Critical Hardy inequality on the half-space via the harmonic transplantation

Megumi Sano[a,1], Futoshi Takahashi[b]

[a] *Laboratory of Mathematics, School of Engineering, Hiroshima University, Higashi-Hiroshima, 739-8527, Japan / Mathematical Institute, Tohoku University, Sendai, 980-8578, Japan*
[b] *Department of Mathematics, Graduate School of Science, Osaka City University, Sumiyoshi-ku, Osaka, 558-8585, Japan*

Abstract

We prove a critical Hardy inequality on the half-space $\mathbb{R}^N_+$ by using the harmonic transplantation for functions in $\dot{W}^{1,N}_0(\mathbb{R}^N_+)$. Also we give an improvement of the subcritical Hardy inequality on $\dot{W}^{1,p}_0(\mathbb{R}^N_+)$ for $p \in [2, N)$, which converges to the critical Hardy inequality when $p \nearrow N$. Sobolev type inequalities are also discussed.

Keywords: Harmonic transplantation, The Hardy inequality
2010 MSC: 35A23, 35J20, 35A08

Contents

Email addresses: smegumi@hiroshima-u.ac.jp (Megumi Sano), futoshi@sci.osaka-cu.ac.jp (Futoshi Takahashi)
[1] Corresponding author.

1. Introduction and main results

Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a domain with $a \in \Omega$ and $1 < p < N$. The Hardy inequality

$$\left(\frac{N-p}{p}\right)^p \int_\Omega \frac{|u|^p}{|x-a|^p} dx \leq \int_\Omega |\nabla u|^p dx, \tag{1.1}$$

holds for all $u \in \dot{W}_0^{1,p}(\Omega)$, where $\dot{W}_0^{1,p}(\Omega)$ is the homogeneous Sobolev space defined as the completion of $C_c^\infty(\Omega)$ with respect to the (semi-)norm $\|\nabla(\cdot)\|_{L^p(\Omega)}$. It is well-known that $(\frac{N-p}{p})^p$ is the best constant and is not attained. Hardy's best constant $\left(\frac{N-p}{p}\right)^p$ plays an important role in investigating qualitative properties of solutions to elliptic or parabolic partial differential equations, such as, stability, instantaneous blow-up, and global-in-time asymptotics, see for example [4, 10].

On the other hand, in the limiting case where $p = N$, the Hardy inequality (1.1) looks degenerate as the best constant vanishes. However in this case, we can obtain the critical Hardy inequality on bounded domains:

$$\left(\frac{N-1}{N}\right)^N \int_\Omega \frac{|u|^N}{|x-a|^N \left(\log \frac{R}{|x-a|}\right)^N} dx \leq \int_\Omega |\nabla u|^p dx \quad \left(u \in \dot{W}_0^{1,N}(\Omega)\right) \tag{1.2}$$

as a limit of the Hardy inequality (1.1) as $p \nearrow N$, where $R = \sup_{x\in\Omega} |x-a|$, see [20] or I.-(ii) in §3.3. It is also known that $(\frac{N-1}{N})^N$ is the best constant and is not attained, see e.g. [1, 21, 9].

In the present paper, we introduce a critical Hardy inequality similar to (1.2) when Ω is unbounded, especially the half-space $\mathbb{R}^N_+ = \{(x, y) \mid x \in \mathbb{R}^{N-1}, y > 0\}$. Note that if $\Omega = \mathbb{R}^N$, this kind of inequality does not hold even if we restrict functions to radially symmetric ones, see Proposition 5.1 in §5.

Our first result is the following.

Theorem 1.1. *(Critical Hardy inequality on the half-space) Let $N \geq 2$. Then the inequality*

$$\left(\frac{N-1}{N}\right)^N \int_{\mathbb{R}^N_+} \frac{|u(x,y)|^N}{(|x|^2 + (1-y)^2)^{\frac{N}{2}} \left(\frac{|x|^2+(1+y)^2}{4}\right)^{\frac{N}{2}} \left(\log \sqrt{\frac{|x|^2+(1+y)^2}{|x|^2+(1-y)^2}}\right)^N} \, dxdy \leq \int_{\mathbb{R}^N_+} |\nabla u(x,y)|^N \, dxdy \tag{1.3}$$

holds for any $u \in \dot{W}_0^{1,N}(\mathbb{R}^N_+)$. *Furthermore,* $(\frac{N-1}{N})^N$ *is the best constant and is not attained.*

Remark 1.2. *(Asymptotic behavior of the potential function) Set*

$$V_N(x,y) := \frac{1}{(|x|^2 + (1-y)^2)\left(\frac{|x|^2+(1+y)^2}{4}\right)\left(\log \sqrt{\frac{|x|^2+(1+y)^2}{|x|^2+(1-y)^2}}\right)^2}. \tag{1.4}$$

Then the inequality (1.3) is of the form

$$\left(\frac{N-1}{N}\right)^N \int_{\mathbb{R}^N_+} V_N(x,y)^{\frac{N}{2}} |u(x,y)|^N \, dxdy \le \int_{\mathbb{R}^N_+} |\nabla u(x,y)|^N \, dxdy.$$

The inequality (1.3) has two aspects: one is the critical Hardy inequality on bounded domains and the other is the geometric Hardy inequality on $\mathbb{R}^N_+$, *which involves the distance from the boundary* $\partial\mathbb{R}^N_+$. *Indeed, the potential function* $V_N(x,y)^{\frac{N}{2}}$ *behaves like* $\left(|x|^2 + (1-y)^2\right)^{-\frac{N}{2}} \left(\log \frac{2}{\sqrt{|x|^2+(1-y)^2}}\right)^{-N}$ *when* (x,y) *is near to* $e_N = (0,1) \in \mathbb{R}^N_+$, *which is similar to the critical Hardy potential on the ball* $B_2(e_N)$ *with radius* 2 *and center* e_N. *Also* $V_N(x,y)^{\frac{N}{2}}$ *behaves like* $y^{-N} = \mathrm{dist}((x,y), \partial\mathbb{R}^N_+)^{-N}$ *near the boundary* $\partial\mathbb{R}^N_+$ *or* ∞, *which is similar to the geometric Hardy potential on* $\mathbb{R}^N_+$. *In fact, since* $Y := \frac{|x|^2+(1+y)^2}{|x|^2+(1-y)^2} = 1 + o(1)$ *as* $|x|^2 + (y-1)^2 \to \infty$ *or* $y \to 0$ *and* $\log Y = Y - 1 + o(1)$ *as* $Y \to 1$, *we have*

$$\begin{aligned}
V_N(x,y)^{\frac{N}{2}} &= \frac{1}{(|x|^2 + (1-y)^2)^{\frac{N}{2}} \left(\frac{|x|^2+(1+y)^2}{4}\right)^{\frac{N}{2}} \left(\log \sqrt{Y}\right)^N} \\
&= \frac{1}{(|x|^2 + (1-y)^2)^{\frac{N}{2}} \left(\frac{|x|^2+(1+y)^2}{4}\right)^{\frac{N}{2}} \left(\frac{Y-1}{2}\right)^N} + o(1) \\
&= \frac{\left(|x|^2 + (1-y)^2\right)^{\frac{N}{2}}}{y^N \left(|x|^2 + (1+y)^2\right)^{\frac{N}{2}}} + o(1) \\
&= O(y^{-N})
\end{aligned}$$

as $|x|^2 + (y-1)^2 \to \infty$ *or* $y \to 0$.

Next, we give an improvement of (1.1) which yields (1.3) as $p \nearrow N$. Improvements of the Hardy and the Hardy-Sobolev inequalities on balls are studied for radially symmetric functions in [15, 20, 30]. However, on the half-space $\mathbb{R}^N_+$, we cannot consider radial symmetry since radial functions which are zero on the boundary $\partial\mathbb{R}^N_+$ must be identically zero. Instead of radial symmetry, we introduce the following new symmetry for functions $u = u(x, y)$ on $\mathbb{R}^N_+$:

Put

$$U_p(x,y) = \begin{cases} \frac{p-1}{N-p}\omega_{N-1}^{-\frac{1}{p-1}}\left[\left(|x|^2 + (1-y)^2\right)^{-\frac{N-p}{2(p-1)}} - \left(|x|^2 + (1+y)^2\right)^{-\frac{N-p}{2(p-1)}}\right] & \text{if } p \in (1, N), \\ \omega_{N-1}^{-\frac{1}{N-1}} \log \sqrt{\frac{|x|^2+(1+y)^2}{|x|^2+(1-y)^2}} & \text{if } p = N, \end{cases} \tag{1.5}$$

where ω_{N-1} is the area of the unit sphere $\mathbb{S}^{N-1}$ in $\mathbb{R}^N$. Note that the function $U_p(x, y)$ is obtained from the fundamental solution of the p-Laplacian on $\mathbb{R}^N$ with singularity $e_N = (0, 1)$, by "reflecting the singularity" with respect to the boundary $\partial\mathbb{R}^N_+$. So $U_p \equiv 0$ on $\partial\mathbb{R}^N_+$. However, it is different from p-Green's function $G_{\mathbb{R}^N_+,e_N}(x, y)$ on $\mathbb{R}^N_+$ when $p \neq 2$ and $p \neq N$, see §2.

We consider functions on $\mathbb{R}^N_+$ of the form

$$u(x, y) = \tilde{u}(s), \text{ where } s = U_p(x, y), \quad (x, y) \in \mathbb{R}^N_+, \tag{1.6}$$

for some function $\tilde{u}$ on $\mathbb{R}$ with the property $\tilde{u}(0) = 0$. In the following, with some ambiguity, we identify $\tilde{u}$ as u and write, for example, $u(x, y) = u(s)$, $s = U_p(x, y)$ for $(x, y) \in \mathbb{R}^N_+$. Thus a function of the form (1.6) has the same value on each level set of U_p and vanishes on $\partial\mathbb{R}^N_+$.

Our second result is an improvement of the Hardy inequality (1.1) on $\mathbb{R}^N_+$ for functions with the symmetry (1.6).

Theorem 1.3. *(Improved Hardy inequality for $p \geq 2$) Let $2 \leq p < N$. Then the inequality*

$$\left(\frac{N-p}{p}\right)^p \int_{\mathbb{R}^N_+} \frac{V_p(x,y)^{\frac{p}{2}}}{\left(|x|^2 + (1-y)^2\right)^{\frac{p}{2}}} |u(x,y)|^p \, dxdy \leq \int_{\mathbb{R}^N_+} |\nabla u(x,y)|^p \, dxdy \tag{1.7}$$

holds for any $u \in \dot{W}^{1,p}_0(\mathbb{R}^N_+)$ of the form (1.6), where

$$\begin{cases} V_p(x,y) = \frac{1+X^{\frac{N-1}{p-1}} - 2X^{\frac{N-p}{2(p-1)}}\left(|x|^2+(1+y)^2\right)^{-1}(|x|^2+y^2-1)}{\left[1-X^{\frac{N-p}{2(p-1)}}\right]^2}, \\ X = \frac{|x|^2+(1-y)^2}{|x|^2+(1+y)^2} \in [0,1). \end{cases} \tag{1.8}$$

Furthermore, $(\frac{N-p}{p})^p$ is the best constant and is not attained.

Remark 1.4. *Actually, the inequality (4.10) holds for functions without the symmetry (1.6) by combining Proposition 2.1 and a result in [14], see Theorem 4.6 in §4. Our method is based on the harmonic transplantation.*

Remark 1.5. *($V_p \geq 1$) We remark here that (4.10) is an improvement of the Hardy inequality (1.1) with $a = e_N$ since $V_p(x,y) \geq 1$. In fact, for any $(x,y) \in \mathbb{R}^N_+ \cap \overline{B^N_1}$, we have $V_p(x,y) \geq 1 + X^{\frac{N-1}{p-1}} \geq 1$. Also, for any $(x,y) \in \mathbb{R}^N_+ \setminus \overline{B^N_1}$, we have*

$$
\begin{aligned}
V_p(x,y) &= 1 + \frac{X^{\frac{N-p}{p-1}}}{(1 - X^{\frac{N-p}{2(p-1)}})^2} \left[X - 1 + X^{-\frac{N-p}{2(p-1)}} \frac{4(y+1)}{|x|^2 + (1+y)^2} \right] \\
&= 1 + \frac{X^{\frac{N-p}{p-1}}}{(1 - X^{\frac{N-p}{2(p-1)}})^2} \left[\left\{ X^{-\frac{N-p}{2(p-1)}} - 1 \right\} \frac{4y}{|x|^2 + (1+y)^2} + X^{-\frac{N-p}{2(p-1)}} \frac{4}{|x|^2 + (1+y)^2} \right] \\
&\geq 1
\end{aligned}
$$

since $X \in [0,1)$.

Remark 1.6. *Unlike (1.1), it is possible to take the limit $p \nearrow N$ in the improved Hardy inequality (4.10). Thus we obtain Theorem 1.1 (for functions with symmetry* (1.6)*) from Theorem 1.3 in this way.*

In fact, since $1 - X^s = s \log \frac{1}{X} + o(s)$ as $s \to 0$, taking $s = \frac{N-p}{2(p-1)}$, we see that V_p in (1.8) *satisfies*

$$
\begin{aligned}
&\left(\frac{N-p}{p} \right)^p \frac{V_p(x,y)^{\frac{p}{2}}}{(|x|^2 + (1-y)^2)^{\frac{p}{2}}} \\
&= \left(\frac{N-p}{p} \right)^p \frac{\left\{ 1 + X - 2\left(|x|^2 + (y+1)^2 \right)^{-1} (|x|^2 + y^2 - 1) \right\}^{\frac{N}{2}}}{(|x|^2 + (1-y)^2)^{\frac{N}{2}} \left[\frac{N-p}{2(p-1)} \log \frac{1}{X} \right]^p} + o(1) \\
&= \left(\frac{N-1}{N} \right)^N V_N(x,y)^{\frac{N}{2}} + o(1) \quad (p \nearrow N)
\end{aligned}
$$

where V_N is defined in (1.4)*. Therefore, we obtain (1.3) as a limit of (4.10) as $p \nearrow N$.*

However, note that, Theorem 1.1 is proved by another method in §4 and valid for functions without any symmetry.

This paper is organized as follows: In §2, we show propositions about the function $U_p(x,y)$ in (1.5), which coincides with p-Green's function $G_{\mathbb{R}^N_+, e_N}(x,y)$

when $p = 2$ or N. Although U_p is different from $G_{\mathbb{R}^N_+, e_N}$ for $p \in (2, N)$, we can prove that U_p is superharmonic for $p \in (2, N)$ on $\mathbb{R}^N_+ \setminus \{e_N\}$. This is a key point of the proof of Theorem 1.3. In §3, we recall the Möbius transformation and the harmonic transplantation proposed by Hersch [18]. We point out that various transformations so far appeared in references can be understood as a special or a general case of harmonic transplantation. Also, we explain the difference between two transformations. In §4, we show Theorem 1.1 by exploiting the Möbius transformation. Due to the lack of the explicit form of p-Green's function $G_{\mathbb{R}^N_+, e_N}$ for $p \in (2, N)$, it seems difficult to apply the original harmonic transplantation which exploits the p-Green's functions, to obtain an improvement of the Hardy inequality on the half-space $\mathbb{R}^N_+$, see Theorem 4.2 in §3. We use U_p in (1.5) instead of p-Green's function $G_{\mathbb{R}^N_+, e_N}$ to define a modified version of the harmonic transplantation. By usng this new transformation, we show Theorem 1.3. In the last of §4, we mention that these transformations can be also applicable to Sobolev type inequalities. In §5, we show several propositions related to main theorems and give an application of a special type of harmonic transplantation.

We fix several notations: B_R or B_R^N denotes the N-dimensional ball centered 0 with radius R. As a matter of convenience, we set $B_\infty^N = \mathbb{R}^N$ and $\frac{1}{\infty} = 0$. ω_{N-1} denotes the area of the unit sphere $\mathbb{S}^{N-1}$ in $\mathbb{R}^N$. $[f > \varepsilon]$ denotes the set $\{(x, y) \in \mathbb{R}^N_+ \mid f(x, y) > \varepsilon\}$. $|A|$ denotes the Lebesgue measure of a set $A \subset \mathbb{R}^N$.

2. Green's function on the half-space

Let $G_{\Omega,a} = G_{\Omega,a}(z) : \Omega \setminus \{a\} \to \mathbb{R}$ be the p-Green function with singularity at $a \in \Omega$ associated with p-Laplace operator $\Delta_p(\cdot) = \mathrm{div}(|\nabla(\cdot)|^{p-2}\nabla(\cdot))$. Namely, $G_{\Omega,a}(z)$ satisfies

$$\begin{cases} -\Delta_p G_{\Omega,a}(z) = \delta_a(z), & z \in \Omega, \\ \quad G_{\Omega,a}(z) = 0 & z \in \partial\Omega, \end{cases} \tag{2.1}$$

where δ_a is the Dirac measure giving unit mass to a point $a \in \Omega$. When $\Omega = \mathbb{R}^N_+$ and $a = e_N = (0, 1)$, we have

$$G_{\mathbb{R}^N_+, e_N}(x, y) = \begin{cases} \frac{p-1}{N-p}\omega_{N-1}^{-\frac{1}{p-1}} \left[\left(|x|^2 + (1-y)^2\right)^{-\frac{N-p}{2(p-1)}} - \psi_p(x, y) \right] & \text{if } p \in (1, N), \\ \omega_{N-1}^{-\frac{1}{N-1}} \log \sqrt{\frac{|x|^2 + (1+y)^2}{|x|^2 + (1-y)^2}} & \text{if } p = N, \end{cases} \tag{2.2}$$

where ψ_p is a function with $\psi_p \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N_+)$, $\lim_{|x|^2 + (y-1)^2 \to 0} \left(|x|^2 + (1-y)^2\right)^{\frac{N-1}{2(p-1)}} \nabla\psi_p(x, y) = 0$, and $\psi_2(x, y) = \left(|x|^2 + (1+y)^2\right)^{-\frac{N-2}{2}}$ (see [22]). Note that U_p in (1.5) coincides with $G_{\mathbb{R}^N_+,e_N}$ for $p = 2$ or $p = N$. To the best of our knowledge, we do not know the explicit form of ψ_p when $p \neq 2$ and $p \neq N$. This fact causes some difficulty in the application of harmonic transplantation in §3 on $\mathbb{R}^N_+$, since we need the explicit form of Green's function in the use of harmonic transplantation. However, fortunately, we see that U_p is a super (or sub) solution of (2.1) in the distributional sense according to the range of p as follows. This fact enables us to use U_p instead of $G_{\mathbb{R}^N_+,e_N}$ in the proof of Theorem 1.3.

Proposition 2.1. *Let* $1 < p \leq N$ *and let* U_p *be as in* (1.5). *Then for any* $\phi \in C_c^\infty(\mathbb{R}^N_+)$ *with* $\phi \geq 0$,

$$\int_{\mathbb{R}^N_+} |\nabla U_p|^{p-2}\nabla U_p \cdot \nabla\phi \, dxdy = \phi(0,1) + \int_{\mathbb{R}^N_+} (-\Delta_p U)\, \phi \, dxdy \tag{2.3}$$

$$\begin{cases} \leq \phi(0,1) & \text{if } p \in (1,2], \\ \geq \phi(0,1) & \text{if } p \in [2,N), \\ = \phi(0,1) & \text{if } p = N. \end{cases}$$

Proposition 2.1 follows from Proposition 2.2.

Proposition 2.2. *Let* $1 < p \leq N$ *and let* U_p *be as in* (1.5). *Then for* $(x,y) \in \mathbb{R}^N_+ \setminus \{(0,1)\}$, *we have the followings:*

(I) Let $1 < p < N$. *Then*

$$-\Delta_p U_p = \frac{(N-p)(p-2)}{(p-1)^2\, \omega_{N-1}^{\frac{2}{p-1}}} |\nabla U_p|^{p-4} U_p \left[|x|^2 + (y-1)^2\right]^{-\frac{N-p}{2(p-1)}-1} \left[|x|^2 + (y+1)^2\right]^{-\frac{N-p}{2(p-1)}-1}$$
$$\times \left[N - p + (N+p-2)\frac{(|x|^2 + y^2 - 1)^2}{\{|x|^2 + (y-1)^2\}\{|x|^2 + (y+1)^2\}}\right].$$

(II) $-\Delta_N U_N = 0$.

Especially, we see that the pointwise estimates

$$\begin{cases} -\Delta_p U_p \leq 0 & \text{on } \mathbb{R}^N_+ \setminus \{e_N\}, \quad (1 < p \leq 2), \\ -\Delta_p U_p \geq 0 & \text{on } \mathbb{R}^N_+ \setminus \{e_N\}, \quad (2 \leq p < N), \\ -\Delta_p U_p = 0 & \text{on } \mathbb{R}^N_+ \setminus \{e_N\}, \quad (p = N) \end{cases}$$

hold.

Proof. (Proof of Proposition 2.2)

(I) For $(x, y) \in \mathbb{R}^N_+ \setminus \{(0, 1)\}$, we have

$$\omega_{N-1}^{\frac{1}{p-1}} \nabla U_p = -\left[|x|^2 + (y-1)^2\right]^{-\frac{N-p}{2(p-1)}-1} \begin{pmatrix} x \\ y-1 \end{pmatrix} + \left[|x|^2 + (y+1)^2\right]^{-\frac{N-p}{2(p-1)}-1} \begin{pmatrix} x \\ y+1 \end{pmatrix}$$

which implies that

$$\begin{aligned} |\nabla U_p|^2 = \omega_{N-1}^{-\frac{2}{p-1}} \Bigg[& \left\{|x|^2 + (y-1)^2\right\}^{-\frac{N-p}{p-1}-1} + \left\{|x|^2 + (y+1)^2\right\}^{-\frac{N-p}{p-1}-1} \\ & - 2\left\{|x|^2 + (y-1)^2\right\}^{-\frac{N-p}{2(p-1)}-1} \left\{|x|^2 + (y+1)^2\right\}^{-\frac{N-p}{2(p-1)}-1} \left\{|x|^2 + y^2 - 1\right\} \Bigg]. \end{aligned}$$

We put $V = |\nabla U_p|^2$. Then we have

$$\begin{aligned} &\operatorname{div}(|\nabla U_p|^{p-2} \nabla U_p) = \operatorname{div}(V^{\frac{p-2}{2}} \nabla U_p) = V^{\frac{p-4}{2}} \left[V \Delta U_p + \frac{p-2}{2} \nabla V \cdot \nabla U_p \right], \\ &\Delta U_p = \operatorname{div}(\nabla U_p) = -\frac{(N-1)(p-2)}{(p-1)\, \omega_{N-1}^{\frac{1}{p-1}}} \left[\left\{|x|^2 + (y-1)^2\right\}^{-\frac{N-p}{2(p-1)}-1} - \left\{|x|^2 + (y+1)^2\right\}^{-\frac{N-p}{2(p-1)}-1} \right]. \end{aligned}$$

Also, we have

$$\begin{aligned} &\omega_{N-1}^{\frac{2}{p-1}} \nabla V \\ &= -2\,\frac{N-1}{p-1} \left\{|x|^2 + (y-1)^2\right\}^{-\frac{N-p}{p-1}-2} \begin{pmatrix} x \\ y-1 \end{pmatrix} - 2\,\frac{N-1}{p-1} \left\{|x|^2 + (y+1)^2\right\}^{-\frac{N-p}{p-1}-2} \begin{pmatrix} x \\ y+1 \end{pmatrix} \\ &- 4\left\{|x|^2 + (y-1)^2\right\}^{-\frac{N-p}{2(p-1)}-1} \left\{|x|^2 + (y+1)^2\right\}^{-\frac{N-p}{2(p-1)}-1} \begin{pmatrix} x \\ y \end{pmatrix} \\ &+ 2\left(\frac{N-p}{p-1} + 2\right) \left\{|x|^2 + (y-1)^2\right\}^{-\frac{N-p}{2(p-1)}-2} \left\{|x|^2 + (y+1)^2\right\}^{-\frac{N-p}{2(p-1)}-1} (|x|^2 + y^2 - 1) \begin{pmatrix} x \\ y-1 \end{pmatrix} \\ &+ 2\left(\frac{N-p}{p-1} + 2\right) \left\{|x|^2 + (y-1)^2\right\}^{-\frac{N-p}{2(p-1)}-1} \left\{|x|^2 + (y+1)^2\right\}^{-\frac{N-p}{2(p-1)}-2} (|x|^2 + y^2 - 1) \begin{pmatrix} x \\ y+1 \end{pmatrix} \end{aligned}$$

which implies that

$$\frac{p-2}{2}\nabla V\cdot\nabla U_p$$

$$= \frac{p-2}{2}\omega_{N-1}^{-\frac{3}{p-1}}\left[\left[|x|^2+(y-1)^2\right]^{-\frac{N-p}{2(p-1)}-1}\begin{pmatrix}x\\ y-1\end{pmatrix}-\left[|x|^2+(y+1)^2\right]^{-\frac{N-p}{2(p-1)}-1}\begin{pmatrix}x\\ y+1\end{pmatrix}\right]\cdot$$

$$\left[2\,\frac{N-1}{p-1}\left\{|x|^2+(y-1)^2\right\}^{-\frac{N-p}{p-1}-2}\begin{pmatrix}x\\ y-1\end{pmatrix}+2\,\frac{N-1}{p-1}\left\{|x|^2+(y+1)^2\right\}^{-\frac{N-p}{p-1}-2}\begin{pmatrix}x\\ y+1\end{pmatrix}\right.$$

$$+4\left\{|x|^2+(y-1)^2\right\}^{-\frac{N-p}{2(p-1)}-1}\left\{x|^2+(y+1)^2\right\}^{-\frac{N-p}{2(p-1)}-1}\begin{pmatrix}x\\ y\end{pmatrix}$$

$$-2\left(\frac{N-p}{p-1}+2\right)\left\{|x|^2+(y-1)^2\right\}^{-\frac{N-p}{2(p-1)}-2}\left\{|x|^2+(y+1)^2\right\}^{-\frac{N-p}{2(p-1)}-1}(|x|^2+y^2-1)\begin{pmatrix}x\\ y-1\end{pmatrix}$$

$$\left.-2\left(\frac{N-p}{p-1}+2\right)\left\{|x|^2+(y-1)^2\right\}^{-\frac{N-p}{2(p-1)}-1}\left\{|x|^2+(y+1)^2\right\}^{-\frac{N-p}{2(p-1)}-2}(|x|^2+y^2-1)\begin{pmatrix}x\\ y+1\end{pmatrix}\right].$$

Therefore, we have

$$\frac{p-2}{2}\nabla V\cdot\nabla U_p$$

$$= \frac{p-2}{2}\omega_{N-1}^{-\frac{3}{p-1}}\left[2\,\frac{N-1}{p-1}\left[|x|^2+(y-1)^2\right]^{-\frac{3(N-p)}{2(p-1)}-2}\right.$$

$$+2\,\frac{N-1}{p-1}\left[|x|^2+(y-1)^2\right]^{-\frac{N-p}{2(p-1)}-1}\left[|x|^2+(y+1)^2\right]^{-\frac{N-p}{p-1}-2}(|x|^2+y^2-1)$$

$$+4\left[|x|^2+(y-1)^2\right]^{-\frac{N-p}{p-1}-2}\left[|x|^2+(y+1)^2\right]^{-\frac{N-p}{2(p-1)}-1}(|x|^2+y^2-y)$$

$$-2\left(\frac{N-1}{p-1}+1\right)\left\{|x|^2+(y-1)^2\right\}^{-\frac{N-p}{p-1}-2}\left\{|x|^2+(y+1)^2\right\}^{-\frac{N-p}{2(p-1)}-1}(|x|^2+y^2-1)$$

$$-2\left(\frac{N-1}{p-1}+1\right)\left\{|x|^2+(y-1)^2\right\}^{-\frac{N-p}{p-1}-2}\left\{|x|^2+(y+1)^2\right\}^{-\frac{N-p}{2(p-1)}-2}(|x|^2+y^2-1)^2$$

$$-2\,\frac{N-1}{p-1}\left\{|x|^2+(y-1)^2\right\}^{-\frac{N-p}{p-1}-2}\left\{|x|^2+(y+1)^2\right\}^{-\frac{N-p}{2(p-1)}-1}(|x|^2+y^2-1)$$

$$-2\,\frac{N-1}{p-1}\left[|x|^2+(y+1)^2\right]^{-\frac{3(N-p)}{2(p-1)}-2}+4\left[|x|^2+(y-1)^2\right]^{-\frac{N-p}{2(p-1)}-1}\left[|x|^2+(y+1)^2\right]^{-\frac{N-p}{2(p-1)}-2}(|x|^2+y^2+y)$$

$$+2\left(\frac{N-1}{p-1}+1\right)\left\{|x|^2+(y-1)^2\right\}^{-\frac{N-p}{2(p-1)}-2}\left\{|x|^2+(y+1)^2\right\}^{-\frac{N-p}{p-1}-2}(|x|^2+y^2-1)^2$$

$$\left.+2\left(\frac{N-1}{p-1}+1\right)\left\{|x|^2+(y-1)^2\right\}^{-\frac{N-p}{2(p-1)}-1}\left\{|x|^2+(y+1)^2\right\}^{-\frac{N-p}{p-1}-2}(|x|^2+y^2-1)\right].$$

Since

$$
\begin{aligned}
V\Delta U_p = -\omega_{N-1}^{-\frac{3}{p-1}} \frac{p-2}{2} \Bigg[& 2\,\frac{N-1}{p-1} \left\{|x|^2 + (y-1)^2\right\}^{-\frac{3(N-p)}{2(p-1)}-2} \\
& - 2\,\frac{N-1}{p-1} \left\{|x|^2 + (y-1)^2\right\}^{-\frac{N-p}{p-1}-1} \left\{|x|^2 + (y+1)^2\right\}^{-\frac{N-p}{2(p-1)}-1} \\
& + 2\,\frac{N-1}{p-1} \left\{|x|^2 + (y-1)^2\right\}^{-\frac{N-p}{2(p-1)}-1} \left\{|x|^2 + (y+1)^2\right\}^{-\frac{N-p}{p-1}-1} \\
& - 2\,\frac{N-1}{p-1} \left\{|x|^2 + (y+1)^2\right\}^{-\frac{3(N-p)}{2(p-1)}-2} \\
& - 4\,\frac{N-1}{p-1} \left\{|x|^2 + (y-1)^2\right\}^{-\frac{N-p}{p-1}-2} \left\{x|^2 + (y+1)^2\right\}^{-\frac{N-p}{2(p-1)}-1} (|x|^2 + y^2 - 1) \\
& + 4\,\frac{N-1}{p-1} \left\{|x|^2 + (y-1)^2\right\}^{-\frac{N-p}{2(p-1)}-1} \left\{|x|^2 + (y+1)^2\right\}^{-\frac{N-p}{p-1}-2} (|x|^2 + y^2 - 1) \Bigg],
\end{aligned}
$$

we have

$$
\begin{aligned}
& \operatorname{div}(|\nabla U_p|^{p-2}\nabla U_p) V^{-\frac{p-4}{2}} \omega_{N-1}^{\frac{3}{p-1}} \frac{2}{p-2} = \left[V\Delta U_p + \frac{p-2}{2} \nabla V \cdot \nabla U_p \right] \omega_{N-1}^{\frac{3}{p-1}} \frac{2}{p-2} \\
& = -2\,\frac{N-1}{p-1} \left\{|x|^2 + (y-1)^2\right\}^{-\frac{N-p}{2(p-1)}-1} \left\{|x|^2 + (y+1)^2\right\}^{-\frac{N-p}{2(p-1)}-1} U_p \frac{N-p}{p-1}\, \omega_{N-1}^{\frac{1}{p-1}} \\
& + 2 \left\{|x|^2 + (y-1)^2\right\}^{-\frac{N-p}{2(p-1)}-1} \left\{|x|^2 + (y+1)^2\right\}^{-\frac{N-p}{2(p-1)}-1} \Bigg[\left\{|x|^2 + (y+1)^2\right\}^{-\frac{N-p}{2(p-1)}-1} (|x|^2 + y^2 - 1) \\
& - \left\{|x|^2 + (y-1)^2\right\}^{-\frac{N-p}{2(p-1)}-1} (|x|^2 + y^2 - 1) + 2 \left\{|x|^2 + (y-1)^2\right\}^{-\frac{N-p}{2(p-1)}-1} (|x|^2 + y^2 - y) \\
& - 2 \left\{|x|^2 + (y+1)^2\right\}^{-\frac{N-p}{2(p-1)}-1} (|x|^2 + y^2 + y) \Bigg] \\
& - 2 \left(\frac{N-1}{p-1} + 1 \right) \left\{|x|^2 + (y-1)^2\right\}^{-\frac{N-p}{2(p-1)}-2} \left\{|x|^2 + (y+1)^2\right\}^{-\frac{N-p}{2(p-1)}-2} (|x|^2 + y^2 - 1)^2 U_p \\
& = -\frac{2(N-p)\,\omega_{N-1}^{\frac{1}{p-1}}}{(p-1)^2} U_p \left\{|x|^2 + (y-1)^2\right\}^{-\frac{N-p}{2(p-1)}-1} \left\{|x|^2 + (y+1)^2\right\}^{-\frac{N-p}{2(p-1)}-1} \\
& \left[N - p + (N+p-2) \frac{(|x|^2 + y^2 - 1)^2}{\{|x|^2 + (y-1)^2\}\{|x|^2 + (y+1)^2\}} \right]
\end{aligned}
$$

which implies Proposition 2.2 (I).
(II) The proof is done by direct calculation in the same way as (I). We omit the proof here.

$\square$

Proof. (Proof of Proposition 2.1) Let $B_\varepsilon(e_N)$ be the ball with center e_N and radius ε. For any $\phi \in C_c^\infty(\mathbb{R}^N_+)$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N_+} |\nabla U_p|^{p-2}\nabla U_p \cdot \nabla\phi\, dxdy \\
&= \int_{\partial B_\varepsilon(e_N)} |\nabla U_p|^{p-2}(\nabla U_p \cdot \nu)\, \phi\, dS + \int_{\mathbb{R}^N_+\setminus B_\varepsilon(e_N)} (-\Delta_p U_p)\, \phi\, dxdy + \int_{B_\varepsilon(e_N)} |\nabla U_p|^{p-2}\nabla U_p \cdot \nabla\phi\, dxdy.
\end{aligned} \tag{2.4}$$

where $\nu = -(x, y-1)^T \left\{|x|^2 + (y-1)^2\right\}^{-\frac{1}{2}}$. We claim

$$\int_{\partial B_\varepsilon(e_N)} |\nabla U_p|^{p-2}(\nabla U_p \cdot \nu)\, \phi\, dS = \phi(0,1) + o(1) \quad (\varepsilon \to 0), \tag{2.5}$$

$$\int_{B_\varepsilon(e_N)} |\nabla U_p|^{p-2}\nabla U_p \cdot \nabla\phi\, dxdy = o(1) \quad (\varepsilon \to 0), \tag{2.6}$$

Indeed, a direct calculation shows that

$$\begin{aligned}
&\int_{\partial B_\varepsilon(e_N)} |\nabla U_p|^{p-2}(\nabla U_p \cdot \nu)\, \phi\, dS \\
&= \omega_{N-1}^{-1} \int_{\partial B_\varepsilon} \Bigg[\left\{|x|^2 + (y-1)^2\right\}^{-\frac{N-p}{p-1}-1} + \left\{|x|^2 + (y+1)^2\right\}^{-\frac{N-p}{p-1}-1} - 2\left\{|x|^2 + (y-1)^2\right\}^{-\frac{N-p}{2(p-1)}-1} \\
&\left\{|x|^2 + (y+1)^2\right\}^{-\frac{N-p}{2(p-1)}-1} \{|x|^2 + y^2 - 1\}\Bigg]^{\frac{p-2}{2}} \Bigg[\left\{|x|^2 + (y-1)^2\right\}^{-\frac{N-p}{2(p-1)}} - \left\{|x|^2 + (y+1)^2\right\}^{-\frac{N-p}{2(p-1)}-1} \\
&\{|x|^2 + y^2 - 1\}\Bigg]\left\{|x|^2 + (y-1)^2\right\}^{-\frac{1}{2}}\ \phi\, dS \\
&= \omega_{N-1}^{-1}\varepsilon^{-1-\frac{N-p+(N-1)(p-2)}{p-1}} \int_{\partial B_\varepsilon(e_N)} \Bigg[1 + \left\{\frac{\varepsilon^2}{|x|^2 + (y+1)^2}\right\}^{\frac{N-1}{p-1}} - 2\varepsilon^{-\frac{N-1}{p-1}-1}\left\{|x|^2 + (y+1)^2\right\}^{-\frac{N-p}{2(p-1)}-1} \\
&\{|x|^2 + y^2 - 1\}\Bigg]^{\frac{p-2}{2}} \Bigg[1 - \left\{\frac{\varepsilon^2}{|x|^2 + (y+1)^2}\right\}^{\frac{N-p}{2(p-1)}} \frac{|x|^2 + y^2 - 1}{|x|^2 + (y+1)^2}\Bigg]\phi\, dS \\
&= \phi(0,1) + o(1) \quad (\varepsilon \to 0)
\end{aligned}$$

which implies (2.5).

On the other hand, we see

$$\left| \int_{B_\varepsilon(e_N)} |\nabla U_p|^{p-2} \nabla U_p \cdot \nabla\phi \, dxdy \right| \le C \, \|\nabla\phi\|_\infty \int_{B_\varepsilon(e_N)} \left\{ |x|^2 + (y-1)^2 \right\}^{-\frac{N-1}{2}} \, dxdy$$
$$= C \, \|\nabla\phi\|_\infty \, \omega_{N-1} \int_0^\varepsilon \, dr = o(1) \quad (\varepsilon \to 0),$$

which proves (2.6).

Finally, we check that the second term in (2.4) satisfies

$$\int_{\mathbb{R}^N_+ \setminus B_\varepsilon(e_N)} (-\Delta_p U_p) \, \phi \, dxdy \to \int_{\mathbb{R}^N_+} (-\Delta_p U_p) \, \phi \, dxdy \tag{2.7}$$

as $\varepsilon \to 0$. Actually, we have

$$|\Delta_p U_p| = O\left(|\nabla U_p|^{p-4} U_p (|x|^2 + (y-1)^2)^{-\frac{N-p}{2(p-1)} - 1} \right),$$
$$|\nabla U_p|^{p-4} = O\left((|x|^2 + (y-1)^2)^{(-\frac{N-p}{p-1} - 1)(\frac{p-4}{2})} \right),$$
$$|U_p| = O\left((|x|^2 + (y-1)^2)^{-\frac{N-p}{2(p-1)}} \right)$$

near $(x, y) = (0, 1)$ by Proposition 2.2. Thus we have

$$|\Delta_p U_p| = O\left((|x|^2 + (y-1)^2)^{(-\frac{N-p}{p-1} - 1)(\frac{p-4}{2}) - \frac{N-p}{2(p-1)} - \frac{N-p}{2(p-1)} - 1} \right)$$
$$= O\left(\left(\sqrt{|x|^2 + (y-1)^2} \right)^{(-\frac{N-p}{p-1} - 1)(p-4) - \frac{N-p}{(p-1)} - \frac{N-p}{(p-1)} - 2} \right)$$
$$= O\left(\left(\sqrt{|x|^2 + (y-1)^2} \right)^{-\frac{(N-1)(p-2)}{p-1}} \right)$$

near $(x, y) = (0, 1)$. This is locally integrable if

$$\frac{(N-1)(p-2)}{p-1} < N$$

which always holds for $p \in (1, N]$. Thus (2.7) follows from Lebesgue's dominated convergence theorem.

Returning to (2.4) with (2.5), (2.6), and (2.7), we obtain (2.3). □

3. Möbius transformation and harmonic transplantation

In this section, we recall Möbius transformation and harmonic transplantation. Both transformations preserve the norm $\|\nabla(\cdot)\|_p$ and coincide in the critical case $p = N$. However in the subcritical case $p < N$, these transformations are different from each other.

3.1. Möbius transformation

First, we recall the definition of Möbius transformation and its properties.

Definition 3.1. *(Möbius transformation) For $b \in \mathbb{R}^N$, $\lambda > 0$, $R \in O(N)$, where $O(N)$ is the orthogonal group in $\mathbb{R}^N$, set*

$$\begin{aligned} T_b(z) &= z + b \quad (translation), \\ S_\lambda(z) &= \lambda z \quad (scaling), \\ R(z) &= Rz \quad (rotation), \\ J(z) &= z^* = \frac{z}{|z|^2} \quad (reflection). \end{aligned}$$

A Möbius transformation $M : \mathbb{R}^N \to \mathbb{R}^N$ is a finite composition of T_b, S_λ, R and J. Also, the group of Möbius transformations is denoted by $M(\mathbb{R}^N)$.

Remark 3.2. *Set $u \otimes v = (u_i v_j)_{1\le i,j\le N}$ for $u = (u_1, \cdots, u_N)^T$ and $v = (v_1, \cdots, v_N)^T$. The differential and the Jacobian of each transformation are as follows.*

$$\begin{aligned} (T_b)'(z) &= I, \ \det(T_b)'(z) = 1 \\ (S_\lambda)'(z) &= \lambda I, \ \det(S_\lambda)'(z) = \lambda^N \\ R'(z) &= R, \ \det R'(z) = \det R = \pm 1 \\ J'(z) &= \frac{1}{|z|^2}\left(I - 2\frac{z}{|z|} \otimes \frac{z}{|z|}\right), \ \det J'(z) = \frac{(-1)}{|z|^{2N}} \end{aligned}$$

where I is the identity matrix on $\mathbb{R}^N$. For the proof of the last one, see Proposition 5.3.

For a function $f : \mathbb{R}^N \to \mathbb{R}$, we set

$$(M^\# f)(z) = |\det M'(z)|^{\frac{N-p}{Np}} f(M(z)) \tag{3.1}$$

for $z \in \mathbb{R}^N$. We call $M^\sharp f$ is also the Möbius transformation of the function f. Then we see that the transformation $M^\#$ preserves several quantities as follows.

Proposition 3.3. *Let* $1 \le p \le N$ *and* $p^* = \frac{Np}{N-p}$ *for* $p < N$*. If* $M \in M(\mathbb{R}^N)$*, then*

$$\int_{\mathbb{R}^N} |\nabla (M^\# f)(z)|^p \, dz = \int_{\mathbb{R}^N} |\nabla f(w)|^p \, dw \quad \textit{for } p = 2 \textit{ or } N, \tag{3.2}$$

$$\int_{\mathbb{R}^N} |(M^\# f)(z)|^{p^*} \, dz = \int_{\mathbb{R}^N} |f(w)|^{p^*} \, dw \quad \textit{for } p < N, \tag{3.3}$$

$$\int_{\mathbb{R}^N} \frac{|(M^\# f)(z)|^p}{|z|^p} \, dz = \int_{\mathbb{R}^N} \frac{|f(w)|^p}{|w|^p} \, dw \tag{3.4}$$

hold for any $f \in C_c^1(\mathbb{R}^N \setminus \{0\})$*.*

Remark 3.4. *From Remark 3.2, we have*

$$\begin{aligned}
(T_b^\# f)(z) &= f(z+b), \\
(S_\lambda^\# f)(z) &= \lambda^{\frac{N-p}{p}} f(\lambda z), \\
(R^\# f)(z) &= f(Rz), \\
(J^\# f)(z) &= |z|^{\frac{2}{p}(p-N)} f\left(\frac{z}{|z|^2}\right).
\end{aligned}$$

The last transformation is called the Kelvin transformation when $p = 2$ *or* N*. In the case where* $p \neq 2$ *and* $p \neq N$*, there is no radial function* $\rho = \rho(|z|) \not\equiv 0$ *such that*

$$\int_{\mathbb{R}^N} |\nabla g(z)|^p \, dz = \int_{\mathbb{R}^N} |\nabla f(w)|^p \, dw \textit{ holds for } g(z) = \rho(|z|) f\left(\frac{z}{|z|^2}\right),$$

see the proof below. Furthermore, all transformations except for $J^\#$ *above preserve the p-harmonicity of functions:* $\Delta_p f = 0$ *implies* $\Delta_p(M^\# f) = 0$*, where M is one of* T_b*,* S_λ*, and R. Also* $J^\#$ *preserves the p−harmonicity of functions when* $p = 2$ *or* $p = N$*. When* $p \neq 2$ *and* $p \neq N$*, it is shown in [24] that there is no radial function* ρ *such that* $\Delta_p g = 0$*, g as above, for any function f satisfying* $\Delta_p f = 0$*.*

Proof. (Proof of Proposition 3.3) We can easily show (3.3) and (3.4). We show (3.2) only. First, we claim that (3.2) holds for each transformation T_b, S_λ, R, J. We shall show (3.2) only for $M = J$. We use the polar coordinate $z = r\omega, r = |z|, \omega \in \mathbb{S}^{N-1}$. Then we have $w := Jz = s\omega$, $|w| = s = r^{-1}$, and

$$(J^\# f)(r\omega) = \rho(r) f(s\omega), \quad \text{where } \rho(r) = r^{\frac{2}{p}(p-N)}.$$

Therefore, we have

$$
\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla(J^\# f)(z)|^p \, dz \\
&= \int_0^\infty \int_{\mathbb{S}^{N-1}} \left[\left| \frac{\partial(J^\# f)}{\partial r} \right|^2 + \frac{1}{r^2} |\nabla_{\mathbb{S}^{N-1}}(J^\# f)|^2 \right]^{\frac{p}{2}} r^{N-1} \, drdS_\omega \\
&= \iint \left[\left| -\frac{\partial f}{\partial s} \rho(r) r^{-2} + \rho'(r) f \right|^2 + \rho(r)^2 r^{-2} |\nabla_{\mathbb{S}^{N-1}} f|^2 \right]^{\frac{p}{2}} r^{N-1} \, drdS_\omega \\
&= \iint \left[\left| \frac{\partial f}{\partial s} - \frac{\rho'(r) r^2}{\rho(r)} f \right|^2 + r^2 |\nabla_{\mathbb{S}^{N-1}} f|^2 \right]^{\frac{p}{2}} \rho(r)^p r^{-2p+N-1} \, drdS_\omega \\
&= \iint \left[\left| \frac{\partial f}{\partial s} \right|^2 - \frac{\rho'(r)}{\rho(r) s^2} \frac{\partial}{\partial s}(f^2) + \frac{|\rho'(r)|^2}{\rho(r)^2 s^4} f^2 + \frac{1}{s^2} |\nabla_{\mathbb{S}^{N-1}} f|^2 \right]^{\frac{p}{2}} \rho(r)^p r^{2(N-p)} s^{N-1} \, dsdS_\omega
\end{aligned}
$$

where $r = s^{-1}$. Since $\rho(r)^p r^{2(N-p)} = 1$ and $\rho(r) = 1$ for $p = N$, we obtain (3.2) for $p = N$. In the case where $p = 2$, by the integration by parts, we have

$$
\begin{aligned}
\int_{\mathbb{R}^N} |\nabla(J^\# f)(z)|^2 \, dz &= \int_{\mathbb{R}^N} |\nabla f(w)|^2 \, dw + (N-2)^2 \iint \left(\frac{\partial}{\partial s} \left(\frac{1}{s} \right) f^2 + \frac{1}{s^2} f^2 \right) s^{N-1} \, dsdS_\omega \\
&= \int_{\mathbb{R}^N} |\nabla f(w)|^2 \, dw.
\end{aligned}
$$

Therefore, we obtain the claim. Let $A, B \in \{T_b, S_\lambda, R, J\}$. Since

$$
\begin{aligned}
(A \circ B)^\#(z) &= |\det(A \circ B)'(z)|^{\frac{N-p}{Np}} f((A \circ B)(z)) \\
&= |\det A'(B(z)) \cdot \det B'(z)|^{\frac{N-p}{Np}} f(A(B(z))) \\
&= |\det B'(z)|^{\frac{N-p}{Np}} (A^\# f)(B(z)) \\
&= [(B^\# \circ A^\#) f](z),
\end{aligned}
$$

we have

$$
(A \circ B)^\# = B^\# \circ A^\#.
$$

From this and the claim, we have

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla[(A\circ B)^\# f](z)|^p\,dz &= \int_{\mathbb{R}^N} |\nabla[(B^\# \circ A^\#) f](z)|^p\,dz \\ &= \int_{\mathbb{R}^N} |\nabla(A^\# f)(w)|^p\,dw \\ &= \int_{\mathbb{R}^N} |\nabla f(\xi)|^p\,d\xi\end{aligned}$$

for $p = 2$ or N, where $w = B(z)$ and $\xi = A(w) = (A\circ B)(z)$. Since Möbius transformation is a finite composition of T_b, S_λ, R, J, we obtain (3.2) for any $M \in M(\mathbb{R}^N)$ by induction. □

For more information on Möbius transformation, see e.g. [2, 5].

3.2. An example of Möbius transformation: Cayley type transformation

Let $N \geq 2$ and $p = 2$ or N. Consider the transformation $\mathbf{B}$ from $\mathbb{R}^N$ to $\mathbb{R}^N$ as follows (Ref. [7]).

$$(\tilde{x}, \tilde{y}) = \mathbf{B}(x, y) = \left(\frac{2x,\ 1 - |x|^2 - y^2}{(1+y)^2 + |x|^2}\right) \quad (x, y) \in \mathbb{R}^N,\ (\tilde{x}, \tilde{y}) \in \mathbb{R}^N. \tag{3.5}$$

We see

$$|\mathbf{B}(x, y)|^2 = \frac{|x|^2 + (y-1)^2}{|x|^2 + (y+1)^2}, \tag{3.6}$$

thus if we restrict $\mathbf{B}$ on $\mathbb{R}^N_+$, then $\mathbf{B}$ maps $\mathbb{R}^N_+$ to the unit ball $B^N_1 \subset \mathbb{R}^N$. Also $|\mathbf{B}(x, y)| = 1$ if and only if $y = 0$, thus $\mathbf{B}(\partial\mathbb{R}^N_+) = \partial B^N_1$. We can check that the inverse function $\mathbf{B}^{-1}$ is the same as $\mathbf{B}$, that is

$$(x, y) = \mathbf{B}^{-1}(\tilde{x}, \tilde{y}) = \left(\frac{2\tilde{x},\ 1 - |\tilde{x}|^2 - \tilde{y}^2}{(1+\tilde{y})^2 + |\tilde{x}|^2}\right).$$

Note that the transformation (3.5) is a Möbius transformation:$\mathbf{B} \in M(\mathbb{R}^N)$. In fact, we see that

$$\mathbf{B}(z) = R\circ J\circ T_{e_N}\circ S_2\circ J\circ T_{-e_N}(z), \text{where } R = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & -1 \end{pmatrix},\ e_N = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \tag{3.7}$$

see [2] p.34 or Proposition 5.4 in §5. Therefore, we obtain the following.

Proposition 3.5. *Let* $\mathbf{B} : \mathbb{R}^N \to B_1^N$ *be given by (3.5) and let* $z = (x, y)$. *Then*

$$\det \mathbf{B}'(z) = -\left\{ \frac{2}{(1+y)^2 + |x|^2} \right\}^N .$$

Proof. From (3.7) and Remark 3.2, we have

$$\begin{aligned}
\det \mathbf{B}'(z) = &\underbrace{\det R}_{=-1} \cdot \det J'\left(T_{e_N} \circ S_2 \circ J \circ T_{-e_N}(z)\right) \cdot \underbrace{\det (T_{e_N})'\left(S_2 \circ J \circ T_{-e_N}(z)\right)}_{=1} \\
&\cdot \underbrace{\det S_2'\left(J \circ T_{-e_N}(z)\right)}_{=2^N} \cdot \det J'(T_{-e_N}(z)) \cdot \det \underbrace{(T_{-e_N})'(z)}_{=1} \\
&= (-1) \cdot \frac{(-1)}{|T_{e_N} \circ S_2 \circ J \circ T_{-e_N}(z)|^{2N}} \cdot 2^N \cdot \frac{(-1)}{|T_{-e_N}(z)|^{2N}} \\
&= (-1) \frac{1}{|e_N + 2(z - e_N)^*|^{2N}} \cdot \frac{1}{|z - e_N|^{2N}} \cdot 2^N \\
&= (-1) \frac{1}{|e_N + 2\frac{(z-e_N)}{|z-e_N|^2}|^{2N}} \cdot \frac{1}{|z - e_N|^{2N}} \cdot 2^N = (-1) \frac{2^N}{\{|x|^2 + (1+y)^2\}^N}.
\end{aligned}$$

□

3.3. Harmonic transplantation

Harmonic transplantation was first proposed by J. Hersch [18] in the attempt to extend several isoperimetric problems on two-dimensional simply-connected domains to higher connectivity and higher dimensions, see also [17, 3]. Here, we recall the original harmonic transplantation from B_1^N to $\Omega \subset \mathbb{R}^N$.

For $v \in \dot{W}^{1,p}_{0,\mathrm{rad}}(B_1^N)$ and $a \in \Omega$, define $H_a(v) = u : \Omega \setminus \{a\} \to \mathbb{R}$ by

$$u(y) = H_a(v)(y) = v\left(\left(G_{B_1^N, O} \right)^{-1} \left(G_{\Omega, a}(y) \right) \right), \tag{3.8}$$

where $G_{B_1^N,O}$ and $G_{\Omega,a}$ are p-Green's functions on the ball B_1^N with the pole O and on Ω with the pole $a \in \Omega$, respectively. In the case $p \in (1, N)$, we have

$$G_{B_1^N,O}(z) = \frac{p-1}{N-p} \omega_{N-1}^{-\frac{1}{p-1}} \left[|z|^{-\frac{N-p}{p-1}} - 1 \right] \tag{3.9}$$

which implies that

$$\left(G_{B_1^N,O} \right)^{-1} \left(G_{\Omega,a}(y) \right) = \left[\frac{N-p}{p-1} \omega_{N-1}^{\frac{1}{p-1}} G_{\Omega,a}(y) + 1 \right]^{-\frac{p-1}{N-p}} .$$

Also, we can rewrite the transformation (3.8) to

$$u(y) = v(z), \text{ where } G_{\Omega,a}(y) = G_{B_1^N,O}(z). \tag{3.10}$$

Hereafter, we call the transformed function $u = H_a(v)$ on $\Omega \setminus \{a\}$ via (3.10) *the harmonic transplantation* of a function $v \in \dot{W}^{1,p}_{0,rad}(B_1^N)$. We see that harmonic transplantation (3.10) preserves $\|\nabla(\cdot)\|_{L^p}$ for $p \in (1, N]$. A proof of this fact is shown for the sake of reader's convenience.

Lemma 3.6. *([17] Theorem 10.3, [13] Lemma 22) Let* $v \in \dot{W}^{1,p}_{0,\mathrm{rad}}(B_1^N)$ *and* $1 < p \le N$. *Then* $H_a(v) \in \dot{W}^{1,p}_0(\Omega)$ *and* $\|\nabla\left(H_a(v)\right)\|_{L^p(\Omega)} = \|\nabla v\|_{L^p(B_1^N)}$.

Proof. In the case where $p = N$, see [13] Lemma 22. Let $1 < p < N$. We write $G = G_{\Omega,a}$. Let h be defined by

$$h(y) = \left(G_{B_1^N,O}\right)^{-1}(G(y)) = \left[\frac{N-p}{p-1}\omega_{N-1}^{\frac{1}{p-1}} G(y) + 1\right]^{-\frac{p-1}{N-p}}, \quad y \in \Omega,$$

and hence $u(y) = v(h(y))$. In particular, $\nabla u(y) = v'(h(y))\nabla h(y)$. Note that since $G \ge 0$ in Ω, we get that $0 < h(y) \le 1$ on $\overline{\Omega}$ and if $y \in h^{-1}(\{t\}) \cap \Omega$, then $t \in [0, 1]$. Thus the coarea formula gives that

$$\begin{aligned}
\int_\Omega |\nabla u|^p &= \int_\Omega |v'(h(y))|^p |\nabla h(y)|^{p-1} |\nabla h(y)|\, dy \\
&= \int_0^1 \left[\int_{h^{-1}(\{t\})\cap\Omega} |v'(h(y))|^p |\nabla h(y)|^{p-1}\, d\mathcal{H}^{N-1}(y)\right] dt.
\end{aligned}$$

Using $|\nabla h| = \omega_{N-1}^{\frac{1}{p-1}} h(y)^{\frac{N-1}{p-1}} |\nabla G(y)|$, we have

$$\int_\Omega |\nabla u|^p = \int_0^1 \omega_{N-1} t^{N-1} |v'(t)|^p \left[\int_{h^{-1}(\{t\})\cap\Omega} |\nabla G(y)|^{p-1}\, d\mathcal{H}^{N-1}(y)\right] dt.$$

Note that $h^{-1}(\{t\}) \cap \Omega$ is also a level set of G. Since

$$\int_{\{G<t\}} |\nabla G(y)|^p\, dy = t, \qquad \int_{\{G=t\}} |\nabla G(y)|^{p-1}\, d\mathcal{H}^{N-1}(y) = 1.$$

for any $t \in [0, \infty)$ (Ref. [17] Lemma 9.1, or [13] Proposition 4), we obtain

$$\int_{h^{-1}(\{t\})\cap\Omega} |\nabla G(y)|^{p-1}\, d\mathcal{H}^{N-1}(y) = 1 \quad ({}^\forall t \in (0, 1)),$$

which implies that

$$\int_{\Omega} |\nabla u|^p = \int_0^1 \omega_{N-1} t^{N-1} |v'(t)|^p \, dt = \int_{B_1^N} |\nabla v|^p.$$

$\square$

Up to now, various transformations are found in literature so far. These transformations can be understood as a variant of harmonic transplantation. Here, we classify these transformations into the following three types:

I. Domains of two Green's functions in (3.10) are different from each other.
II. Operators of two Green's functions in (3.10) are different from each other.
III. Dimensions of two Green's functions in (3.10) are different from each other.

Original harmonic transplantation (3.10) is type I. For reader's convenience, we unify these transformations in the form of (3.10) and summarize their properties briefly. In the present paper, we use harmonic transplantation (3.10) in I.-(ii) below.

I.-(i): Critical case: $1 < p = N$
If $\Omega = B_R^N, a = O$, then the harmonic transplantation $u = H_a(v)$ in (3.10) becomes

$$u(y) = v(z), \text{ where } G_{B_1^N, O}(y) = \omega_{N-1}^{-\frac{1}{N-1}} \log \frac{1}{|y|} = \omega_{N-1}^{-\frac{1}{N-1}} \log \frac{R}{|z|} = G_{B_R^N, O}(z)$$

which coincides with the scaling $z = S_R(y) = Ry$. On the other hand, if $\Omega = \mathbb{R}^N_+$, $a = (0, 1)$, then the harmonic transplantation $u = H_a(v)$ in (3.10) coincides with the function $\mathbf{B}^\sharp v$ by the Cayley type transformation $\mathbf{B}$ in (3.5), see §3.2. Similar to the various rearrangement techniques, harmonic transplantation (3.10) enables us to construct appropriate test functions for various minimization or maximization problems: we refer the readers to the application of harmonic transplantation to the study of the Trudinger-Moser maximization problem on general bounded domain Ω (Ref. [16, 12, 23, 13]).

I.-(ii): Subcritical case: $1 < p < N$
If $\Omega = B_R^N$ (let $B_\infty^N = \mathbb{R}^N$ and $\frac{1}{\infty} = 0$), $a = O$, then the harmonic transplantation

$u = H_a(v)$ in (3.10) becomes

$$u(y) = v(z), \text{ where } G_{B_1^N,O}(y) = \frac{p-1}{N-p}\omega_{N-1}^{-\frac{1}{p-1}}\left[|y|^{-\frac{N-p}{p-1}} - 1 \right]$$
$$= \frac{p-1}{N-p}\omega_{N-1}^{-\frac{1}{p-1}}\left[|z|^{-\frac{N-p}{p-1}} - R^{-\frac{N-p}{p-1}} \right] = G_{B_R^N,O}(z)$$

which does not coincide with $S_R^{\sharp}(v)$, here S_R is the dilation $z = S_R(y) = Ry\,(R < \infty)$, unlike I.-(i).

We can obtain improved Hardy-Sobolev inequalities on B_R^N via (3.10), which are equivalent to the Hardy-Sobolev inequalities on $\mathbb{R}^N$ (Ref. [20]. See also [31, 30]). Not only the improvement of the inequalities, but also a limit of the improved Hardy-Sobolev inequalities on B_R^N as $p \nearrow N$ can be considered, unlike the classical cases. For the subcritical Rellich inequality, a part of this argument still holds, see §2 in [28]. A limit of the Hardy-Sobolev and the Poincaré inequalities (in some sense) can be considered, see [20] and [6] for taking a limit $p \nearrow N$ or $N \nearrow \infty$ in the Sobolev inequality respectively, [33] for $p \nearrow N$ in the Hardy inequality, and $|\Omega| \searrow 0$ in the Poincaré inequality. Also see [32] for a survey.

In the present paper, we consider the harmonic transplantation $u = H_a(v)$, $v \in \dot{W}_{0,rad}^{1,p}(B_1^N)$, in (3.10) for $\Omega = \mathbb{R}_+^N$, $a = e_N = (0, 1)$, and $p \in (1, N)$. Namely,

$$u(x, y) = v(\tilde{x}, \tilde{y}), \text{ where}$$
$$G_{\mathbb{R}_+^N,(0,1)}(x, y) = \frac{p-1}{N-p}\omega_{N-1}^{-\frac{1}{p-1}}\left[\left(|x|^2 + (1-y)^2\right)^{-\frac{N-p}{2(p-1)}} - \psi_p(x, y) \right]$$
$$= \frac{p-1}{N-p}\omega_{N-1}^{-\frac{1}{p-1}}\left[\left(|\tilde{x}|^2 + |\tilde{y}|^2\right)^{-\frac{N-p}{2(p-1)}} - 1 \right] = G_{B_1^N,O}(\tilde{x}, \tilde{y}) \qquad (3.11)$$

where ψ_p is as in (2.2).

Remark 3.7. *We point out that, in the case $2 = p < N$, there are at least two transformations u of $v \in \dot{W}_{0,rad}^{1,2}(B_1^N)$, by which $\|\nabla u\|_{L^2(\Omega)} = \|\nabla v\|_{L^2(B_1^N)}$ holds. Indeed, when $\Omega = B_R^N$ for $R < \infty$, the harmonic transplantation $u = H_0(v)$ and the Möbius transformation $u = S_R^{\#}(v)$ from $\dot{W}_{0,rad}^{1,2}(B_1^N)$ to $\dot{W}_0^{1,2}(B_R^N)$ preserve the L^2 norm of the gradient. Also when $\Omega = \mathbb{R}_+^N$, the harmonic transplantation $u = H_{e_N}(v)$ and the Möbius transformation $u = \mathbf{B}^{\#}(v)$ via the Cayley type transformation (3.5) from $\dot{W}_{0,rad}^{1,2}(B_1^N)$ to $\dot{W}_0^{1,2}(\mathbb{R}_+^N)$ have the same property, see §3.2.*

II.: From weighted problem to unweighted problem

Let $1 < p < N$ and $\tilde{G}_{B_1^N,O}$ be Green's function with singularity at O associated with the weighted p-Laplace operator $\operatorname{div}(|y|^{p-N}|\nabla(\cdot)|^{p-2}\nabla(\cdot))$. Define $u : B_1^N \to \mathbb{R}$ by

$$u(y) = v(z), \text{where } \tilde{G}_{B_1^N,O}(y) = \omega_{N-1}^{-\frac{1}{p-1}} \log \frac{1}{|y|} = \frac{p-1}{N-p}\omega_{N-1}^{-\frac{1}{p-1}} |z|^{-\frac{N-p}{p-1}} = G_{\mathbb{R}^N,O}(z),$$
$$y \in B_1^N,\ z \in \mathbb{R}^N.$$

Then we have $\|\nabla u\|_{L^p(B_1^N;\,|y|^{p-N}\,dy)} = \|\nabla v\|_{L^p(\mathbb{R}^N)}$ (Ref. [37, 19]). We can remove the weight $|y|^{p-N}$ thanks to the above transformation.

III.-(i): From higher dimensions to one dimension
Consider the Moser transformation

$$u(y) = v(z), \text{ where } G_{B_R^N,O}(y) = \omega_{N-1}^{-\frac{1}{N-1}} \log \frac{R}{|y|} = z. \quad y \in B_R^N, z \in \mathbb{R}_+.$$

Then we have $\|\nabla u\|_{L^N(B_R^N)} = \|v'\|_{L^N(\mathbb{R}_+)}$. The Moser transformation is used to reduce the Trudinger-Moser maximization problem on $\dot{W}^{1,N}_{0,\mathrm{rad}}(B_R^N)$ to the one-dimensional problem (Ref. [26]). On the other hand, if we consider the Moser transformation on the subcritical Sobolev spaces $\dot{W}^{1,p}_{0,\mathrm{rad}}(B_R^N)$ $(p < N)$, then we have

$$u(y) = v(z), \text{ where } G_{B_R^N,O}(y) = \frac{p-1}{N-p}\omega_{N-1}^{-\frac{1}{p-1}} \left[|y|^{-\frac{N-p}{p-1}} - R^{-\frac{N-p}{p-1}} \right] = z.$$

Then again we have $\|\nabla u\|_{L^p(B_R^N)} = \|v'\|_{L^p(\mathbb{R}_+)}$. For an application of these transformations, see Proposition 5.5 in §5.

III.-(ii): Relation between the critical and the subcritical Sobolev spaces
Let $p = N < m$. If we consider the relation

$$\begin{aligned} u(y) = v(z), \text{ where } G_{B_R^N,O}(y) &= \omega_{N-1}^{-\frac{1}{N-1}} \log \frac{R}{|y|} \\ &= \frac{N-1}{m-N}\omega_{m-1}^{-\frac{1}{N-1}} \left[|z|^{-\frac{m-N}{N-1}} - R^{-\frac{m-N}{N-1}} \right] = G_{B_R^m,O}(z), \end{aligned} \tag{3.12}$$

for $u \in \dot{W}^{1,N}_{0,rad}(B_R^N)$, $v \in \dot{W}^{1,N}_{0,rad}(B_R^m)$. then we have $\|\nabla u\|_{L^N(B_R^N)} = \|\nabla v\|_{L^N(B_R^m)}$. Namely, we obtain the equality between two norms of the critical Sobolev spaces $\dot{W}^{1,N}_{0,\mathrm{rad}}(B_R^N)$

and the higher dimensional subcritical Sobolev spaces $\dot{W}^{1,p}_{0,\mathrm{rad}}(B^m_R)$ (Ref. [34]). This transformation (3.12) gives a direct relation between the subcritical Sobolev embeddings

$$\dot{W}^{1,p}_{0,\mathrm{rad}} \hookrightarrow L^{p^*,p} \hookrightarrow L^{p^*,q} \hookrightarrow L^{p^*,\infty}$$

where $p < q$, and the critical Sobolev embeddings

$$\dot{W}^{1,N}_{0,\mathrm{rad}} \hookrightarrow L^{\infty,N}(\log L)^{-1} \hookrightarrow L^{\infty,q}(\log L)^{-1+\frac{1}{N}-\frac{1}{q}} \hookrightarrow L^{\infty,\infty}(\log L)^{-1+\frac{1}{N}} = \mathrm{ExpL}^{\frac{N}{N-1}}.$$

For the subcritical and the critical Sobolev embeddings, see e.g. [33] §1.

III.-(iii): An infinite dimensional form of the Sobolev inequality
Let $p < N < m$. If we consider

$$u(y) = v(z), \text{where } G_{\mathbb{R}^N,O}(y) = \frac{p-1}{N-p}\omega_{N-1}^{-\frac{1}{p-1}}|y|^{-\frac{N-p}{p-1}} = \frac{p-1}{m-p}\omega_{m-1}^{-\frac{1}{p-1}}|z|^{-\frac{m-p}{p-1}} = G_{\mathbb{R}^m,O}(z),$$
$$y \in \mathbb{R}^N,\ z \in \mathbb{R}^m$$

then we have $\|\nabla u\|_{L^p(\mathbb{R}^N)} = \|\nabla v\|_{L^p(\mathbb{R}^m)}$. Namely, we can reduce the m-dimensional Sobolev inequality: $S_{m,p}\|v\|^p_{L^{p^*}(\mathbb{R}^m)} \le \|\nabla v\|^p_{L^p(\mathbb{R}^m)}$ to an N-dimensional inequality for u, which involves m as a parameter and m can be arbitrarily large. Therefore, we can take a limit of the m-dimensional Sobolev inequality as $m \nearrow \infty$ in this sense. As a consequence, we can obtain the N-dimensional Hardy inequality: $\left(\frac{N-p}{p}\right)^p \int_{\mathbb{R}^N} \frac{|u|^p}{|y|^p}\,dy \le \int_{\mathbb{R}^N} |\nabla u|^p\,dy$ as an infinite dimensional form of the Sobolev inequality (Ref. [31]).

4. Proof of Theorems

First, we show Theorem 1.1.

Proof. (Proof of Theorem 1.1) Let $p = N$. We will "transplant" the critical Hardy inequality on B^N_1 to $\mathbb{R}^N_+$ by the Cayley type transformation $\mathbf{B}$ (3.5) in §3.2.

By (3.6), we consider $\mathbf{B}$ maps $\mathbb{R}^N_+$ to B^N_1. Let $u \in C^1_c(\mathbb{R}^N_+)$ and put $v(\tilde{x},\tilde{y}) = (\mathbf{B}^{-1})^\#(u)(\tilde{x},\tilde{y})$ for $(\tilde{x},\tilde{y}) \in B^N_1$. Then we see $u(x,y) = \mathbf{B}^\#(v)(x,y) = v(\mathbf{B}(x,y))$ for $(x,y) \in \mathbb{R}^N_+$. From the fact that $\mathbf{B} \in M(\mathbb{R}^N)$ and Proposition 3.3, we have

$$\int_{\mathbb{R}^N_+} |\nabla u(x,y)|^N\,dxdy = \int_{B^N_1} |\nabla v(\tilde{x},\tilde{y})|^N\,d\tilde{x}\,d\tilde{y}$$

where $(\tilde{x}, \tilde{y}) = \mathbf{B}(x, y)$. Since $|\tilde{x}|^2 + \tilde{y}^2 = \frac{|x|^2+(y-1)^2}{|x|^2+(y+1)^2}$ by (3.6), we have

$$\begin{aligned}
&\int_{B_1^N} \frac{|v(\tilde{x}, \tilde{y})|^N}{\{|\tilde{x}|^2 + |\tilde{y}|^2\}^{\frac{N}{2}} \left(\log \frac{1}{\sqrt{|\tilde{x}|^2+|\tilde{y}|^2}}\right)^N} \, d\tilde{x}\, d\tilde{y} \\
&= \int_{\mathbb{R}_+^N} \frac{|u(x, y)|^N}{|\mathbf{B}(x, y)|^{\frac{N}{2}} \left(\log \frac{1}{|\mathbf{B}(x,y)|}\right)^N} |\det \mathbf{B}'(x, y)| \, dx\, dy \\
&= \int_{\mathbb{R}_+^N} \frac{|u(x, y)|^N}{\{|x|^2 + (1-y)^2\}^{\frac{N}{2}} \left\{\frac{|x|^2+(1+y)^2}{4}\right\}^{\frac{N}{2}} \left(\log \sqrt{\frac{|x|^2+(1+y)^2}{|x|^2+(1-y)^2}}\right)^N} \, dxdy.
\end{aligned}$$

Thus, we obtain (1.3) by the critical Hardy inequality on the unit ball for v (Ref. [1, 21, 35]):

$$\left(\frac{N-1}{N}\right)^N \int_{B_1^N} \frac{|v(\tilde{x}, \tilde{y})|^N}{\{|\tilde{x}|^2 + |\tilde{y}|^2\}^{\frac{N}{2}} \left(\log \frac{1}{\sqrt{|\tilde{x}|^2+|\tilde{y}|^2}}\right)^N} \, d\tilde{x}\, d\tilde{y} \le \int_{B_1^N} |\nabla v(\tilde{x}, \tilde{y})|^N \, d\tilde{x}\, d\tilde{y}.$$

Optimality and the non-attainability of the constant $\left(\frac{N-1}{N}\right)^N$ in (1.3) follows from results for the critical Hardy inequality on the unit ball also. □

In the same way as above, we also obtain a Trudinger-Moser type inequality on the half-space from the result on balls

Theorem 4.1. *Let $N \ge 2$. Then*

$$\sup \left\{ \int_{\mathbb{R}_+^N} e^{\alpha |u(x,y)|^{\frac{N}{N-1}}} \frac{2^N \, dxdy}{\{|x|^2 + (y+1)^2\}^N} \,\middle|\, \|\nabla u\|_{L^N(\mathbb{R}_+^N)} \le 1,\ u \in \dot{W}_0^{1,N}(\mathbb{R}_+^N) \right\}$$

is finite if and only if $\alpha \le N\omega_{N-1}^{\frac{1}{N-1}}$. Moreover, the above maximization problem is attained for any $\alpha \le N\omega_{N-1}^{\frac{1}{N-1}}$.

Proof. Again, we use the transformation $v = (\mathbf{B}^{-1})^\#(u)$, $u \in \dot{W}_0^{1,N}(\mathbb{R}_+^N)$. Since $|\det \mathbf{B}'(x, y)| = \frac{2^N}{\{|x|^2+(y+1)^2\}^N}$, the theorem follows from the Trudinger-Moser inequality on B_1^N and its attainability: see [11]. □

Next, we show Theorem 1.3.

Before that, we claim the following Theorem:

Theorem 4.2. *Let $1 < p < N$. Then the Hardy type inequality*

$$\left(\frac{N-p}{p}\right)^p \int_{\mathbb{R}^N_+} \frac{W_p(x,y)^{\frac{p}{2}}}{(|x|^2+(1-y)^2)^{\frac{p}{2}}} |u(x,y)|^p \, dxdy \le \int_{\mathbb{R}^N_+} |\nabla u(x,y)|^p \, dxdy,$$

holds for any $u \in C^1_c(\mathbb{R}^N_+)$ of the form $u(x,y) = u\left(G_{\mathbb{R}^N_+, e_N}(x,y)\right)$, where ψ_p is given in §2 and for $(x,y) \in \mathbb{R}^N_+$,

$$W_p(x,y) = \frac{1 + \left(|x|^2+(1-y)^2\right)^{\frac{N-1}{p-1}} |\nabla \psi_p|^2 - 2\left(|x|^2+(1-y)^2\right)^{\frac{N-p}{2(p-1)}+1} \nabla \psi_p \cdot \begin{pmatrix} x \\ y-1 \end{pmatrix}}{\left[1 - \tilde{X}^{\frac{N-p}{2(p-1)}}\right]^2},$$

$$\tilde{X} = \psi_p(x,y)^{\frac{2(p-1)}{N-p}} \left\{|x|^2 + (1-y)^2\right\}.$$

The proof of Theorem 4.2 consists of the use of the harmonic transplantation between $\mathbb{R}^N_+$ and $\mathbb{R}^N$, the Hardy inequality (1.1), and Lemma 3.6. Since the proof of Theorem 4.2 is almost the same as that of Theorem 1.3 below, we omit the proof.

As we mentioned in §2, we do not know the explicit form of ψ_p when $2 \neq p < N$. Due to the lack of the explicit form of ψ_p, we cannot check that the inequality in Theorem 4.2 is improved, i.e., $W_p(x,y) \ge 1$. Therefore, we consider a modification of harmonic transplantation by using U_p in (1.5) instead of $G_{\mathbb{R}^N_+, e_N}$ in (3.11). This is the main idea we have invented in the proof of Theorem 1.3.

Consider the following modified transformation for radial functions $v = v(z) = v(t)$, $(t = |z| \in [0,1])$ on B^N_1, or $w = w(\tilde{z}) = w(r)$, $(r = |\tilde{z}| \in [0,+\infty))$ on $\mathbb{R}^N$:

$$\begin{cases} u(x,y) = v(t), \text{ where } U_p(x,y) = G_{B^N_1, O}(t), & (x,y) \in \mathbb{R}^N_+, \\ u(x,y) = w(r), \text{ where } U_p(x,y) = G_{\mathbb{R}^N, O}(r), & (x,y) \in \mathbb{R}^N_+. \end{cases} \tag{4.1}$$

We call the function u on $\mathbb{R}^N_+$ in (4.1) *the generalized harmonic transplantation* of v (or w).

We obtain the following Lemma instead of Lemma 3.6.

Lemma 4.3. *Let $1 < p < N, v, w$ be radial functions on $B^N_1, \mathbb{R}^N$, and u be given by (4.1). Then we have*

$$\int_{\mathbb{R}^N_+} |\nabla u|^p = \int_{B^N_1} |\nabla v|^p + \int_0^1 \omega_{N-1} t^{N-1} |v'(t)|^p F_p\left(G_{B^N_1, O}(t)\right) dt$$
$$= \int_{\mathbb{R}^N} |\nabla w|^p + \int_0^\infty \omega_{N-1} r^{N-1} |w'(r)|^p F_p\left(G_{\mathbb{R}^N, O}(r)\right) dr$$

where

$$F_p(s) = \int_{[U_p>s]} (-\Delta_p U_p)\, dxdy. \tag{4.2}$$

Especially, if $p \geq 2$ (resp. $p \leq 2$), then $F_p(s) \geq 0$ (resp. $F_p(s) \leq 0$) and $\|\nabla u\|_p \geq \|\nabla v\|_p$, $\|\nabla u\|_p \geq \|\nabla w\|_p$ (resp. $\|\nabla u\|_p \leq \|\nabla v\|_p$, $\|\nabla u\|_p \leq \|\nabla w\|_p$) holds.

Proof. We prove the first equality only, since the proof of the second equality is similar. Also we note that the proof below is an analogue to that of Lemma 3.6. Let h be defined by

$$h(x,y) = \left(G_{B_1^N,O}\right)^{-1}\left(U_p(x,y)\right) = \left[\frac{N-p}{p-1}\omega_{N-1}^{\frac{1}{p-1}} U_p(x,y) + 1\right]^{-\frac{p-1}{N-p}}, \quad (x,y) \in \mathbb{R}^N_+.$$

Thus $t = h(x,y)$ is equivalent to $U_p(x,y) = G_{B_1^N,O}(t)$ and $u(x,y) = v(h(x,y))$. In particular, $\nabla u(x,y) = v'(h(x,y))\nabla h(x,y)$. Note that since $U_p \geq 0$ in $\mathbb{R}^N_+$, we get that $0 < h(x,y) \leq 1$ for $(x,y) \in \overline{\mathbb{R}^N_+}$ and if $(x,y) \in h^{-1}(\{t\}) \cap \mathbb{R}^N_+$, Thus, the coarea formula gives that

$$\begin{aligned}
\int_{\mathbb{R}^N_+} |\nabla u|^p &= \int_{\mathbb{R}^N_+} |v'(h(x,y))|^p |\nabla h(x,y)|^{p-1} |\nabla h(x,y)|\, dxdy \\
&= \int_0^1 \left[\int_{h^{-1}(\{t\})\cap\mathbb{R}^N_+} |v'(h(x,y))|^p |\nabla h(x,y)|^{p-1}\, d\mathcal{H}^{N-1}(x,y)\right] dt.
\end{aligned}$$

Inserting $|\nabla h| = \omega_{N-1}^{\frac{1}{p-1}} h(y)^{\frac{N-1}{p-1}} |\nabla U_p(x,y)|$, we have

$$\int_{\mathbb{R}^N_+} |\nabla u|^p = \int_0^1 \omega_{N-1} t^{N-1} |v'(t)|^p \left[\int_{h^{-1}(\{t\})\cap\mathbb{R}^N_+} |\nabla U_p(x,y)|^{p-1}\, d\mathcal{H}^{N-1}(x,y)\right] dt. \tag{4.3}$$

Note that $h^{-1}(\{t\}) \cap \mathbb{R}^N_+$ is a level set of U_p. Applying $\phi_t = \min\{t, U_p\}$, $(t > 0)$ as a test function of (2.3) in Proposition 2.1, we have

$$\int_{[U_p<t]} |\nabla U_p|^p\, dxdy = t + \int_{[U_p<t]} (-\Delta_p U_p)\, U_p\, dxdy + \int_{[U_p>t]} (-\Delta_p U_p)\, t\, dxdy.$$

If we differentiate the above with respect to t, then we have

$$\begin{aligned}
&\int_{[U_p=t]} |\nabla U_p|^{p-1}\, d\mathcal{H}^{N-1}(x,y)\\
&= 1 + \int_{[U_p=t]} (-\Delta_p U_p)\, \frac{U_p}{|\nabla U_p|}\, d\mathcal{H}^{N-1}(x,y) - \int_{[U_p=t]} (-\Delta_p U_p)\, \frac{t}{|\nabla U_p|}\, d\mathcal{H}^{N-1}(x,y)\\
&\qquad + \int_{[U_p>t]} (-\Delta_p U_p)\, dxdy\\
&= 1 + \int_{[U_p>t]} (-\Delta_p U_p)\, dxdy
\end{aligned}$$

thanks to the coarea formula. Therefore, replacing t by $G_{B_1^N,O}(t)$ for any $t \in (0,1)$, we have

$$\int_{h^{-1}(\{t\})\cap\mathbb{R}^N_+} |\nabla U_p(x,y)|^{p-1}\, d\mathcal{H}^{N-1}(x,y) = 1 + \int_{\left[U_p>G_{B_1^N,O}(t)\right]} (-\Delta_p U_p)\, dxdy. \tag{4.4}$$

Inserting (4.4) into (4.3), we obtain

$$\begin{aligned}
\int_{\mathbb{R}^N_+} |\nabla u|^p &= \int_0^1 \omega_{N-1} t^{N-1} |v'(t)|^p \left(1 + \int_{[U_p>t]} (-\Delta_p U_p)\, dxdy\right) dt\\
&= \int_{B_1^N} |\nabla v|^p + \int_0^1 \omega_{N-1} t^{N-1} |v'(t)|^p F_p\left(\omega_{N-1}^{-\frac{1}{p-1}} \frac{p-1}{N-p}\left(t^{-\frac{N-p}{p-1}} - 1\right)\right) dt.
\end{aligned}$$

□

Lemma 4.4. *Let* $1 < p < N$, $0 \le s \le p$, *v and w be radial functions on* B_1^N *and* $\mathbb{R}^N$ *respectively, and let u be given by (4.1). Then we have*

$$\begin{aligned}
&\int_{\mathbb{R}^N_+} \frac{V_p(x,y)^{\frac{p}{2}}}{(|x|^2 + (1-y)^2)^{\frac{p}{2}}} |u(x,y)|^p\, dxdy\\
&= \int_{B_1^N} \frac{|v|^p}{|z|^p \left[1 - |z|^{\frac{N-p}{p-1}}\right]^p}\, dz + \omega_{N-1} \int_0^1 |v|^p t^{N-1-p} \left[1 - t^{\frac{N-p}{p-1}}\right]^{-p} F_p\left(G_{B_1^N,O}(t)\right) dr\\
&= \int_{\mathbb{R}^N} \frac{|w|^p}{|\tilde{z}|^p}\, d\tilde{z} + \omega_{N-1} \int_0^\infty |w|^p r^{N-1-p} F_p\left(G_{\mathbb{R}^N,O}(r)\right) dr.
\end{aligned}$$

where $F_p(s)$ is defined in (4.2)*, and for* $(x,y) \in \mathbb{R}^N_+$*, V_p and X are defined in* (1.8).

Proof. We prove the second equality for v only. The first equality is similar. For $(x,y)\in\mathbb{R}^N_+$, define $h(x,y)$ by the relation

$$U_p(x,y)=G_{\mathbb{R}^N,O}(h(x,y))=\frac{p-1}{N-p}\omega_{N-1}^{-\frac{1}{p-1}}h(x,y)^{-\frac{N-p}{p-1}},$$

that is,

$$h(x,y)=\left[\left(|x|^2+(1-y)^2\right)^{-\frac{N-p}{2(p-1)}}-\left(|x|^2+(1+y)^2\right)^{-\frac{N-p}{2(p-1)}}\right]^{\frac{p-1}{p-N}}.$$

As in (4.4), we can obtain

$$\int_{h^{-1}(\{t\})\cap\mathbb{R}^N_+}|\nabla U_p(x,y)|^{p-1}\,d\mathcal{H}^{N-1}(x,y)=1+\int_{\left[U_p>G_{\mathbb{R}^N,O}(t)\right]}(-\Delta_pU_p)\,dxdy. \tag{4.5}$$

Thus by the coarea formula and (4.5), we have

$$\int_{\mathbb{R}^N_+}|u|^p\frac{|\nabla h(x,y)|^p}{h(x,y)^p}\,dxdy=\int_0^\infty\int_{h^{-1}(\{r\})\cap\mathbb{R}^N_+}|w|^pr^{-p}|\nabla h(x,y)|^{p-1}\,d\mathcal{H}^{N-1}(x,y)\,dr \tag{4.6}$$

$$\begin{aligned}
&=\omega_{N-1}\int_0^\infty|w|^pr^{N-1-p}\int_{h^{-1}(\{r\})\cap\mathbb{R}^N_+}|\nabla U_p(x,y)|^{p-1}\,d\mathcal{H}^{N-1}(x,y)\,dr\\
&\overset{(4.5)}{=}\omega_{N-1}\int_0^\infty|w|^pr^{N-1-p}\,dr+\omega_{N-1}\int_0^\infty|w|^pr^{N-1-p}F_p\left(G_{\mathbb{R}^N,O}(r)\right)\,dr\\
&=\int_{\mathbb{R}^N}\frac{|w|^p}{|\tilde z|^p}\,d\tilde z+\omega_{N-1}\int_0^\infty|w|^pr^{N-1-p}F_p\left(G_{\mathbb{R}^N,O}(r)\right)\,dr,
\end{aligned}$$

where $|\tilde z|=r=h(x,y)$. On the other hand, we have

$$\nabla h(x,y)=h(x,y)^{\frac{N-1}{p-1}}\left[\left(|x|^2+(1-y)^2\right)^{\frac{p-N}{2(p-1)}-1}\begin{pmatrix}x\\y-1\end{pmatrix}-\left(|x|^2+(1+y)^2\right)^{\frac{p-N}{2(p-1)}-1}\begin{pmatrix}x\\y+1\end{pmatrix}\right]$$

which implies that

$$\frac{|\nabla h(x,y)|^p}{h(x,y)^p} = h(x,y)^{\frac{N-p}{p-1}p} \left| \left(|x|^2 + (1-y)^2\right)^{\frac{p-N}{2(p-1)}-1} \begin{pmatrix} x \\ y-1 \end{pmatrix} - \left(|x|^2 + (1+y)^2\right)^{\frac{p-N}{2(p-1)}-1} \begin{pmatrix} x \\ y+1 \end{pmatrix} \right|^p$$

$$= \frac{\left[\left(|x|^2 + (1-y)^2\right)^{\frac{1-N}{p-1}} + \left(|x|^2 + (1+y)^2\right)^{\frac{1-N}{p-1}} - 2\left(|x|^2 + (1-y)^2\right)^{\frac{2-p-N}{2(p-1)}} \left(|x|^2 + (1+y)^2\right)^{\frac{2-p-N}{2(p-1)}} (|x|^2 + y^2 - 1) \right]^{\frac{p}{2}}}{\left[(|x|^2 + (1-y)^2)^{-\frac{N-p}{2(p-1)}} - (|x|^2 + (1+y)^2)^{-\frac{N-p}{2(p-1)}} \right]^p}$$

$$= \frac{\left[1 + X^{\frac{N-1}{p-1}} - 2X^{\frac{N-p}{2(p-1)}} \left(|x|^2 + (1+y)^2\right)^{-1} (|x|^2 + y^2 - 1)\right]^{\frac{p}{2}}}{[|x|^2 + (1-y)^2]^{\frac{p}{2}} \left[1 - X^{\frac{N-p}{2(p-1)}}\right]^p}$$

$$= \frac{V_p(x,y)^{\frac{p}{2}}}{[|x|^2 + (1-y)^2]^{\frac{p}{2}}}.$$

Inserting this in the left hand-side of (4.6), we obtain the second equality of Lemma 4.4. □

Now, we prove Theorem 1.3. First, we claim the next lemma.

Lemma 4.5. *Let* $2 \le p \le N$ *and let* F_p *be defined in* (4.2). *Then*

$$\left(\frac{N-p}{p}\right)^p \int_0^\infty |w|^p r^{N-1-p} F_p\left(G_{\mathbb{R}^N,O}(r)\right) dr \le \int_0^\infty |w'|^p r^{N-1} F_p\left(G_{\mathbb{R}^N,O}(r)\right) dr \quad (4.7)$$

holds for any radial function $w = w(r) \in C_c^1(\mathbb{R}^N)$.

Proof. Since for $p \ge 2$,

$$F_p'(s) = -\int_{[U_p=s]} \frac{-\Delta_p U_p}{|\nabla U_p|} d\mathcal{H}^{N-1}(x,y) \le 0$$

by Proposition 2.2 and $G'_{\mathbb{R}^N,O}(r) \le 0$, we have

$$\int_0^\infty |w|^p r^{N-1-p} F_p\left(G_{\mathbb{R}^N,O}(r)\right) dr$$

$$= -\frac{p}{N-p} \int_0^\infty |w|^{p-2} w w' r^{N-p} F_p\left(G_{\mathbb{R}^N,O}(r)\right) dr - \frac{1}{N-p} \int_0^\infty |w|^p r^{N-p} F_p'\left(G_{\mathbb{R}^N,O}(r)\right) G'_{\mathbb{R}^N,O}(r)\, dr$$

$$\le -\frac{p}{N-p} \int_0^\infty |w|^{p-2} w w' r^{N-p} F_p\left(G_{\mathbb{R}^N,O}(r)\right) dr$$

$$\le \frac{p}{N-p} \left(\int_0^\infty |w'|^p r^{N-1} F_p\left(G_{\mathbb{R}^N,O}(r)\right) dr \right)^{\frac{1}{p}} \left(\int_0^\infty |w|^p r^{N-1-p} F_p\left(G_{\mathbb{R}^N,O}(r)\right) dr \right)^{\frac{p-1}{p}}.$$

Therefore, we obtain (4.7). □

Proof. (Proof of Theorem 1.3) From Lemma 4.4, (4.7), the classical Hardy inequality (1.1) on $\mathbb{R}^N$, and Lemma 4.3, we have

$$\begin{aligned}
&\left(\frac{N-p}{p}\right)^p \int_{\mathbb{R}^N_+} \frac{|u(x,y)|^p}{(|x|^2+(1-y)^2)^{\frac{p}{2}}} V_p(x,y)^{\frac{p}{2}}\, dxdy \\
&\overset{\text{Lemma 4.4}}{=} \left(\frac{N-p}{p}\right)^p \int_{\mathbb{R}^N} \frac{|w|^p}{|\tilde{z}|^p}\, d\tilde{z} + \left(\frac{N-p}{p}\right)^p \omega_{N-1} \int_0^\infty |w|^p r^{N-1-p} F_p\left(G_{\mathbb{R}^N,O}(r)\right) dr \\
&\overset{(4.7)}{\leq} \left(\frac{N-p}{p}\right)^p \int_{\mathbb{R}^N} \frac{|w|^p}{|\tilde{z}|^p}\, d\tilde{z} + \int_0^\infty |w'|^p r^{N-1} F_p\left(G_{\mathbb{R}^N,O}(r)\right) dr \\
&\overset{(1.1)}{<} \int_{\mathbb{R}^N} |\nabla w|^p\, d\tilde{z} + \int_0^\infty |w'|^p r^{N-1} F_p\left(G_{\mathbb{R}^N,O}(r)\right) dr \\
&\overset{\text{Lemma 4.3}}{=} \int_{\mathbb{R}^N_+} |\nabla u(x,y)|^p\, dxdy.
\end{aligned}$$

Thus the inequality (4.10)

$$\left(\frac{N-p}{p}\right)^p \int_{\mathbb{R}^N_+} \frac{|u(x,y)|^p}{(|x|^2+(1-y)^2)^{\frac{p}{2}}} V_p(x,y)^{\frac{p}{2}}\, dxdy < \int_{\mathbb{R}^N_+} |\nabla u(x,y)|^p\, dxdy.$$

is proven.

The remaining is to show the optimality of the constant $\left(\frac{N-p}{p}\right)^p$ in the above inequality. For large $M > 0$ and small $\varepsilon > 0$, consider the following test function:

$$u_{\varepsilon,M}(x,y) = U_p(x,y)^{\frac{p-1}{p} - \frac{p-1}{N-p}\varepsilon} \psi_M\left(U_p(x,y)\right)$$

where $U_p(x,y)$ is in (1.5). Put

$$C(N,p) = \frac{p-1}{N-p} \omega_{N-1}^{-\frac{1}{p-1}}$$

and define $\psi_M \in C^\infty(0,\infty)$, $0 \leq \psi_M \leq 1$, $\psi_M(s) = 1$ for $s \geq M$, $\psi_M(s) = 0$ for $s \leq \frac{M}{2}$. Let $\delta_1 = \delta_1(M)$, $\delta_2 = \delta_2(M) > 0$ satisfy

$$\delta_1^{-\frac{N-p}{p-1}} - (2-\delta_1)^{-\frac{N-p}{p-1}} = C(N,p)^{-1} M, \quad \delta_2 = \left(\frac{M}{C(N,p)}\right)^{\frac{p-1}{N-p}}.$$

Then we have $B_{\delta_1} = B_{\delta_1}(0,1) \subset [U_p \geq M] \subset B_{\delta_2}$. Then we have

$$
\begin{aligned}
&\int_{\mathbb{R}^N_+} \frac{|u_{\varepsilon,M}(x,y)|^p V_p(x,y)^{\frac{p}{2}}}{(|x|^2+(1-y)^2)^{\frac{p}{2}}}\,dxdy \geq \int_{[U_p\geq M]} \frac{|U_p|^{p-1-\frac{p-1}{N-p}p\varepsilon}}{(|x|^2+(1-y)^2)^{\frac{p}{2}}}\,dxdy \\
&\geq C(N,p)^{p-1-\frac{p-1}{N-p}p\varepsilon} \int_{B_{\delta_1}} \left(|x|^2+(1-y)^2\right)^{-\frac{N}{2}+\frac{p\varepsilon}{2}} \left[1-\left(\frac{|x|^2+(y-1)^2}{|x|^2+(y+1)^2}\right)^{\frac{N-p}{2(p-1)}}\right]^{p-1} dxdy \\
&\geq C(N,p)^{p-1-\frac{p-1}{N-p}p\varepsilon} \int_{B_{\delta_1}} \left(|x|^2+(1-y)^2\right)^{-\frac{N}{2}+\frac{p\varepsilon}{2}} \left[1-(p-1)\left(\frac{|x|^2+(y-1)^2}{|x|^2+(y+1)^2}\right)^{\frac{N-p}{2(p-1)}}\right] dxdy \\
&\geq C(N,p)^{p-1-\frac{p-1}{N-p}p\varepsilon}\omega_{N-1}\left[\int_0^{\delta_1} r^{-1+p\varepsilon}\,dr - \frac{p-1}{(2-\delta_1)^{\frac{N-p}{p-1}}}\int_0^{\delta_1} r^{-1+p\varepsilon+\frac{N-p}{p-1}}\,dr\right] \\
&= \frac{C(N,p)^{p-1}}{p}\omega_{N-1}\varepsilon^{-1} + o(\varepsilon^{-1}) \quad (\varepsilon \to 0). \qquad (4.8)
\end{aligned}
$$

Since

$$
\nabla u_{\varepsilon,M} = \left(\frac{p-1}{p} - \frac{p-1}{N-p}\varepsilon\right) U_p^{-\frac{1}{p}-\frac{p-1}{N-p}\varepsilon} (\nabla U_p)\, \psi_M + \psi_M^{'}\, U_p^{1-\frac{1}{p}-\frac{p-1}{N-p}\varepsilon} (\nabla U_p)
$$

and $(a+b)^p \leq a^p + pa^{p-1}b$ for $a, b \geq 0$, we have

$$
\begin{aligned}
&|\nabla u_{\varepsilon,M}|^p \\
&\leq \left(\frac{p-1}{p} - \frac{p-1}{N-p}\varepsilon\right)^p U_p^{-1-\frac{p-1}{N-p}p\varepsilon} |\nabla U_p|^p + p\left(\frac{p-1}{p}\right)^{p-1} \psi_M^{'}\, U_p^{-\frac{p-1}{N-p}\varepsilon-\frac{(p-1)^2}{N-p}\varepsilon} |\nabla U_p|^p.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
&\int_{\mathbb{R}^N_+} |\nabla u_{\varepsilon,M}(x,y)|^p \, dxdy \\
&\le \int_{[U_p \ge M]} \left(\frac{p-1}{p} - \frac{p-1}{N-p}\varepsilon \right)^p U_p^{-1-\frac{p-1}{N-p}p\varepsilon} |\nabla U_p|^p \, dxdy + \int_{[\frac{M}{2} \le U_p \le M]} |\nabla u_{\varepsilon,M}|^p \, dxdy \\
&\le \left(\frac{p-1}{p} \right)^p C(N,p)^{-1-\frac{p-1}{N-p}p\varepsilon} \omega_{N-1}^{-\frac{p}{p-1}} \int_{B_{\delta_2}} \left(|x|^2 + (1-y)^2 \right)^{-\frac{N}{2}+\frac{p\varepsilon}{2}} \times \\
&\quad \left[1 + \left\{ \frac{|x|^2 + (y-1)^2}{(2-\delta_2)^2} \right\}^{\frac{N-p}{p-1}+1} + 2 \left\{ \frac{|x|^2 + (y-1)^2}{(2-\delta_2)^2} \right\}^{\frac{N-p}{2(p-1)}} \frac{\left(|x|^2 + y^2 - 1 \right)_-}{(2-\delta_2)^2} \right] dxdy + o(\varepsilon^{-1}) \\
&\le \left(\frac{p-1}{p} \right)^p C(N,p)^{-1-\frac{p-1}{N-p}p\varepsilon} \omega_{N-1}^{-\frac{p}{p-1}+1} \int_0^{\delta_2} r^{-1+p\varepsilon} \, dr + o(\varepsilon^{-1}) \\
&= \left(\frac{p-1}{p} \right)^p \frac{C(N,p)^{-1}}{p} \omega_{N-1}^{-\frac{1}{p-1}} \varepsilon^{-1} + o(\varepsilon^{-1}) \quad (\varepsilon \to 0), \qquad (4.9)
\end{aligned}
$$

where $(f(x))_- := \max\{0, -f(x)\}$. From (4.8) and (4.9), we have

$$
\begin{aligned}
\frac{\int_{\mathbb{R}^N_+} |\nabla u_{\varepsilon,M}(x,y)|^p \, dxdy}{\int_{\mathbb{R}^N_+} \frac{|u_{\varepsilon,M}(x,y)|^p V_p(x,y)^{\frac{p}{2}}}{\left(|x|^2+(1-y)^2\right)^{\frac{p}{2}}} \, dxdy} &\le \frac{\left(\frac{p-1}{p}\right)^p \frac{C(N,p)^{-1}}{p} \omega_{N-1}^{-\frac{1}{p-1}} \varepsilon^{-1} + o(\varepsilon^{-1})}{\frac{C(N,p)^{p-1}}{p} \omega_{N-1} \varepsilon^{-1} + o(\varepsilon^{-1})} \\
&= \left(\frac{p-1}{p\,C(N,p)} \right)^p \omega_{N-1}^{-\frac{p}{p-1}} + o(1) \\
&= \left(\frac{N-p}{p} \right)^p + o(1) \quad (\varepsilon \to 0).
\end{aligned}
$$

Therefore, the constant $\left(\frac{N-p}{p}\right)^p$ in the inequality (4.10) is optimal. □

As we mention in Remark 1.4, the improved inequality (4.10) is valid for functions without any symmetry by using Proposition 2.1 and a result in [14].

Theorem 4.6. *Let $2 \le p < N$. Then the inequality*

$$
\left(\frac{N-p}{p} \right)^p \int_{\mathbb{R}^N_+} \frac{V_p(x,y)^{\frac{p}{2}}}{\left(|x|^2 + (1-y)^2\right)^{\frac{p}{2}}} |u(x,y)|^p \, dxdy \le \int_{\mathbb{R}^N_+} |\nabla u(x,y)|^p \, dxdy \qquad (4.10)
$$

holds for any $u \in \dot{W}^{1,p}_0(\mathbb{R}^N_+)$, where V_p and X are defined in (1.8). *Furthermore, $(\frac{N-p}{p})^p$ is the best constant and is not attained.*

Proof. U_p is a nonnegative function and Proposition 2.1 implies that $-\Delta_p U_p \geq 0$ in weak sense for $p \in [2, N)$. Substituting U_p for ρ in [14]:Theorem 2.1, we have the inequality (4.10) for any functions $\dot{W}_0^{1,p}(\mathbb{R}_+^N)$, since

$$\begin{aligned}\left(\frac{p-1}{p}\right)^p \frac{|\nabla \rho|^p}{\rho^p} &= \left(\frac{p-1}{p}\right)^p \frac{|\nabla U_p|^p}{U_p^p} \\ &= \left(\frac{p-1}{p}\right)^p \left(\frac{N-p}{p-1}\right)^p \frac{|\nabla h(x,y)|^p}{h(x,y)^p} \\ &= \left(\frac{N-p}{p}\right)^p \frac{V_p(x,y)^{\frac{p}{2}}}{\left[|x|^2 + (1-y)^2\right]^{\frac{p}{2}}},\end{aligned}$$

where $h(x, y)$ is given by the proof of Lemma 4.4. The optimality of the constant $(\frac{N-p}{p})^p$ in the inequality (4.10) follows from Theorem 1.3 and the non-attainability follows from [14]:Theorem 4.1. □

In the last of this section, we give an improved Hardy-Sobolev inequality on the half-space for $p = 2$. The proof is simpler than the that of Theorem 1.3. We omit it here.

Theorem 4.7. *(Improved Hardy-Sobolev inequality for $p = 2$) Let $p = 2 < N$, $0 \leq s < 2$, and $2^*(s) = \frac{2(N-s)}{N-2}$. Then the inequality*

$$S_{N,2,s}\left(\int_{\mathbb{R}_+^N} \frac{|u(x,y)|^{2^*(s)}}{(|x|^2 + (1-y)^2)^{\frac{s}{2}}} \frac{V_2(x,y)}{\left[1 - X^{\frac{N-2}{2}}\right]^{\frac{2-s}{N-2}}}\, dxdy\right)^{\frac{2}{2^*(s)}} \leq \int_{\mathbb{R}_+^N} |\nabla u(x,y)|^2\, dxdy, \tag{4.11}$$

holds for any $u \in \dot{W}_0^{1,2}(\mathbb{R}_+^N)$ of the form $u(x, y) = \tilde{u}(G_{\mathbb{R}_+^N, e_N}(x, y))$ for some function $\tilde{u}$ on $[0, +\infty)$, where $V_2(x, y)$ and X is given by (1.8) in Theorem 1.3, and $S_{N,2,s}$ is the Hardy-Sobolev best constant, i.e.,

$$S_{N,2,s} = \inf_{u \in C_c^\infty(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2\, dx}{\left(\int_{\mathbb{R}^N} \frac{|u|^{2^*(s)}}{|x|^s}\, dx\right)^{\frac{2}{2^*(s)}}}.$$

Remark 4.8. *The inequality (4.11) does not hold for functions without the symmetry (1.6), see Proposition 5.2.*

5. Appendix

First, we show that the radial critical Sobolev space $\dot{W}^{1,N}_{0,\mathrm{rad}}(\mathbb{R}^N)$ cannot be embedded to any weighted Lebesgue space $L^q(\mathbb{R}^N; g(x)\,dx)$ for $q \in [1, \infty)$ and for $g > 0$.

Proposition 5.1. *There is no weight function $g > 0$ such that the inequality*

$$C\left(\int_{\mathbb{R}^N} |u|^q g(x)\,dx\right)^{\frac{N}{q}} \leq \int_{\mathbb{R}^N} |\nabla u|^N\,dx$$

holds for any $u \in C^1_{c,\mathrm{rad}}(\mathbb{R}^N)$ for some $C > 0$.

Proof. Consider the radial test function

$$\phi_R(|x|) = \begin{cases} 1 & \text{if } |x| \leq 1, \\ \frac{\log \frac{R}{|x|}}{\log R} & \text{if } 1 < |x| < R, \\ 0 & \text{if } |x| \geq R. \end{cases}$$

Direct calculation shows that

$$\int_{\mathbb{R}^N} |\phi_R|^q g(x)\,dx \geq \int_{B_1} g(x)\,dx > 0,$$
$$\int_{\mathbb{R}^N} |\nabla \phi_R|^N\,dx = \omega_{N-1} (\log R)^{1-N} \to 0 \quad (R \to \infty).$$

Though ϕ_R is not C^1, we can mollify it as in [19]:Lemma 8.1, to obtain a $C^1_{rad}(\mathbb{R}^N)$ function with the same property. Therefore, we obtain Proposition 5.1. □

Next, we show that the improved inequalities (4.10), (4.11) in Theorem 1.3 and Theorem 4.7 do not hold without the symmetry (1.6).

Proposition 5.2. *Let $1 < p < N$, $0 \leq s < p$, $p^*(s) = \frac{p(N-s)}{N-p}$, and V_p, X be given in* (1.8). *Then*

$$S := \inf_{u \in C^1_c(\mathbb{R}^N_+)\setminus\{0\}} \frac{\int_{\mathbb{R}^N_+} |\nabla u(x,y)|^p\,dxdy}{\left(\int_{\mathbb{R}^N_+} \frac{|u(x,y)|^{p^*(s)}}{\left(|x|^2+(1-y)^2\right)^{\frac{s}{2}}} \frac{V_p(x,y)^{\frac{p}{2}}}{\left[1-X^{\frac{N-p}{2(p-1)}}\right]^{\frac{(p-1)(p-s)}{N-p}}}\,dxdy\right)^{\frac{p}{p^*(s)}}} = 0.$$

Proof. We use the same test function as it in [31]:Propositon 2. Let $z = (x, y) \in \mathbb{R}^N_+$ and $z_\varepsilon = (0, \varepsilon)$. Note that $X = \frac{|x|^2+(1-y)^2}{|x|^2+(1+y)^2} \to 1$ and

$$V_p(x, y) = \left(\frac{4(p-1)}{N-p}\right)^2 (1-X)^{-2} + o\left((1-X)^{-2}\right)$$
$$= \left(\frac{p-1}{N-p}\right)^2 y^{-2} + o\left(y^{-2}\right)$$

as $|z| = \sqrt{|x|^2 + y^2} \to 0$. For small $\varepsilon > 0$, we define u_ε as follows:

$$u_\varepsilon(z) = \begin{cases} v\left(\frac{|z-z_\varepsilon|}{\varepsilon}\right) & \text{if } z \in B_\varepsilon(z_\varepsilon), \\ 0 & \text{if } z \in \mathbb{R}^N_+ \setminus B_\varepsilon(z_\varepsilon), \end{cases} \quad \text{where } v(t) = \begin{cases} 1 & \text{if } 0 \le t \le \frac{1}{2}, \\ 2(1-t) & \text{if } \frac{1}{2} < t \le 1. \end{cases}$$

Then we have

$$\int_{\mathbb{R}^N_+} |\nabla u_\varepsilon(z)|^p \, dz = \varepsilon^{N-p} \int_{B_1} |\nabla v(|z|)|^p \, dz = C\varepsilon^{N-p},$$
$$\int_{\mathbb{R}^N_+} \frac{|u_\varepsilon(x,y)|^{p^*(s)}}{(|x|^2 + (1-y)^2)^{\frac{s}{2}}} \frac{V_p(x,y)^{\frac{p}{2}}}{\left[1 - X^{\frac{N-p}{2(p-1)}}\right]^{\frac{(p-1)(p-s)}{N-p}}} \, dxdy$$
$$\ge C\,\varepsilon^{-p} \int_{B_{\varepsilon/2}(z_\varepsilon)} \varepsilon^{-\frac{(p-1)(p-s)}{N-p}} dz = O\left(\varepsilon^{N-p-\frac{(p-1)(p-s)}{N-p}}\right) \quad (\varepsilon \to 0).$$

Applying u_ε as a test function for S, we see

$$S \le C\,\varepsilon^{(N-p)\left(1-\frac{p}{p^*(s)}\right)+\frac{(p-1)(p-s)p}{(N-p)p^*(s)}} = O\left(\varepsilon^{\frac{N-1}{N-s}(p-s)}\right) \to 0 \text{ as } \varepsilon \to 0.$$

□

Next proposition is a fact from linear algebra.

Proposition 5.3. *Let $v \in \mathbb{R}^N$, $|v| = 1$, $t \in \mathbb{R}$, $A = I + t\, v \otimes v$, where I is the identity matrix on $\mathbb{R}^N$. Then A has two eigenvalues* 1 *and* $1 + t$*. The multiplicity of* 1 *is* $N - 1$*, and the multiplicity of* $1 + t$ *is* 1*. Especially,*

$$\det A = 1 + t.$$

If $t \neq -1$, then there exists the inverse matrix

$$A^{-1} = I - \frac{t}{t+1}\, v \otimes v. \tag{5.1}$$

Proof. Let $u = (u_1, \dots, u_N)^T$ satisfy $u \cdot v = 0$. Then we have

$$((v \otimes v)u)_i = \sum_{j=1}^{N} (v \otimes v)_{i,j} u_j = \sum_{j=1}^{N} v_i v_j u_j = \left(\sum_{j=1}^{N} v_j u_j \right) v_i = 0.$$

Therefore, $Au = (I + tv \otimes v)u = u$ which means that u is the eigenvector of the eigenvalue 1 of A. Note that there are $N - 1$ such linearly independent u, thus the multiplicity of the eigenvalue 1 is $N - 1$. Also since $|v| = 1$, for any $i = 1, \dots, N$, we have

$$((v \otimes v)v)_i = \sum_{j=1}^{N} v_i v_j v_j = \left(\sum_{j=1}^{N} v_j v_j \right) v_i = v_i$$

which implies that $(v \otimes v)v = v$. Therefore, $Av = (I + tv \otimes v)v = (1 + t)v$ which means that v is the eigenvector of the eigenvalue $1 + t$ of A. Hence, the multiplicity of $1 + t$ is 1 and $\det(I + tv \otimes v) = 1 + t$.

Next, we show (5.1). Since

$$\begin{aligned} \left((v \otimes v)^2\right)_{ik} &= \sum_{j=1}^{N} (v \otimes v)_{i,j} (v \otimes v)_{jk} = \sum_{j=1}^{N} v_i v_j v_j v_k \\ &= \left(\sum_{j=1}^{N} v_j^2 \right) v_i v_k = (v \otimes v)_{ik}, \end{aligned}$$

we have $(v \otimes v)^2 = (v \otimes v)$. Therefore, we have

$$(I + tv \otimes v)(I - sv \otimes v) = I + (t - s - st)v \otimes v.$$

If $t - s - st = 0$, then the right-hand side is I. Therefore, $A^{-1} = I - \frac{t}{t+1} v \otimes v$. □

Proposition 5.4. *Let J, T_b, S_λ be given by Definition 3.1 and $\mathbf{B}$ be given by (3.5). Then*

$$\mathbf{B}(z) = R \circ J \circ T_{e_N} \circ S_2 \circ J \circ T_{-e_N}(z), \text{ where } R = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & -1 \end{pmatrix}, \; e_N = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

for any $z \in \mathbb{R}^N$.

Proof. Direct calculation shows that

$$\begin{aligned}
J \circ T_{e_N} \circ S_2 \circ J \circ T_{-e_N}(z) &= [e_N + 2(z-e_N)^*]^* = \frac{e_N + 2(z-e_N)^*}{|e_N + 2(z-e_N)^*|^2} \\
&= \frac{e_N + \frac{2(z-e_N)}{|z-e_N|^2}}{\left|e_N + \frac{2(z-e_N)}{|z-e_N|^2}\right|^2} \\
&= \frac{e_N + \frac{2(z-e_N)}{|z-e_N|^2}}{1 + \frac{4}{|z-e_N|^2} + \frac{4e_N\cdot(z-e_N)}{|z-e_N|^2}} \\
&= \frac{e_N|z-e_N|^2 + 2(z-e_N)}{|z-e_N|^2 + 4z_N} \\
&= \frac{e_N(|z|^2 - 2z_N + 1) + 2(z-e_N)}{|z|^2 + 2z_N + 1} \\
&= \frac{(2x, |z|^2 - 1)}{|x|^2 + (1+y)^2} \\
&= \frac{(2x, |x|^2 + y^2 - 1)}{|x|^2 + (1+y)^2}.
\end{aligned}$$

Therefore, we have

$$R \circ J \circ T_{e_N} \circ S_2 \circ J \circ T_{-e_N}(z) = \frac{(2x, 1 - |x|^2 - y^2)}{|x|^2 + (1+y)^2} = \mathbf{B}(z).$$

□

Finally, we describe an application of the transformations in III.-(i) in §3.3. For more general case, see [19]. It is well-known that the Sobolev inequality

$$S_{N,p}\left(\int_{\mathbb{R}^N} |u|^{p^*}\,dx\right)^{\frac{p}{p^*}} \leq \int_{\mathbb{R}^N} |\nabla u|^p\,dx \tag{5.2}$$

for any $u \in \dot{W}_0^{1,p}(\mathbb{R}^N)$, $1 < p < N$, with the best constant

$$S_{N,p} = \pi^{\frac{p}{2}} N \left(\frac{N-p}{p-1}\right)^{p-1} \left(\frac{\Gamma(\frac{N}{p})\Gamma(1+N-\frac{N}{p})}{\Gamma(1+\frac{N}{2})\Gamma(N)}\right)^{\frac{p}{N}},$$

follows from a one-dimensional inequality obtained by Bliss [8]: Let $v : [0,+\infty) \to \mathbb{R}$ be an absolutely continuous function on $(0,+\infty)$ such that $v' \in L^p(0,+\infty)$,

$v(0) = 0$. Put $q > p > 1$. Then the inequality

$$C(p,q)\left(\int_0^\infty \frac{|v(t)|^q}{t^{1+q(\frac{p-1}{p})}}dt\right)^{1/q} \le \left(\int_0^\infty |v'(t)|^p dt\right)^{1/p} \tag{5.3}$$

holds where

$$C(p,q) = \left(\frac{\Gamma\left(\frac{q}{q-p}\right)\Gamma\left(\frac{p(q-1)}{q-p}\right)}{\Gamma\left(\frac{pq}{q-p}\right)}\right)^{1/p-1/q} \left(\frac{q(p-1)}{p}\right)^{1/q}.$$

See Maz'ya [25], pp. 274, the equation (4.6.4). Also see [27] for a new proof of this classical inequality. In the following, we consider radial functions only. To obtain the best Sobolev inequality (5.2) for radial functions $u(r)$, $r = |x|$, we change the variables

$$v(t) = u(r), \text{ where } t = \frac{p-1}{N-p}\omega_{N-1}^{-\frac{1}{p-1}} r^{-\frac{N-p}{p-1}}$$

and apply the Bliss inequality (5.3) for $q = p^* > p > 1$. Note that the condition $v(0) = 0$ is satisfied if $u(r) \to 0$ as $r \to \infty$.

Instead, let us change the variables

$$v(t) = u(r), \text{ where } t = \omega_{N-1}^{-\frac{1}{N-1}} \log\frac{1}{r}$$

and put $q > p = N$. Then the usual computation shows that

$$\omega_{N-1}^{1-\frac{q}{N}} \int_0^\infty \frac{|v(t)|^q}{t^{1+q(\frac{N-1}{N})}}dt = \omega_{N-1}\int_0^1 \frac{|u(r)|^q}{r\left(\log\frac{1}{r}\right)^{1+q(\frac{N-1}{N})}}dr = \int_{B_1^N} \frac{|u(x)|^q}{|x|^N\left(\log\frac{1}{|x|}\right)^{1+q(\frac{N-1}{N})}}dx,$$

$$\int_0^\infty |v'(t)|^N dt = \int_{B_1^N} |\nabla u|^N dx.$$

Also in this case, the condition $v(0) = 0$ is equivalent to $u(1) = 0$. Inserting these identities into (5.3), we obtain

$$C(q)\left(\int_{B_1^N} \frac{|u(x)|^q}{|x|^N\left(\log\frac{1}{|x|}\right)^{1+q(\frac{N-1}{N})}}dx\right)^{N/q} \le \int_{B_1^N} |\nabla u|^N dx \tag{5.4}$$

for any $u \in \dot{W}^{1,N}_{0,rad}(B^N_1)$ where

$$C(q) = \omega_{N-1}^{1-N/q} C(N,q)^{-N} = \omega_{N-1}^{1-N/q} \left(\frac{\Gamma\left(\frac{q}{q-N}\right)\Gamma\left(\frac{N(q-1)}{q-N}\right)}{\Gamma\left(\frac{Nq}{q-N}\right)} \right)^{1-N/q} \left(\frac{q(N-1)}{N} \right)^{N/q}. \quad (5.5)$$

For the inequality (5.4), see e.g. [29]. Now, we check that $\lim_{q \to N+0} C(q) = \left(\frac{N-1}{N}\right)^N$. Recall the Stirling formula

$$\Gamma(s) = \sqrt{2\pi} s^{s-1/2} e^{-s} + o(1), \quad \text{as} \quad s \to +\infty$$

and put $s = \frac{q}{q-N}$. Then, we see

$$\begin{aligned} \frac{\Gamma\left(\frac{q}{q-N}\right)\Gamma\left(\frac{N(q-1)}{q-N}\right)}{\Gamma\left(\frac{Nq}{q-N}\right)} &= \frac{\Gamma(s)\Gamma(\frac{(q-1)N}{q}s)}{\Gamma(Ns)} \sim \frac{\Gamma(s)\Gamma((N-1)s)}{\Gamma(Ns)} \\ &\sim \sqrt{2\pi} \frac{s^{s-1/2}e^{-s}((N-1)s))^{(N-1)s-1/2}e^{-(N-1)s}}{(Ns)^{Ns-1/2}e^{-Ns}} = \sqrt{2\pi} \frac{(N-1)^{(N-1)s-1/2}}{N^{Ns-1/2}} s^{-1/2} \end{aligned}$$

as $q \to N+0$ (which is equivalent to $s = \frac{q}{q-N} \to \infty$), and for $C(q)$ in (5.5), we have

$$\begin{aligned} C(q) &\sim \left(\frac{\Gamma(s)\Gamma(\frac{N(q-1)}{q}s)}{\Gamma(Ns)} \right)^{1/s} \left(\frac{q(N-1)}{N} \right)^{N/q} \\ &\sim \left(\sqrt{2\pi} \frac{(N-1)^{(N-1)s-1/2}}{N^{Ns-1/2}} s^{-1/2} \right)^{1/s} (N-1) \\ &\sim \frac{(N-1)^{N-1}}{N^N} (N-1) s^{-1/2s} \to \left(\frac{N-1}{N} \right)^N \quad \text{as} \quad s \to +\infty. \end{aligned}$$

Thus if we take a limit $q \to N+0$ in the inequality (5.4), we have the critical Hardy inequality

$$\left(\frac{N-1}{N} \right)^N \int_{B^N_1} \frac{|u(x)|^N}{|x|^N \left(\log \frac{1}{|x|}\right)^N} dx \leq \int_{B^N_1} |\nabla u|^N dx$$

on a unit ball for any $u \in \dot{W}^{1,N}_{0,\mathrm{rad}}(B^N_1)$.

In conclusion, we obtain the following.

Proposition 5.5. *The Bliss inequality (5.3) yields both the best Sobolev inequality (5.2) and the generalized critical Hardy inequality (5.4) for radially symmetric functions.*

Acknowledgment

The first author (M.S.) was supported by JSPS KAKENHI Early-Career Scientists, No. JP19K14568. The second author (F.T.) was supported by JSPS Grant-in-Aid for Scientific Research (B), No. JP19136384. This work was partly supported by Osaka City University Advanced Mathematical Institute (MEXT Joint Usage/Research Center on Mathematics and Theoretical Physics) JPMXP0619217849.

References

[1] Adimurthi, Sandeep, K., *Existence and non-existence of the first eigenvalue of the perturbed Hardy-Sobolev operator*, Proc. Roy. Soc. Edinburgh Sect. A **132** (2002), No.5, 1021-1043.

[2] Ahlfors, L. V., *Möbius transformations in several dimensions*, Ordway Professorship Lectures in Mathematics. University of Minnesota, School of Mathematics, Minneapolis, Minn., (1981).

[3] Bandle, C., Brillard, A., Flucher, M., *Green's function, harmonic transplantation, and best Sobolev constant in spaces of constant curvature*, Trans. Amer. Math. Soc. 350 (1998), no. 3, 1103-1128.

[4] Baras, P., Goldstein, J. A., *The heat equation with a singular potential*, Trans. Amer. Math. Soc., 284 (1984), 121-139.

[5] Beardon, A. F., *The geometry of discrete groups*, Corrected reprint of the 1983 original. Graduate Texts in Mathematics, 91. Springer-Verlag, New York, 1995.

[6] Beckner, W., Pearson, M., *On sharp Sobolev embedding and the logarithmic Sobolev inequalitiy*, Bull. London Math. Soc., 30 (1998), 80-84.

[7] Benguria, R. D., Frank, R. L., Loss, M., *The sharp constant in the Hardy-Sobolev-Maz'ya inequality in the three dimensional upper half-space*, Math. Res. Lett. 15 (2008), no. 4, 613-622.

[8] Bliss, G. A., *An Integral Inequality*, J. London Math. Soc. 5 (1930), no. 1, 40-46.

[9] Byeon, J., Takahashi, F., *Hardy's inequality in a limiting case on general bounded domains*, Commun. Contemp. Math. 21 (2019), no. 8, 1850070, 24 pp.

[10] Brezis, H., Vázquez, J. L., *Blow-up solutions of some nonlinear elliptic problems*, Rev. Mat. Univ. Complut. Madrid 10 (1997), No. 2, 443-469.

[11] Carleson, L., Chang, S.-Y. A., *On the existence of an extremal function for an inequality of J. Moser*, Bull. Sci. Math. (2) 110 (1986), no. 2, 113-127.

[12] Csató, G., Roy, P., *Extremal functions for the singular Moser-Trudinger inequality in* 2 *dimensions*, Calc. Var. Partial Differential Equations 54 (2015), no. 2, 2341-2366.

[13] Csató, G., Nguyen, Van Hoang, Roy, P., *Extremals for the singular Moser-Trudinger inequality via $n-$harmonic transplantation*, J. Differential Equations 270 (2021), 843-882.

[14] D'Ambrosio, L., Dipierro, S., *Hardy inequalities on Riemannian manifolds and applications*, Ann. Inst. H. Poincaré Anal. Non Linéaire 31 (2014), no. 3, 449-475.

[15] A. Fabricant, N. Kutev, and T. Rangelov, *Hardy-type inequality with double singular kernels*, Centr. Eur. J. Math., 11(9):1689-1697, 2013.

[16] Flucher, M., *Extremal functions for the Trudinger-Moser inequality in 2 dimensions*, Comment. Math. Helv. 67 (1992), no. 3, 471-497.

[17] Flucher, M., *Variational problems with concentration*, Progress in Nonlinear Differential Equations and their Applications, 36. Birkhäuser Verlag, Basel, 1999. viii+163 pp.

[18] Hersch, J., *Transplantation harmonique, transplantation par modules, et théorèmes isopérimétriques. (French. English summary)*, Comment. Math. Helv. 44 (1969), 354-366.

[19] Horiuchi, T., Kumlin, P., *On the Caffarelli-Kohn-Nirenberg-type inequalities involving critical and supercritical weights*, Kyoto J. Math. 52 (2012), no. 4, 661-742.

[20] Ioku, N., *Attainability of the best Sobolev constant in a ball*, Math. Ann. 375 (2019), no. 1-2, 1-16.

[21] Ioku, N., Ishiwata, M., *A Scale Invariant Form of a Critical Hardy Inequality*, Int. Math. Res. Not. IMRN (2015), no. 18, 8830-8846.

[22] Kichenassamy, S., Véron, L., *Singular solutions of the $p-$Laplace equation*, Math. Ann. 275 (1986), no. 4, 599-615.

[23] Lin, K. C., *Extremal functions for Moser's inequality*, Trans. Am. Math. Soc., 348 (1996), pp. 2663-2671.

[24] Lindqvist, P., *A Remark on the Kelvin Transform for a Quasilinear Equation*, arXiv: 1606.02563v1, (2016).

[25] V. Maz'ya: *Sobolev spaces with applications to elliptic partial differential equations. Second, revised and augmented edition,* Grundlehren der Mathematischen Wissenschaften, **342**. Springer, Heidelberg, 2011. xxviii+866 pp.

[26] Moser, J., *A sharp form of an inequality by N. Trudinger*, Indiana Univ. Math. J. 20 (1970/71), 1077-1092.

[27] Osękowski, A., *A new approach to Hardy-type inequalities*, Arch. Math (Basel). **104** (2015), 165-176.

[28] Sano, M., *Explicit optimal constants of two critical Rellich inequalities for radially symmetric functions*, arXiv: 2002.04768v2.

[29] Sano, M., *Extremal functions of generalized critical Hardy inequalities*, J. Differential Equations 267 (2019), no. 4, 2594-2615.

[30] Sano, M., *Improvements and generalizations of two Hardy type inequalities and their applications to the Rellich type inequalities*, arXiv: 2104.01737.

[31] Sano, M., *Minimization problem associated with an improved Hardy-Sobolev type inequality*, Nonlinear Anal. 200 (2020), 111965, 16 pp.

[32] Sano, M., *Two limits on Hardy and Sobolev inequalities*, RIMS Kôkyûroku 2172 , (2020), 105-119.

[33] Sano, M., Sobukawa, T., *Remarks on a limiting case of Hardy type inequalities*, Math. Inequal. Appl. 23 (2020), no. 4, 1425-1440.

[34] Sano, M., Takahashi, F., *Scale invariance structures of the critical and the subcritical Hardy inequalities and their improvements*, Calc. Var. Partial Differential Equations 56 (2017), no. 3, Art. 69, 14 pp.

[35] Takahashi, F., *A simple proof of Hardy's inequality in a limiting case*, Arch. der Math. 104 (2015), no. 1, 77-82.

[36] Trudinger, N. S., *On imbeddings into Orlicz spaces and some applications*, J. Math. Mech. 17 (1967), 473-483.

[37] Zographopoulos, N. B. *Existence of extremal functions for a Hardy-Sobolev inequality*, J. Funct. Anal. 259 (2010), no. 1, 308-314.