we can also regard the image of $\widetilde{X}(x_1+jx_4)|_{\{|x_1|<|x_4|\}}$ as an "analytic"extension of the other parts in the sense of Theorem 9.1.

Figure 9.1 shows $\widetilde{X}|_{\mathbf{C}(i_2)}$ and $\widetilde{X}|_{\mathbf{D}}$. Each thick line means a null curve of type-changing.

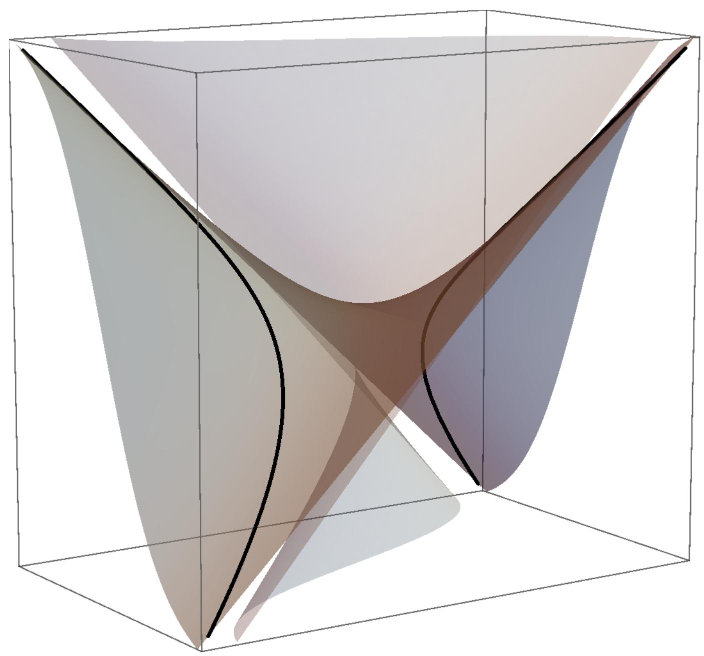

FIGURE 9.1.

Example 9.3. Let us observe the bicomplex extension of a spacelike maximal surface in $\mathbf{R}^{2,1}$ which has a simple end of zero flux on a fold singularity (cf. [10], [11]). This is a typical case of Theorem 3.3. We may assume that both the end and the stereographic image of the limit normal at the end are 0 without loss of generality. Then the Weierstrass data of the surface around the end is of the following form in general:

$$\widetilde{g} \;=\; \widetilde{z}^{m+2}\widetilde{g}_{\mathrm{hol}}, \qquad \widetilde{f} \;=\; \frac{\alpha}{\widetilde{z}^2} + \widetilde{f}_{\mathrm{hol}},$$

where $\widetilde{g}_{\mathrm{hol}}$ and $\widetilde{f}_{\mathrm{hol}}$ are bicomplex holomorphic functions on a domain Ω_∞ including 0 satisfying (5.1), (5.4) and (5.7), $m \in \mathbf{N} \cup \{0\}$, $\alpha \in \mathbf{R} \setminus \{0\}$ and $\beta = \widetilde{g}_{\mathrm{hol}}(0) \in \mathbf{R} \setminus \{0\}$. By direct computation, we have

$$\begin{aligned}
\Phi(\widetilde{z}) \;&=\; \int^{\widetilde{z}} {}^t(1 - \widetilde{z}^{2m+4}\widetilde{g}_{\mathrm{hol}}^2, 2\widetilde{z}^{m+2}\widetilde{g}_{\mathrm{hol}}, 1 + \widetilde{z}^{2m+4}\widetilde{g}_{\mathrm{hol}}^2)\left(\frac{\alpha}{\widetilde{z}^2} + \widetilde{f}_{\mathrm{hol}}\right) d\widetilde{z} \\
&=\; \int^{\widetilde{z}} \Big\{ {}^t\left(\frac{\alpha}{\widetilde{z}^2},\ 0,\ \frac{\alpha}{\widetilde{z}^2}\right) + {}^t(\widetilde{f}_{\mathrm{hol}}, 2\alpha\widetilde{z}^m\widetilde{g}_{\mathrm{hol}}, \widetilde{f}_{\mathrm{hol}}) \\
&\qquad + \widetilde{z}^{m+2}\ {}^t(-\widetilde{z}^m\widetilde{g}_{\mathrm{hol}}^2(\alpha + \widetilde{z}^2\widetilde{f}_{\mathrm{hol}}), 2\widetilde{g}_{\mathrm{hol}}\widetilde{f}_{\mathrm{hol}}, \widetilde{z}^m\widetilde{g}_{\mathrm{hol}}^2(\alpha + \widetilde{z}^2\widetilde{f}_{\mathrm{hol}}))\Big\}\, d\widetilde{z} \\
&=\; {}^t\left(-\frac{\alpha}{\widetilde{z}},\ 0,\ -\frac{\alpha}{\widetilde{z}}\right) + \Phi_{\mathrm{hol}}(\widetilde{z}),
\end{aligned}$$

where Φ_{hol} is a bicomplex holomorphic map satisfying (3.2). Hence

$$\widetilde{X}(\zeta_1 + i_2\zeta_2) \;=\; {}^t\left(-\frac{\alpha\zeta_1}{{\zeta_1}^2 + {\zeta_2}^2},\ 0,\ -\frac{\alpha\zeta_1}{{\zeta_1}^2 + {\zeta_2}^2}\right) + \widetilde{X}_{\mathrm{hol}}(\zeta_1 + i_2\zeta_2),$$

where $\widetilde{X}_{\rm hol} := (\Phi_{\rm hol})_1 + i_1(\Phi_{\rm hol})_2$ is holomorphic with respect to (ζ_1, ζ_2), and even with respect to ζ_2. Then its spacelike maximal part and timelike minimal parts are given by the following:

$$\begin{aligned}
\widetilde{X}(x_1+i_2x_3) &= {}^t\left(-\frac{\alpha x_1}{{x_1}^2+{x_3}^2},\ 0,\ -\frac{\alpha x_1}{{x_1}^2+{x_3}^2}\right)+\widetilde{X}_{\rm hol}(x_1+i_2x_3)\\
&\qquad\qquad ((x_1,x_3)\neq(0,0), x_1+i_2x_3\in\Omega_\infty),\\
\widetilde{X}(x_1+jx_4) &= {}^t\left(-\frac{\alpha x_1}{{x_1}^2-{x_4}^2},\ 0,\ -\frac{\alpha x_1}{{x_1}^2-{x_4}^2}\right)+\widetilde{X}_{\rm hol}(x_1+jx_4)\\
&\qquad\qquad (|x_1|\neq|x_4|, x_1+jx_4\in\Omega_\infty).
\end{aligned}$$

It is clear that the image of $\widetilde{X}(x_1+jx_4)|_{\{|x_1|>|x_4|\}}$ is an analytic extension of that of $\widetilde{X}(x_1+i_2x_3)|_{\{(x_1,x_3)\neq(0,0)\}}$ in the sense that $\widetilde{X}_{\rm fld}(s,t)$ defined by (3.3) with $F=\widetilde{X}$ is analytic with respect to (s,t). Also in this case, we can regard the image of $\widetilde{X}(x_1+jx_4)|_{\{|x_1|<|x_4|\}}$ as an "analytic"extension of the other parts in the sense of Theorem 9.1.

On the other hand, for any $s\in\mathbf{R}$, it holds that

$$\begin{aligned}
\widetilde{X}(s{x_3}^2+i_2x_3) &= {}^t\left(-\frac{\alpha s}{s^2{x_3}^2+1},\ 0,\ -\frac{\alpha s}{s^2{x_3}^2+1}\right)+\widetilde{X}_{\rm hol}(s{x_3}^2+i_2x_3)\\
&\qquad\qquad (x_3\neq0, s{x_3}^2+i_2x_3\in\Omega_\infty),\\
\widetilde{X}(-s{x_4}^2+jx_4) &= {}^t\left(-\frac{\alpha s}{-s^2{x_4}^2+1},\ 0,\ -\frac{\alpha s}{-s^2{x_4}^2+1}\right)+\widetilde{X}_{\rm hol}(-s{x_4}^2+jx_4)\\
&\qquad\qquad (|sx_4|\neq1, -s{x_4}^2+jx_4\in\Omega_\infty).
\end{aligned}$$

Note here that

$$\begin{aligned}
\widetilde{X}(s{x_3}^2+i_2x_3) &= {}^t(-\alpha s,0,-\alpha s)+\widetilde{X}_{\rm hol}(0)+{x_3}^2C(s)+O({x_3}^4),\\
\widetilde{X}(-s{x_4}^2+jx_4) &= {}^t(-\alpha s,0,-\alpha s)+\widetilde{X}_{\rm hol}(0)-{x_4}^2C(s)+O({x_4}^4)
\end{aligned}$$

hold for $C(s)={}^t(C^1(s),C^2(s),C^3(s))\in\mathbf{R}^3$, where

$$\begin{aligned}
C^1(s) &= C^3(s) = \alpha s^3+\widetilde{f}_{\rm hol}(0)s-\frac{1}{2}(\widetilde{f}_{\rm hol})_{\widetilde{z}}(0),\\
C^2(s) &= \begin{cases} \alpha\{2\widetilde{g}_{\rm hol}(0)s-(\widetilde{g}_{\rm hol})_{\widetilde{z}}(0)\} & (m=0),\\ -\alpha\widetilde{g}_{\rm hol}(0) & (m=1),\\ 0 & (m\geq2).\end{cases}
\end{aligned}$$

Now, if $m = 0$ or 1, then the image of $\widetilde{X}(x_1 + jx_4)|_{\{|x_1|<|x_4|\}}$ can be regarded as an analytic extension of that of $\widetilde{X}(x_1 + i_2 x_3)|_{\{(x_1,x_3)\neq(0,0)\}}$ across the subset of a lightlike line $\{{}^t(-\alpha s, 0, -\alpha s) \mid s \in \mathbf{R}, C^2(s) \neq 0\}$ in the sense that $\widetilde{X}_{\mathrm{end}}(s,t)$ defined by (3.4) with $F = \widetilde{X}$ is analytic with respect to (s,t) and nondegenerate on the subset above.

The most simple example is given by $(\widetilde{g}_{\mathrm{hol}}, \widetilde{f}_{\mathrm{hol}}) = (-1, 0)$ and $\alpha = -1$. In this case, if $m = 0$, then we have

$$\begin{aligned}
\widetilde{X}_{\mathrm{fld}}(s,t) &= {}^t\left(\frac{s}{s^2+t} + \frac{1}{3}(s^3 - 3st),\ 2s,\ \frac{s}{s^2+t} - \frac{1}{3}(s^3 - 3st)\right) \quad (s^2 + t > 0),\\
\widetilde{X}_{\mathrm{end}}(s,t) &= {}^t\left(\frac{s}{s^2t+1} + \frac{1}{3}(s^3t^3 - 3st^2),\ 2st,\ \frac{s}{s^2t+1} - \frac{1}{3}(s^3t^3 - 3st^2)\right) \quad (s^2t + 1 > 0),
\end{aligned}$$

and if $m = 1$, then we have

$$\begin{aligned}
\widetilde{X}_{\mathrm{fld}}(s,t) &= {}^t\left(\frac{s}{s^2+t} + \frac{1}{5}(s^5 - 10s^3t + 5st^2), s^2 - t, \frac{s}{s^2+t} - \frac{1}{5}(s^5 - 10s^3t + 5st^2)\right)\\
&\qquad (s^2 + t > 0),\\
\widetilde{X}_{\mathrm{end}}(s,t) &= {}^t\left(\frac{s}{s^2t+1} + \frac{1}{5}(s^5t^5 - 10s^3t^4 + 5st^3), s^2t^2 - t,\right.\\
&\qquad \left.\frac{s}{s^2t+1} - \frac{1}{5}(s^5t^5 - 10s^3t^4 + 5st^3)\right) \quad (s^2t + 1 > 0),
\end{aligned}$$

where we set $\widetilde{X}_{\mathrm{hol}}(0) := {}^t(0,0,0)$.

Figure 9.2 shows $\widetilde{X}|_{\mathbf{C}(i_2)}$ and $\widetilde{X}|_{\mathbf{D}}$ with $(\widetilde{g}_{\mathrm{hol}}, \widetilde{f}_{\mathrm{hol}}) = (-1, 0)$ and $\alpha = -1$. Each thick line means a null curve of type-changing. Each slit means a lightlike line segment of incomplete end.

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$m = 0$ $m = 1$

Figure 9.2.

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