ON WEAKLY REFLECTIVE SUBMANIFOLDS IN COMPACT ISOTROPY IRREDUCIBLE RIEMANNIAN HOMOGENEOUS SPACES

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Abstract. We show that for any weakly reflective submanifold of a compact isotropy irreducible Riemannian homogeneous space its inverse image under the parallel transport map is an infinite dimensional weakly reflective PF submanifold of a Hilbert space. This is an extension of the author's previous result in the case of compact irreducible Riemannian symmetric spaces. We also give a characterization of so obtained weakly reflective PF submanifolds.

Introduction

A submanifold N immersed in a Riemannian manifold M is called *weakly reflective* ([3]) if for each normal vector ξ at each $p \in N$ there exists an isometry ν_ξ of M satisfying the conditions $\nu_\xi(p) = p$, $d\nu_\xi(\xi) = -\xi$ and $\nu_\xi(N) = N$. We call such an isometry ν_ξ a reflection with respect to ξ. If every ν_ξ can be chosen from a particular subgroup S of the isometry group $I(M)$ then we call N *S-weakly reflective*. By definition weakly reflective submanifolds are *austere* ([2]): for each normal vector ξ the set of eigenvalues with multiplicities of the shape operator A_ξ is invariant under the multiplication by (-1). Thus weakly reflective submanifolds are minimal submanifolds. A singular orbit of a cohomogeneity one action is a typical example of a weakly reflective submanifold ([8], [3]). It is an interesting problem to determine weakly reflective orbits in an isometric action of a Lie group (e.g. [3], [1]).

Recently the author [5] introduced the concept of weakly reflective submanifolds into a class of proper Fredholm (PF) submanifolds in Hilbert spaces (Terng [10]) and showed that if N is a weakly reflective submanifold of an irreducible Riemannian symmetric space G/K of compact type then its inverse image $\Phi_K^{-1}(N)$ under the parallel transport map $\Phi_K : V_\mathfrak{g} \to G/K$ ([12]) is an infinite dimensional weakly reflective PF submanifold of the Hilbert space $V_\mathfrak{g} := L^2([0,1], \mathfrak{g})$ consisting of all L^2-paths from $[0,1]$ to the Lie algebra $\mathfrak{g}$ of G ([5, Theorem 8]). Using this result many examples of infinite dimensional weakly reflective PF submanifolds were obtained from finite dimensional weakly reflective submanifolds in G/K. The purpose of this paper is to extend that result to the case G/K is a compact isotropy irreducible Riemannian homogeneous space and to give a characterization of so obtained weakly reflective PF submanifolds in terms of S-weak reflectivity defined above. The results are summarized in Section 2. Their proofs are given in Section 3. To do these, in

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Section 1 we prepare the setting of $P(G,H)$-actions and the parallel transport map over a compact Lie group G which is *not* necessarily connected. This generalized setting is important in the formulation of our results.

1. Preliminaries

Let G be a compact Lie group with Lie algebra $\mathfrak{g}$. Denote by G_0 its identity component. Choose an $\mathrm{Ad}(G)$-invariant inner product of $\mathfrak{g}$ and equip the corresponding bi-invariant inner product with G. Denote by $\mathcal{G} := H^1([0,1],G)$ the Hilbert Lie group of all Sobolev H^1-paths from $[0,1]$ to G and by $V_{\mathfrak{g}} := L^2([0,1],\mathfrak{g})$ the Hilbert space of all L^2-paths from $[0,1]$ to $\mathfrak{g}$. For each $a \in G$ (resp. $x \in \mathfrak{g}$) we denote by $\hat{a} \in \mathcal{G}$ (resp. $\hat{x} \in V_{\mathfrak{g}}$) the constant path which values at a (resp. x). Define the $\mathcal{G}$-action on $V_{\mathfrak{g}}$ via the gauge transformations:

$$g * u := gug^{-1} - g'g^{-1}, \quad g \in \mathcal{G},\ u \in V_{\mathfrak{g}}, \tag{1.1}$$

where g' denotes the weak derivative of g. We know that this action is isometric, transitive, proper and Fredholm ([7, Theorem 5.8.1]).

Let H be a closed subgroup of $G \times G$. Then H acts on G isometrically by

$$(b_1, b_2) \cdot a := b_1 a b_2^{-1}, \quad a, b_1, b_2 \in G. \tag{1.2}$$

Define a Lie subgroup $P(G,H)$ of $\mathcal{G}$ by

$$P(G,H) := \{g \in \mathcal{G} \mid (g(0), g(1)) \in H\}.$$

The induced action of $P(G,H)$ on $V_{\mathfrak{g}}$ is called the *$P(G,H)$-action*. We know that the $P(G,H)$-action is isometric, proper and Fredholm ([11, p. 132]).

The *parallel transport map* ([4], [11]) $\Phi : V_{\mathfrak{g}} \to G_0$ is a Riemannian submersion defined by

$$\Phi(u) := E_u(1), \quad u \in V_{\mathfrak{g}},$$

where $E_u \in \mathcal{G}$ is the unique solution to the linear ordinary differential equation

$$\begin{cases} E_u^{-1} E_u' = u, \\ E_u(0) = e. \end{cases}$$

Let H be a closed subgroup of $G_0 \times G_0$. Then it follows ([11, Proposition 1.1]) that for each $g \in P(G_0, H)$ and $u \in V_{\mathfrak{g}}$

$$\Phi(g * u) = (g(0), g(1)) \cdot \Phi(u) \quad \text{and} \quad P(G_0, H) * u = \Phi^{-1}(H \cdot \Phi(u)). \tag{1.3}$$

In general, if N is a closed submanifold of G_0 then the inverse image $\Phi^{-1}(N)$ is a PF submanifold of $V_{\mathfrak{g}}$ ([12, Lemma 5.1]).

Let K be a closed subgroup of G_0 with Lie algebra $\mathfrak{k}$. Denote by $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ the orthogonal direct sum decomposition. Restricting the $\mathrm{Ad}(G)$-invariant inner product of $\mathfrak{g}$ to $\mathfrak{m}$ we equip the induced G_0-invariant Riemannian metric with the homogeneous space G_0/K. We call such a metric the normal homogeneous metric and G_0/K the compact normal homogeneous space. Denote by $\pi : G_0 \to G_0/K$ the natural projection, which is a Riemannian submersion with totally geodesic fiber. The *parallel transport map Φ_K over G_0/K* is a Riemannian submersion defined by

$$\Phi_K := \pi \circ \Phi : V_{\mathfrak{g}} \to G_0 \to G_0/K. \tag{1.4}$$

2. Results

Let M be a compact Riemannian homogeneous space, that is, the group of isometries $G := I(M)$ is compact and the identity component $G_0 := I_0(M)$ acts transitively on M. Fix $p \in M$ and denote by $K := I_0(M)_p$ the isotropy subgroup of G_0 at $p \in M$. Then we have a diffeomorphism $G_0/K \cong M$. Choose an inner product of the Lie algebra $\mathfrak{g}$ which is invariant not only under $\mathrm{Ad}(G_0)$ but also under $\mathrm{Ad}(G)$. We equip the corresponding bi-invariant metric with G_0 and the normal homogeneous metric with G_0/K. Here note that M and G_0/K are *not* isometric in general. To avoid this we suppose that M is *isotropy irreducible*: the linear isotropy representation of K on the tangent space T_pM is $\mathbb{R}$-irreducible. From this condition the G_0-invariant Riemannian metric on G_0/K is unique up to scaling and thus we can (and will) always choose an $\mathrm{Ad}(G)$-invariant inner product of $\mathfrak{g}$ so that the normal homogeneous space G_0/K is isometric to M. For the geometry, structure and the classification of isotropy irreducible homogeneous spaces, see [13], [14] and [15].

The following theorem is the main result of this paper:

Theorem 1. *Let M be a compact isotropy irreducible Riemannian homogeneous space and N a closed submanifold of M. Denote by G the group of isometries on M, by G_0 its identity component, by K the isotropy subgroup of G_0 at a fixed $p \in M$ and by $\Phi_K = \pi \circ \Phi : V_{\mathfrak{g}} \to G_0 \to G_0/K = M$ the parallel transport map. Then for a closed subgroup S of G satisfying $aSa^{-1} = S$ for all $a \in G_0$ the following are equivalent:*

(i) *N is an S-weakly reflective submanifold of M,*
(ii) *$\pi^{-1}(N)$ is an $(S \times S_p)$-weakly reflective submanifold of G_0,*
(iii) *$\Phi_K^{-1}(N)$ is a $P(S, S \times S_p)$-weakly reflective PF submanifold of $V_{\mathfrak{g}}$,*

where S_p denotes the isotropy subgroup of S at p.

Remark 1. Strictly speaking "$(S\times S_p)$-weakly reflective" in the statement (ii) above should be written by "$((S\times S_p)\cap I(G_0))$-weakly reflective" because not all elements of $(S \times S_p)$ preserve G_0 and induce isometries on G_0. However we will continue to use such an abbreviation for simplicity.

Remark 2. In the previous paper [5, Theorem 8] M was assumed to be a symmetric space of compact type and only the case $S = G$ was considered. Theorem 1 here does not require such assumptions and moreover it characterizes the weakly reflective PF submanifold $\Phi_K^{-1}(N)$ precisely.

Considering the case $S = G$ we obtain:

Corollary 1. *Let $M = G_0/K$ and N be as above. Then the following are equivalent:*

(i) *N is a weakly reflective submanifold of M,*
(ii) *$\pi^{-1}(N)$ is a $(G \times G_p)$-weakly reflective submanifold of G_0,*
(iii) *$\Phi_K^{-1}(N)$ is a $P(G, G \times G_p)$-weakly reflective PF submanifold of $V_{\mathfrak{g}}$.*

Considering the case $S = G_0$ we obtain:

Corollary 2. *Let $M = G_0/K$ and N be as above. Then the following are equivalent:*

(i) *N is a G_0-weakly reflective submanifold of M,*
(ii) *$\pi^{-1}(N)$ is a $(G_0 \times K)$-weakly reflective submanifold of G_0,*
(iii) *$\Phi_K^{-1}(N)$ is a $P(G_0, G_0 \times K)$-weakly reflective PF submanifold of $V_{\mathfrak{g}}$.*

Next we suppose a certain homogeneous condition for N and the condition $S \subset G_0$. However we do not suppose that S satisfies $aSa^{-1} = S$ for *all* $a \in G_0$:

Theorem 2. *Let $M = G_0/K$ and N be as in Theorem 1. Suppose that N is an orbit of a closed subgroup U of G_0. Then for a closed subgroup S of G_0 satisfying $aSa^{-1} = S$ for all $a \in U$ the following are equivalent:*

(i) *N is an S-weakly reflective submanifold of M,*
(ii) *$\pi^{-1}(N)$ is an $(S \times K)$-weakly reflective submanifold of G_0,*
(iii) *$\Phi_K^{-1}(N)$ is a $P(G_0, S \times K)$-weakly reflective PF submanifold of $V_{\mathfrak{g}}$.*

Considering the case $S = U$ we obtain:

Corollary 3. *Let $M = G_0/K$, N and U be as in Theorem 2. Then the following are equivalent:*

(i) *N is an U-weakly reflective submanifold of M,*
(ii) *$\pi^{-1}(N)$ is an $(U \times K)$-weakly reflective submanifold of G_0,*
(iii) *$\Phi_K^{-1}(N)$ is a $P(G_0, U \times K)$-weakly reflective PF submanifold of $V_{\mathfrak{g}}$.*

Considering the case $S = U_0$ we obtain:

Corollary 4. *Let $M = G_0/K$, N and U be as in Theorem 2. Then the following are equivalent:*

(i) *N is an U_0-weakly reflective submanifold of M,*
(ii) *$\pi^{-1}(N)$ is an $(U_0 \times K)$-weakly reflective submanifold of G_0,*
(iii) *$\Phi_K^{-1}(N)$ is a $P(G_0, U_0 \times K)$-weakly reflective PF submanifold of $V_{\mathfrak{g}}$.*

Note that if N is an orbit $U \cdot aK$ through $aK \in G_0/K$ then we have $\pi^{-1}(N) = (U \times K) \cdot a$ and $\Phi_K^{-1}(N) = P(G_0, U \times K) * u$ for $u \in \Phi^{-1}(a)$ by (1.3). Applying above results to examples of weakly reflective submanifolds in M we obtain examples of infinite dimensional weakly reflective PF submanifolds in Hilbert spaces as follows:

Example 1. Let $M = G_0/K$ be as above and U a closed subgroup of G_0. Suppose that the U-action on M is of cohomogeneity one. Then any singular orbit $N = U \cdot aK$ is a U-weakly reflective submanifold of M ([8], [3]). Applying Corollary 3 to N the orbit $P(G_0, U \times K) * u$ through $u \in \Phi^{-1}(a)$ is a $P(G_0, U \times K)$-weakly reflective PF submanifold of $V_{\mathfrak{g}}$. In particular if $\operatorname{codim} N \geq 2$ it was essentially shown that N is a U_0-weakly reflective submanifold of M. Then by Corollary 4 the orbit $P(G_0, U \times K) * u$ is a $P(G_0, U_0 \times K)$-weakly reflective PF submanifold of $V_{\mathfrak{g}}$.

Example 2. Let $n \in \mathbb{Z}_{\geq 2}$. Set $M := S^{2n-1}(\sqrt{2}) \subset \mathbb{R}^{2n}$ and $N := S^{n-1}(1) \times S^{n-1}(1) \subset M$. It was shown ([3]) that N is an $O(2n)$-weakly reflective submanifold of M. Fix $p \in N$. Set $(G_0, K) := (SO(2n), SO(2n)_p)$ and $U := SO(n) \times SO(n)$ so that $M = G_0/K$ and $N = U \cdot eK$. Applying Corollary 1 to N the orbit $P(G_0, U \times K) * \hat{0}$ is a $P(O(2n), O(2n) \times O(2n)_p)$-weakly reflective PF submanifold of $V_{\mathfrak{g}}$. In particular if n is even then it was essentially shown that N is an $SO(2n)$-weakly reflective submanifold of M. Then by Corollary 2 the orbit $P(G_0, U \times K) * \hat{0}$ is a $P(SO(2n), SO(2n) \times SO(2n)_p)$-weakly reflective PF submanifold of $V_{\mathfrak{g}}$.

Example 3. Consider the symmetric pair $(SO(7), SO(3) \times SO(4))$ and the cohomogeneity one action of the exceptional Lie group G_2 on the symmetric space $M = SO(7)/(SO(3) \times SO(4))$. Then there exists a unique weakly reflective principal orbit $N = G_2 \cdot aK$, which was shown to be $O(7)$-weakly reflective ([1]). Applying

Theorem 1 to N the orbit $P(SO(7), G_2 \times SO(3) \times SO(4)) * u$ through $u \in \Phi^{-1}(a)$ is a $P(O(7), O(7) \times O(3) \times O(4))$-weakly reflective PF submanifold of $V_{\mathfrak{g}}$.

Remark 3. Recently Taketomi [9] introduced a generalized concept of weakly reflective submanifolds, namely arid submanifolds. The theorems and corollaries in this section are still valid in the arid case (see also [6]).

3. Proofs of the theorems

To prove the theorems we first recall several facts. Let G be a compact Lie group with a bi-invariant Riemannian metric and $\Phi : V_{\mathfrak{g}} \to G_0$ the parallel transport map. We know that the $P(G_0, G_0 \times \{e\})$-action on $V_{\mathfrak{g}}$ is simply transitive ([12, Corollary 4.2]). By (1.3) the following diagram commutes for each $g \in P(G_0, G_0 \times \{e\})$:

$$\begin{array}{ccc} V_{\mathfrak{g}} & \xrightarrow{g*} & V_{\mathfrak{g}} \\ {\scriptstyle\Phi}\downarrow & & {\scriptstyle\Phi}\downarrow \\ G_0 & \xrightarrow{(g(0), e)} & G_0. \end{array} \tag{3.1}$$

Let K be a closed subgroup of G_0 and G_0/K the compact normal homogeneous space with projection $\pi : G_0 \to G_0/K$. For each $a \in G_0$ we denote by l_a the left translation by a and L_a the isometry on G_0/K defined by $L_a(bK) := (ab)K$. Then we have the commutative diagram:

$$\begin{array}{ccc} G_0 & \xrightarrow{l_a} & G_0 \\ {\scriptstyle\pi}\downarrow & & {\scriptstyle\pi}\downarrow \\ G_0/K & \xrightarrow{L_a} & G_0/K. \end{array} \tag{3.2}$$

The following lemma was shown in [5, Lemma 2].

Lemma ([5]). *Let $\mathcal{M}$ and $\mathcal{B}$ be Riemannian Hilbert manifolds, $\phi : \mathcal{M} \to \mathcal{B}$ a Riemannian submersion and N a closed submanifold of $\mathcal{B}$. Fix $p \in \phi^{-1}(N)$ and $X \in T_p^{\perp}\phi^{-1}(N)$. Suppose that $\nu_{\mathcal{M}}$ is an isometry of $\mathcal{M}$ fixing p, that $\nu_{\mathcal{B}}$ is an isometry of $\mathcal{B}$ fixing $\phi(p)$ and that the diagram*

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\nu_{\mathcal{M}}} & \mathcal{M} \\ {\scriptstyle\phi}\downarrow & & {\scriptstyle\phi}\downarrow \\ \mathcal{B} & \xrightarrow{\nu_{\mathcal{B}}} & \mathcal{B} \end{array}$$

commutes. Then the following are equivalent:

(i) *$\nu_{\mathcal{M}}$ satisfies $\nu_{\mathcal{M}}(\phi^{-1}(N)) = \phi^{-1}(N)$ and $d\nu_{\mathcal{M}}(X) = -X$,*
(ii) *$\nu_{\mathcal{B}}$ satisfies $\nu_{\mathcal{B}}(N) = N$ and $d\nu_{\mathcal{B}}(d\phi(X)) = -d\phi(X)$.*

We are now in position to prove the theorems.

Proof of Theorem 1. By identification $M = G_0/K$ we have $p = eK$. From the assumption, for each $a \in G_0$ and each $g \in P(G_0, G_0 \times \{e\})$ we have

$$a^{-1}S_{aK}a = S_{eK}, \quad (a, e)^{-1}(S \times S_{eK})_a(a, e) = (S \times S_{eK})_e = \Delta(S_{eK}) \quad \text{and}$$
$$g^{-1}P(S, S \times S_{eK})_{g*\hat{0}}\,g = P(S, S \times S_{eK})_{\hat{0}} = \{\hat{s} \in \hat{S} \mid s \in S_{eK}\}.$$

"(i) $\Rightarrow$ (ii)": Let $a \in \pi^{-1}(N)$ and $w \in T_a^{\perp}\pi^{-1}(N)$. Set $\eta := d\pi(w) \in T_{aK}^{\perp}N$, $N' := L_a^{-1}(N)$ and $\xi := dL_a^{-1}(\eta) \in T_{eK}^{\perp}N'$. Denote by $v \in T_e^{\perp}\pi^{-1}(N')$ the horizontal lift of ξ. From (3.2) we have $l_a(\pi^{-1}(N')) = \pi^{-1}(N)$ and $dl_a(v) = w$. Thus to show the existence of a reflection ν_w with respect to w satisfying $\nu_w \in (S \times S_{eK})_a$ it suffices to construct an reflection ν_v with respect to v satisfying $\nu_v \in (S \times S_{eK})_e$. Take a reflection ν_η with respect to η satisfying $\nu_\eta \in S_{aK}$. Define a reflection ν_ξ with respect to ξ by $\nu_\xi := L_a^{-1} \circ \nu_\eta \circ L_a$. Then $\nu_\xi \in S_{eK}$. Since G_0 is a normal subgroup of G an automorphism $\nu_v : G_0 \to G_0$, $b \mapsto \nu_v(b)$ is well-defined by

$$\nu_v(b) := \nu_\xi \circ b \circ \nu_\xi^{-1},$$

that is,

$$\nu_v = \mathrm{Ad}^G(\nu_\xi)|_{G_0}.$$

Clearly $\nu_v \in (S \times S_{eK})_e$. Note that ν_v is an isometry of G_0 because we fixed a bi-invariant Riemannian metric on G_0 coming from an $\mathrm{Ad}(G)$-invariant inner product of $\mathfrak{g}$. Moreover we have

$$\begin{aligned}(\nu_\xi \circ \pi)(b) &= \nu_\xi(bK) = (\nu_\xi \circ b)(eK) = (\nu_\xi \circ b \circ \nu_\xi^{-1})(eK) \quad \text{and}\\ (\pi \circ \nu_v)(b) &= \nu_v(b)K = \nu_v(b)(eK) = (\nu_\xi \circ b \circ \nu_\xi^{-1})(eK)\end{aligned}$$

for all $b \in G_0$. This shows that the diagram

$$\begin{array}{ccc} G_0 & \xrightarrow{\nu_v} & G_0 \\ {\scriptstyle\pi}\downarrow & & {\scriptstyle\pi}\downarrow \\ G_0/K & \xrightarrow{\nu_\xi} & G_0/K \end{array}$$

commutes. Thus by Lemma, ν_v is a reflection with respect to v. This shows (ii).

"(ii) $\Rightarrow$ (i)": Let $aK \in N$ and $\eta \in T^\perp_{aK}N$. Denote by $w \in T^\perp_a \pi^{-1}(N)$ the horizontal lift of η. Define N', ξ, v as above. Take a reflection ν_w with respect to w satisfying $\nu_w \in (S \times S_{eK})_a$. Then a reflection with respect to v is defined by $\nu_v := (a, e)^{-1} \circ \nu_w \circ (a, e)$ so that $\nu_v \in (S \times S_{eK})_e$. Thus there exists $s \in S_{eK}$ such that $\nu_v = (s, s)$. Define an isometry ν_ξ of G_0/K by $\nu_\xi := s$. Then by Lemma, ν_ξ is a reflection with respect to ξ satisfying $\nu_\xi \in S_{eK}$. Hence a reflection ν_η with respect to η is defined by $\nu_\eta := l_a \circ \nu_\xi \circ l_a^{-1}$ so that $\nu_\eta \in S_{aK}$ and (i) follows.

"(ii) $\Rightarrow$ (iii)": Set $Q := \pi^{-1}(N)$ so that $\Phi_K^{-1}(N) = \Phi^{-1}(Q)$. Let $u \in \Phi^{-1}(Q)$ and $X \in T^\perp_u \Phi^{-1}(Q)$. Take $g \in P(G_0, G_0 \times \{e\})$ so that $u = g * \hat{0}$. Set $a := \Phi(u) = g(0)$, $w := d\Phi(X) \in T^\perp_a Q$, $Q' := a^{-1}Q$ and $v := dl_a^{-1}(w) \in T^\perp_e Q'$. Denote by $\hat{v} \in T^\perp_{\hat{0}} \Phi^{-1}(Q')$ the horizontal lift of v. From (3.1) we have $g * (\Phi^{-1}(Q')) = \Phi^{-1}(Q)$ and $(dg*)\hat{v} = X$. Thus to show the existence of a reflection ν_X with respect to X satisfying $\nu_X \in P(S, S \times S_{eK})_u$ it suffices to construct a reflection $\nu_{\hat{v}}$ with respect to $\hat{v}$ satisfying $\nu_{\hat{v}} \in P(S, S \times S_{eK})_{\hat{0}}$. Take a reflection ν_w with respect to w satisfying $\nu_w \in (S \times S_{eK})_a$. Then a reflection ν_v with respect to v is defined by $\nu_v := (a, e)^{-1} \circ \nu_w \circ (a, e)$ so that $\nu_v \in (S \times S_{eK})_e$. Thus there exists $s \in S_{eK}$ so that $\nu_v = (s, s)$. We define a linear orthogonal transformation $\nu_{\hat{v}}$ of $V_\mathfrak{g}$ by

$$\nu_{\hat{v}}(u) := d\nu_v \circ u = sus^{-1} = \hat{s} * u, \quad u \in V_\mathfrak{g}.$$

Clearly $\nu_{\hat{v}} \in P(S, S \times S_{eK})_{\hat{0}}$. Moreover since ν_v is an automorphism of G_0 we have $\nu_{\hat{v}}(g * \hat{0}) = (\nu_v \circ g) * \hat{0}$ for all $g \in P(G_0, G_0 \times G_0)$. This together with (1.3) implies that the following diagram commutes:

$$\begin{array}{ccc} V_{\mathfrak{g}} & \xrightarrow{\nu_{\hat{v}}} & V_{\mathfrak{g}} \\ {\scriptstyle \Phi}\big\downarrow & & {\scriptstyle \Phi}\big\downarrow \\ G_0 & \xrightarrow{\nu_v} & G_0. \end{array}$$

From Lemma $\nu_{\hat{v}}$ is a refection with respect to $\hat{v}$. This shows (iii).

"(iii) $\Rightarrow$ (ii)": Let $a \in Q$, $w \in T_a^{\perp} Q$ and $u \in \Phi^{-1}(a)$. Denote by $X \in T_u^{\perp} \Phi^{-1}(Q)$ the horizontal lift of w. Take $g \in P(G_0, G_0 \times \{e\})$ so that $g * \hat{0} = u$ and define Q', v, $\hat{v}$ as above. Take a reflection ν_X with respect to X satisfying $\nu_X \in P(S, S \times S_{eK})_u$. Then a reflection $\nu_{\hat{v}}$ with respect to $\hat{v} \in T_{\hat{0}}^{\perp} \Phi^{-1}(Q')$ is defined by $\nu_{\hat{v}} := (g*)^{-1} \circ \nu_X \circ (g*)$ so that $\nu_v \in P(S, S \times S_{eK})_{\hat{0}}$. Thus there exists $s \in S_{eK}$ such that $\nu_{\hat{v}} = \hat{s}$. Define an isometry ν_v of G_0 by $\nu_v := (s, s)$. From Lemma, ν_v is a reflection with respect to v satisfying $\nu_v \in (S \times S_{eK})_e$. Thus a reflection ν_w with respect to w is defined by $\nu_w := l_a \circ \nu_v \circ l_a^{-1}$ so that $\nu_w \in (S \times S_{eK})_a$. This shows (ii). □

Proof of Theorem 2. Take $b \in \pi^{-1}(N)$ and $u \in \Phi^{-1}(b)$. Then $N = U \cdot bK$, $\pi^{-1}(N) = (U \times K) \cdot b$ and $\Phi_K^{-1}(N) = P(G_0, U \times K) * u$. Choose $g \in P(G_0, G_0 \times \{e\})$ so that $u = g * \hat{0}$. Then $b = \Phi(u) = g(0)$. Set $U' := b^{-1} U b$. By (3.1) we have

$$L_b(U' \cdot eK) = U \cdot bK, \quad l_b((U' \times K) \cdot e) = (U \times K) \cdot b \quad \text{and}$$
$$g * (P(G_0, U' \times K) * \hat{0}) = P(G_0, U \times K) * u.$$

Set $S' := b^{-1} S b$. Then $a' S' (a')^{-1} = S'$ for all $a' \in U'$. By definition we have

$$b S' b^{-1} = S, \quad (b, e)(S' \times K)(b, e)^{-1} = S \times K \quad \text{and}$$
$$g P(G_0, S' \times K) g^{-1} = P(G_0, S \times K).$$

Thus we can assume $b = e$ without loss of generality. Since $S \subset G_0$ we have $S \cap K = S_{eK}$. Thus by the assumption, for each $a \in U$, $k \in K$ and $h \in P(G_0, U \times K)$ we have

$$a^{-1} S_{aK} a = S_{eK}, \quad (a, k)^{-1} (S \times K)_{(a,k) \cdot e} (a, k) = (S \times K)_e = \Delta(S_{eK}) \quad \text{and}$$
$$h^{-1} P(G_0, S \times K)_{h * \hat{0}} h = P(G_0, S \times K)_{\hat{0}} = \{\hat{s} \in \hat{S} \mid s \in S_{eK}\}.$$

Thus by homogeneity it suffices to consider normal vectors only at $eK \in G_0/K$, $e \in G_0$ and $\hat{0} \in V_{\mathfrak{g}}$. Take $\xi \in T_{eK}^{\perp}(U \cdot eK)$. Denote by $v \in T_e^{\perp}((U \times K) \cdot e)$ the horizontal lift of ξ and by $\hat{v} \in T_{\hat{0}}^{\perp}(P(G_0, U \times K) * \hat{0})$ the horizontal lift of v. By similar arguments as in the proof of Theorem 1 we can construct a reflection ν_v from ν_ξ and a reflection $\nu_{\hat{\xi}}$ from ν_ξ, and vice versa. These show that the statements (i), (ii) and (iii) in Theorem 2 are equivalent and our claim follows. □

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