Notes on the Lie algebra associated to a finite group of Lie type

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Abstract

This is a detailed exposition of our old paper [KS], in which we introduced a certain finite Lie algebra associated to a finite group of Lie type and exhibited a Lie correspondence between them. The finite Lie algebra is obtained from the Lie algebra of the ambient simple algebraic group of the finite group as the set of fixed points under a linearization of the endomorphism of the algebraic group defining the finite group. We show along the way a bijection between the set of subalgebras of the finite Lie algebra containing a maximal toral subalgebra and the set of subalgebras of the Lie algebra of the ambient algebraic group, stabilized by the linearization of the endomorphism of the algebraic group defining the finite group and contain a maximal toral subalgebra stabilized also by the endomorphism, which is a linearization of the correspondence established earlier for the finite group and the ambient algebraic group by Seitz [S].

1° Introduction

Let $\mathfrak{G}$ be a simple algebraic group scheme over an algebraically closed field $\mathbb{K}$ of positive characteristic p. Let $\bar{G} = \mathfrak{G}(\mathbb{K})$ and σ an endomorphism of $\mathfrak{G}$ such that $\sigma(\mathbb{K})$ is surjective with $G = \bar{G}^{\sigma(\mathbb{K})} = \{g \in \bar{G} | \sigma(\mathbb{K})(g) = g\}$ finite. Thus, G is a finite group of Lie type. We will discuss $\bar{G}$ in the classical algebraic geometry with reduced objects, and abbreviate $\sigma(\mathbb{K})$ simply as σ.

Throughout the paper we assume that $p \geq 5$. Under the assumption $\bar{G}$ admits an $\mathbb{F}_q$-form $\bar{G}_{\mathbb{F}_q}$, q a power of p, and σ decompose into a commuting product of the geometric Frobenius endomorphism F of $\bar{G}$ defined by $\bar{G}_{\mathbb{F}_p}$ and a graph automorphism $\tilde{\tau}$ of $\bar{G}$ stabilizing $\bar{G}_{\mathbb{F}_p}$. Let $\Bbbk = \mathbb{F}_q$, and let $\mathbb{K}[\bar{G}]$ (resp. $\Bbbk[\bar{G}] = \Bbbk[\bar{G}_\Bbbk]$) denote the associated Hopf algebra of $\bar{G}$ (resp. $\bar{G}_\Bbbk$) over $\mathbb{K}$ (resp. $\Bbbk$). Thus,

$$\begin{array}{ccccc}
\mathbb{K} \otimes_\Bbbk \Bbbk[\bar{G}_\Bbbk] & \xrightarrow{\sim} & \mathbb{K}[\bar{G}] & \xleftarrow{\sim} & \mathbb{K} \otimes_\Bbbk \Bbbk[\bar{G}_\Bbbk] \\
\downarrow {\scriptstyle \mathbb{K}\otimes_\Bbbk ?^q} & & {\scriptstyle F^\sharp} \vdots \ \downarrow {\scriptstyle \tilde{\tau}^\sharp} & & \vdots\, {\scriptstyle \mathbb{K}\otimes_\Bbbk \tilde{\tau}^\sharp_\Bbbk} \\
\mathbb{K} \otimes_\Bbbk \Bbbk[\bar{G}_\Bbbk] & \xrightarrow{\sim} & \mathbb{K}[\bar{G}] & \xleftarrow{\sim} & \mathbb{K} \otimes_\Bbbk \Bbbk[\bar{G}_\Bbbk].
\end{array}$$

Let now $\bar{\mathfrak{g}} = \mathrm{Lie}(\bar{G})$ (resp. $\bar{\mathfrak{g}}_\Bbbk = \mathrm{Lie}(\bar{G}_\Bbbk)$) be the Lie algebra of $\bar{G}$ (resp. $\bar{G}_\Bbbk$), and $F' = ?^q \otimes_\Bbbk \bar{\mathfrak{g}}_\Bbbk$ the arithmetic Frobenius endomorphism of $\bar{\mathfrak{g}}$ with respect to $\bar{\mathfrak{g}}_\Bbbk$. We let $\sigma_* = F' \circ \mathrm{d}\tilde{\tau} = \mathrm{d}\tilde{\tau} \circ F'$, and set $\mathfrak{g} = \bar{\mathfrak{g}}^{\sigma_*} = \{x \in \bar{\mathfrak{g}} | \sigma_* x = x\}$ which is equipped with a structure of p-Lie algebra. We call $\mathfrak{g}$ the finite Lie algebra associated to G.

Let $\mathcal{S}(G)$ be the set of subgroups of G generated by maximal tori of G, and $\mathcal{S}(\mathfrak{g})$ the set of subalgebras of $\mathfrak{g}$ containing a maximal toral subalgebra of $\mathfrak{g}$. Assume further that $q \geq 13$. For each $X \in \mathcal{S}(G)$ we define a Lie algebra $\mathcal{L}(X)$ and show that $\mathcal{L} : \mathcal{S}(G) \to \mathcal{S}(\mathfrak{g})$ is a bijection. The definition of $\mathcal{L}$ requires a previous work of Seitz [S], which proved under the additional restriction of q that the fixed point functor $?^\sigma$ by σ gives a bijection from the set $\mathcal{S}(\bar{G})$ of closed connected σ-invariant subgroups of $\bar{G}$ containing a maximal torus of $\bar{G}$ to $\mathcal{S}(G)$. This is a deep theorem exploiting the classification of finite simple groups. We give a linearization that the fixed point functor $?^{\sigma_*}$ under σ_* yields a bijection from the set $\mathcal{S}(\bar{\mathfrak{g}})$ of σ_*-invariant subalgebras of $\bar{\mathfrak{g}}$ containing a σ_*-invariant maximal toral subalgebra of $\bar{\mathfrak{g}}$ to $\mathcal{S}(\mathfrak{g})$. We also show that the functor Lie restricts to a bijection from $\mathcal{S}(\bar{G})$ to $\mathcal{S}(\bar{\mathfrak{g}})$. Composing the 3 obtains $\mathcal{L}$:

$$\begin{array}{ccc} \mathcal{S}(\bar{G}) & \xrightarrow[\sim]{?^\sigma} & \mathcal{S}(G) \\ {\scriptstyle \mathrm{Lie}}\downarrow{\scriptstyle \sim} & & \downarrow{\scriptstyle \mathcal{L}} \\ \mathcal{S}(\bar{\mathfrak{g}}) & \xrightarrow[?^{\sigma_*}]{\sim} & \mathcal{S}(\mathfrak{g}). \end{array}$$

2° A finite Lie algebra associated to a finite group of Lie type

2.1. Let $\bar{G}$ be a simple algebraic group over an algebraically closed field $\mathbb{K}$ of characteristic $p > 0$, and σ a surjective endomorphism of $\bar{G}$ with finite fixed-point group $G = \bar{G}^\sigma$; there should be no confusion of $\bar{G}$ with the closure of G in the Zariski topology. We begin with an elaborate description of the set-up. As the differential $\mathrm{d}\sigma$ vanishes, the fixed-point set by $\mathrm{d}\sigma$ does not lead to a good definition of the Lie algebra associated to G.

We assume throughout the paper that $p \geq 5$. Under the hypothesis we show first

Lemma: *σ is a commuting composite of a geometric Frobenius and a graph automorphism of $\bar{G}$.*

2.2. Fix once and for all a pair $(\bar{B}, \bar{H})$ of σ-stable Borel subgroup $\bar{B}$ of $\bar{G}$ and a σ-stable maximal torus $\bar{H}$ of $\bar{B}$, which exists by Lang's theorem [St67, 10.10]; if X is a Borel subgroup of $\bar{G}$, $\sigma X = gXg^{-1}$ for some $g \in \bar{G}$. Write $g = \sigma(x)^{-1}x$ for some $x \in \bar{G}$. Then $\sigma(xXx^{-1}) = \sigma(x)gXg^{-1}\sigma(x)^{-1} = xXx^{-1}$. Our definition of the Lie algebra associated to G will depend on our choice of the pair $(\bar{B}, \bar{H})$.

Let $\mathbb{G}_m = \mathrm{GL}_1(\mathbb{K})$, $\Lambda = \mathbf{Grp}_{\mathbb{K}}(\bar{H}, \mathbb{G}_m)$ the character group of $\bar{H}$ and put $\Lambda_{\mathbb{R}} = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$. We let σ^* denote the transpose of $\sigma|_{\bar{H}}$ extended to $\Lambda_{\mathbb{R}}$:

$$\sigma^* v = v \circ (\sigma \otimes_{\mathbb{Z}} \mathbb{R}), \quad v \in \Lambda_{\mathbb{R}}. \tag{1}$$

Let $R = R(\bar{G}, \bar{H})$ be the set of roots of $\bar{G}$ with respect to $\bar{H}$, $\bar{N} = \mathrm{N}_{\bar{G}}(\bar{H})$ the normalizer of $\bar{H}$ in $\bar{G}$, and $W = \bar{N}/\bar{H}$ the Weyl group of R. For each $\alpha \in R$ we choose an isomorphism of algebraic groups $x_\alpha : \mathbb{G}_a \to \bar{U}_\alpha$ from the 1-dimensional unipotent group $\mathbb{G}_a$ onto a closed subgroup $\bar{U}_\alpha$ of $\bar{G}$ such that $tx_\alpha(\xi)t^{-1} = x_\alpha(\alpha(t)\xi)$ $\forall t \in \bar{H}$, $\forall \xi \in \mathbb{K}$, and that $x_\alpha(1)x_{-\alpha}(-1)x_\alpha(1) \in \bar{N}$ giving reflection s_α in W, i.e., the $(x_\alpha | \alpha \in R)$ realize R [Sp, 8.1.4]. We call $\bar{U}_\alpha$ the $\bar{H}$-root subgroup of $\bar{G}$ associated to α. For later use put $n_\alpha = x_\alpha(1)x_{-\alpha}(-1)x_\alpha(1)$.

Let R^+ be the positive system of roots consisting of the roots of $\bar{B}$; $\bar{B} = \bar{H} \ltimes \prod_{\alpha \in R^+} \bar{U}_\alpha$. Then [St67, 11.2] there is a permutation τ of R and a power $q(\alpha)$ of p for each $\alpha \in R$ such that

$$\tau R^+ = R^+, \tag{2}$$

$$\forall \alpha \in R, \sigma^* \tau \alpha = q(\alpha)\alpha, \tag{3}$$

$$\forall \alpha \in R, \exists c_\alpha \in \mathbb{K}^\times : \forall \xi \in \mathbb{K}, \sigma(x_\alpha(\xi)) = x_{\tau\alpha}(c_\alpha \xi^{q(\alpha)}). \tag{4}$$

For, as $\bar{H}$ is σ-stable, the $\bar{H}$-root subgroups of $\bar{G}$ are permuted by σ. Define a permutation τ of R by setting $\sigma \bar{U}_\alpha = \bar{U}_{\tau\alpha}$ $\forall \alpha \in R$. Write $\sigma x_\alpha(\xi) = x_{\tau\alpha}(m(\xi))$, $\xi \in \mathbb{K}$. Thus, $m \in \mathbf{Grp}_{\mathbb{K}}(\mathbb{G}_a, \mathbb{G}_a)$. $\forall t \in \bar{H}$,

$$\begin{aligned} x_{\tau\alpha}((\tau\alpha)(\sigma(t))m(\xi)) &= \sigma(t)x_{\tau\alpha}(m(\xi))\sigma(t)^{-1} = \sigma(t)\sigma(x_\alpha(\xi))\sigma(t)^{-1} = \sigma(tx_\alpha(\xi)t^{-1}) \\ &= \sigma(x_\alpha(\alpha(t)\xi)) = x_{\tau\alpha}(m(\alpha(t)\xi)), \end{aligned}$$

and hence $(\tau\alpha)(\sigma(t))m(\xi) = m(\alpha(t)\xi)$. As $\mathbb{K}$ is infinite, m must be homogeneous of degree $q(\alpha)$, say. Then $(\alpha(t)\xi)^{q(\alpha)} = (\tau\alpha)(\sigma(t))\xi^{q(\alpha)}$, and hence $\alpha(t)^{q(\alpha)} = (\tau\alpha)(\sigma(t))$. Written additively, $\sigma^*\tau\alpha = q(\alpha)\alpha$. As m is additive, $q(\alpha)$ must be a power of p [HLAG, 20.3.A].

By [St67, 11.5] on roots of a given length q is constant. If q is not constant with α long and β short, $\frac{(\alpha,\alpha)}{(\beta,\beta)} = \frac{\langle \alpha, \beta^\vee \rangle}{\langle \beta, \alpha^\vee \rangle} = p = \frac{q(\beta)}{q(\alpha)}$, and hence under our standing hypothesis that $p \geq 5$

$$q \text{ is constant on the whole of } R, \tag{5}$$

the value of which we will abbreviate as q. Thus, τ is additive, and preserves the set R^{s} of simple roots. Recall also from [St67, 11.6] that τ preserves the orthogonality and the lengths of roots:

$$(\tau\alpha, \tau\beta) = 0 \text{ iff } (\alpha, \beta) = 0 \quad \forall \alpha, \beta \in R, \tag{6}$$

$$(\tau\alpha, \tau\alpha) = (\alpha, \alpha) \quad \forall \alpha \in R. \tag{7}$$

For let $\alpha \in R$ and s_α the associated reflection. Then $(\sigma^*)^{-1} s_\alpha \sigma^*$ is a reflection in the hyperplane $\{v \in \Lambda_{\mathbb{R}} | \langle \sigma^* v, \alpha^\vee \rangle = 0\} = \{v \in \Lambda_{\mathbb{R}} | (\sigma^* v, \alpha) = 0\}$ with $\tau\alpha \mapsto -\tau\alpha$. Thus, $(\sigma^*)^{-1} s_\alpha \sigma^* = s_\gamma$ for some $\gamma \in R$ [HLART, Lem. 9.1], and hence $\gamma = \pm\tau\alpha$ and

$$(\sigma^*)^{-1} s_\alpha \sigma^* = s_{\tau\alpha}. \tag{8}$$

Then, $\forall \beta \in R$, $(\alpha, \beta) = 0$ iff $s_\alpha s_\beta = s_\beta s_\alpha$ iff $(\sigma^*)^{-1} s_\alpha s_\beta \sigma^* = (\sigma^*)^{-1} s_\beta s_\alpha \sigma^*$ iff $s_{\tau\alpha} s_{\tau\beta} = s_{\tau\beta} s_{\tau\alpha}$ iff $(\tau\alpha, \tau\beta) = 0$, and (6) holds. Assume that $\alpha \in R$ is short, and just suppose $\tau\alpha$ is long. If we take $\beta \in R$ with $\langle \beta, \alpha^\vee \rangle > 1$,

$$\begin{aligned} q\beta - \langle \tau\beta, (\tau\alpha)^\vee \rangle q\alpha &= \sigma^*(\tau\beta - \langle \tau\beta, (\tau\alpha)^\vee \rangle \tau\alpha) = \sigma^* s_{\tau\alpha} \tau\beta \\ &= s_\alpha \sigma^* \tau\beta \quad \text{by (8)} \\ &= s_\alpha q\beta = q(\beta - \langle \beta, \alpha^\vee \rangle \alpha), \end{aligned}$$

and hence $\langle \beta, \alpha^\vee \rangle = \langle \tau\beta, (\tau\alpha)^\vee \rangle = \pm 1$, absurd.

It follows that τ induces an automorphism of Dynkin diagram of R [HLART, p. 57]. Thus, either $\tau = \mathrm{id}$, or $\mathrm{ord}\tau = 2$ and $\bar{G}$ is of type A_l, $l \geq 2$, D_l, $l \geq 4$, E_6, or $\mathrm{ord}\tau = 3$

and $\bar{G}$ is of type D_4. In all cases the isomorphisms $x_\alpha : \mathbb{G}_a \to \bar{U}_\alpha$ may be rearranged so that $c_\alpha = 1$; $\forall \alpha \in R$, $\forall \xi \in \mathbb{K}$,

$$\sigma(x_\alpha(\xi)) = x_{\tau\alpha}(q\xi^q). \tag{9}$$

For let $\alpha \in R$. One now has that $x_{\tau\alpha}(c_\alpha)x_{-\tau\alpha}(-c_{-\alpha})x_{\tau\alpha}(c_\alpha) = \sigma(x_\alpha(1)x_{-\alpha}(-1)x_\alpha(1)) \in \sigma(\bar{N}) = \bar{N}$ while $x_{\tau\alpha}(c_\alpha)x_{-\tau\alpha}(-c_\alpha^{-1})x_{\tau\alpha}(c_\alpha) = (\tau\alpha)^\vee(c_\alpha)n_{\tau\alpha} \in \bar{N}$ by [Sp, 8.1.4.(i)], and hence by the unicity [Sp, 8.1.4.(iii)]

$$c_\alpha c_{-\alpha} = 1. \tag{10}$$

Assume first that $\tau = \mathrm{id}$. Take $d_\alpha \in \mathbb{K}$ with $c_\alpha = d_\alpha^{1-q}$ and define

$$\begin{array}{ccccc} d_\alpha\xi & & \mathbb{G}_a & \xrightarrow{x_\alpha} & \bar{U}_\alpha . \\ \uparrow & & \sim\uparrow & \nearrow_{y_\alpha} & \\ \xi & & \mathbb{G}_a & & \end{array} \tag{11}$$

Then $\sigma(y_\alpha(\xi)) = \sigma(x_\alpha(d_\alpha\xi)) = x_\alpha(c_\alpha(d_\alpha\xi)^q) = x_\alpha(d_\alpha\xi^q) = y_\alpha(\xi^q)$. $\forall t \in \bar{H}$, $ty_\alpha(\xi)t^{-1} = tx_\alpha(d_\alpha\xi)t^{-1} = x_\alpha(\alpha(t)d_\alpha\xi) = y_\alpha(\alpha(t)\xi)$.

Assume next that $\mathrm{ord}\tau = 2$. Partition R into the τ-orbits, and let Ω be an orbit. If $\Omega = \{\alpha\}$, define y_α as in (11) with d_α such that $c_\alpha = d_\alpha^{1-q}$. If $\Omega = \{\alpha, \tau\alpha\}$ is of length 2, take $d_{\tau\alpha} \in \mathbb{K}$ such that $c_\alpha c_{\tau\alpha}^q = d_{\tau\alpha}^{1-q^2}$ and let $d_\alpha = c_{\tau\alpha}d_{\tau\alpha}^q$. Define y_α and $y_{\tau\alpha}$ with d_α and $d_{\tau\alpha}$, resp., as before. Then

$$\begin{aligned} \sigma(y_\alpha(\xi)) &= \sigma(x_\alpha(d_\alpha\xi)) = x_{\tau\alpha}(c_\alpha(d_\alpha\xi)^q) = x_{\tau\alpha}(c_\alpha d_\alpha^q\xi^q) = x_{\tau\alpha}(c_\alpha(c_{\tau\alpha}d_{\tau\alpha}^q)^q\xi^q) \\ &= x_{\tau\alpha}(d_{\tau\alpha}\xi^q) = y_{\tau\alpha}(\xi^q), \\ \sigma(y_{\tau\alpha}(\xi)) &= \sigma(x_{\tau\alpha}(d_{\tau\alpha}\xi)) = x_\alpha(c_{\tau\alpha}(d_{\tau\alpha}\xi)^q) = x_\alpha(c_{\tau\alpha}d_{\tau\alpha}^q\xi^q) = x_\alpha(d_\alpha\xi^q) = y_\alpha(\xi^q). \end{aligned}$$

Thus, $\sigma(y_\alpha(\xi)) = y_{\tau\alpha}(\xi^q)$ $\forall \alpha \in R$.

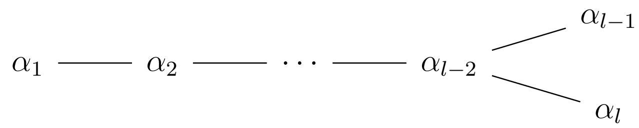
If $\mathrm{ord}\tau = 3$, let Ω be a τ-orbit. If $\Omega = \{\alpha\}$, define y_α with d_α such that $c_\alpha = d_\alpha^{1-q}$, as before. If $\Omega = \{\alpha, \tau\alpha, \tau^2\alpha\}$ is of length 3, take $d_\alpha \in \mathbb{K}$ such that $c_\alpha^{q^2}c_{\tau\alpha}^q c_{\tau^2\alpha} = d_\alpha^{1-q^3}$ and put $d_{\tau\alpha} = c_\alpha d_\alpha^q$, $d_{\tau^2\alpha} = c_{\tau\alpha}d_{\tau\alpha}^q = c_\alpha^q c_{\tau\alpha}d_\alpha^{q^2}$. Define $y_\alpha, y_{\tau\alpha}, y_{\tau^2\alpha}$ with $d_\alpha, d_{\tau\alpha}, d_{\tau^2\alpha}$, resp. as above. Then

$$\begin{aligned} \sigma(y_\alpha(\xi)) &= \sigma(x_\alpha(d_\alpha\xi)) = x_{\tau\alpha}(c_\alpha(d_\alpha\xi)^q) = x_{\tau\alpha}(c_\alpha d_\alpha^q\xi^q) = x_{\tau\alpha}(d_{\tau\alpha}\xi^q) = y_{\tau\alpha}(\xi^q), \\ \sigma(y_{\tau\alpha}(\xi)) &= \sigma(x_{\tau\alpha}(d_{\tau\alpha}\xi)) = x_{\tau^2\alpha}(c_{\tau\alpha}(d_{\tau\alpha}\xi)^q) = x_{\tau^2\alpha}(d_{\tau^2\alpha}\xi^q) = y_{\tau^2\alpha}(\xi^q), \\ \sigma(y_{\tau^2\alpha}(\xi)) &= \sigma(x_{\tau^2\alpha}(d_{\tau^2\alpha}\xi)) = x_\alpha(c_{\tau^2\alpha}(d_{\tau^2\alpha}\xi)^q) = x_\alpha(c_{\tau^2\alpha}d_{\tau^2\alpha}^q\xi^q) \\ &= x_\alpha(c_{\tau^2\alpha}(c_a^q c_{\tau\alpha}d_\alpha^{q^2})^q\xi^q) = x_\alpha(d_\alpha\xi^q) = y_\alpha(\xi^q), \end{aligned}$$

and hence $\sigma(y_\beta(\xi)) = y_{\tau\beta}(\xi^q)$ $\forall \beta \in \mathbb{K}$.

In all cases one can by (10) take $d_{-\alpha}$ so that $d_\alpha d_{-\alpha} = 1$. Then $y_\alpha(1)y_{-\alpha}(-1)y_\alpha(1) = x_\alpha(d_\alpha)x_{-\alpha}(-d_\alpha^{-1})x_\alpha(d_\alpha) \in \bar{N}$. Thus, $(y_\alpha | \alpha \in R)$ realizes R, and (9) holds with the y_α's.

2.3. Keep the notation of 2.2, and assume that $\tau \neq \mathrm{id}$. Let $\tilde{\Lambda} = \{\lambda \in \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q} | \langle\lambda, \alpha\rangle \in \mathbb{Z}\ \forall \alpha \in R\}$ the weight lattice of R. In all cases except in type D_l with l even $\tilde{\Lambda}/\mathbb{Z}$ is cyclic. Assume now that R is of type D_l with l even. Label the Dynkin diagram as



The fundamental weights ϖ_i's are given by

$$\begin{aligned}\varpi_i &= \alpha_1 + 2\alpha_2 + \cdots + (l-1)\alpha_{l-1} + i(\alpha_i + \alpha_{i+1} + \cdots + \alpha_{l-2}) + \frac{i}{2}(\alpha_{l-1} + \alpha_l) \\ &\quad \text{if } i < l-1, \\ \varpi_{l-1} &= \frac{1}{2}(\alpha_1 + 2\alpha_2 + \cdots + (l-2)\alpha_{l-2} + \frac{l}{2}\alpha_{l-1} + \frac{l-2}{2}\alpha_l), \\ \varpi_l &= \frac{1}{2}(\alpha_1 + 2\alpha_2 + \cdots + (l-2)\alpha_{l-2} + \frac{l-2}{2}\alpha_{l-1} + \frac{l}{2}\alpha_l).\end{aligned}$$

Thus, $\tilde{\Lambda}/\mathbb{Z}R \simeq (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$ and the elements of order 2 in $\tilde{\Lambda}/\mathbb{Z}R$ are given by $\varpi_1, \varpi_{l-1}, \varpi_l$ [BLA]. In particular, σ^* would not stabilize Λ if $(\Lambda : \mathbb{Z}R) = 2$ unless $\tau = \mathrm{id}$.

It follows that in our set-up τ induces an automorphism $\tilde{\tau}$ of the root datum of $\bar{G}$. In turn, $\tilde{\tau}$ induces an automorphism of algebraic group $\bar{G}$, denoted still by $\tilde{\tau}$, such that $x_\alpha(\xi) \mapsto x_{\tau\alpha}(\xi)$ $\forall \alpha \in \pm R^{\mathrm{s}}$, $\forall \xi \in \mathbb{K}$ [Sp, 9.6.2]. Then $\sigma \circ \tilde{\tau} = \tilde{\tau} \circ \sigma$ on $\bar{U}_\alpha$ $\forall \alpha \in \pm R^{\mathrm{s}}$, and hence on the whole of $\bar{G}$ as $\bar{G}$ is simple [Sp, 8.1.5]; the morphisms are determined at the points of the domain as it is reduced. As in 2.2 one may rearrange the x_α's such that

(1) $$\tilde{\tau}^{-1} \circ \sigma(x_\alpha(\xi)) = x_\alpha(\xi^q) \quad \forall \alpha \in R, \forall \xi \in \mathbb{K}.$$

Let now $F = \sigma \circ \tilde{\tau}^{-1} = \tilde{\tau}^{-1} \circ \sigma$. Then

(2) $$F n_\alpha = F(x_\alpha(1) x_{-\alpha}(-1) x_\alpha(1)) = n_\alpha \quad \forall \alpha \in R.$$

Recall from [Sp, 8.1.4.(i)] again that $x_\alpha(\zeta) x_{-\alpha}(-\zeta^{-1}) x_\alpha(\zeta) = \alpha^\vee(\zeta) n_\alpha$ $\forall \alpha \in R$, $\forall \zeta \in \mathbb{K}^\times$. Then

$$\begin{aligned}F(\alpha^\vee(\zeta)) n_\alpha &= F(\alpha^\vee(\zeta) n_\alpha) = F(x_\alpha(\zeta) x_{-\alpha}(-\zeta^{-1}) x_\alpha(\zeta)) = x_\alpha(\zeta^q) x_{-\alpha}(-\zeta^{-q}) x_\alpha(\zeta^q) \\ &= \alpha^\vee(\zeta^q) n_\alpha,\end{aligned}$$

and hence $F(\alpha^\vee(\zeta)) = \alpha^\vee(\zeta^q) = \alpha^\vee(\zeta)^q$. As the $\alpha^\vee(\zeta)$'s generate $\bar{H}$ [Sp, 8.1.5],

(3) $$F t = t^q \quad \forall t \in \bar{H}.$$

If $w \in W$, write $w = s_{\alpha_1} \dots s_{\alpha_r}$ for $\alpha_1, \dots, \alpha_r \in R$, and let $n_w = n_{\alpha_1} \dots n_{\alpha_r}$. Letting $\bar{U}^- = \prod_{\alpha \in -R^+} \bar{U}_\alpha$, one has

$$\begin{array}{ccc}(\mathbb{G}_a)^{-R^+} \times (\mathbb{G}_m)^{R^{\mathrm{s}}} \times (\mathbb{G}_a)^{R^+} & \xrightarrow{\sim} & n_w \bar{U}^- \bar{H} \bar{U} \\ \Big\downarrow {?^q} & & \Big\downarrow F \\ (\mathbb{G}_a)^{-R^+} \times (\mathbb{G}_m)^{R^{\mathrm{s}}} \times (\mathbb{G}_a)^{R^+} & \underset{\sim}{\longrightarrow} & n_w \bar{U}^- \bar{H} \bar{U},\end{array}$$

and hence there is induced an $\mathbb{F}_q$-algera $\mathbb{F}_q[n_w\bar{U}^-\bar{H}\bar{U}]$ such that

$$\begin{array}{ccc}\mathbb{K}[n_w\bar{U}^-\bar{H}\bar{U}] & = & \mathbb{K}\otimes_{\mathbb{F}_q}\mathbb{F}_q[n_w\bar{U}^-\bar{H}\bar{U}] \\ {\scriptstyle F^\sharp}\downarrow & & \downarrow{\scriptstyle \mathbb{K}\otimes_{\mathbb{F}_q}?^q} \\ \mathbb{K}[n_w\bar{U}^-\bar{H}\bar{U}] & = & \mathbb{K}\otimes_{\mathbb{F}_q}\mathbb{F}_q[n_w\bar{U}^-\bar{H}\bar{U}].\end{array}$$

Then in $\mathrm{Frac}(\mathbb{K}[\bar{G}])$

$$\begin{aligned}\mathbb{K}[\bar{G}] &= \cap_{w\in W}\mathbb{K}[n_w\bar{U}^-\bar{H}\bar{U}] = \cap_{w\in W}\mathbb{K}\otimes_{\mathbb{F}_q}\mathbb{F}_q[n_w\bar{U}^-\bar{H}\bar{U}] \\ &= \mathbb{K}\otimes_{\mathbb{F}_q}\cap_{w\in W}\mathbb{F}_q[n_w\bar{U}^-\bar{H}\bar{U}] \quad \text{by [BCA, Lem. I.2.6.7]},\end{aligned}$$

and hence, putting $\mathbb{F}_q[\bar{G}] = \cap_{w\in W}\mathbb{F}_q[n_w\bar{U}^-\bar{H}\bar{U}]$ yields

$$\begin{array}{ccc}\mathbb{K}[\bar{G}] & = & \mathbb{K}\otimes_{\mathbb{F}_q}\mathbb{F}_q[\bar{G}] \\ {\scriptstyle F^\sharp}\downarrow & & \downarrow{\scriptstyle \mathbb{K}\otimes_{\mathbb{F}_q}?^q} \\ \mathbb{K}[\bar{G}] & = & \mathbb{K}\otimes_{\mathbb{F}_q}\mathbb{F}_q[\bar{G}].\end{array}$$

Thus, F is the geometric Frobenius $\mathbb{K}\otimes_{\mathbb{F}_q}\mathrm{Fr}_q$ on $\bar{G}$ with respect to an $\mathbb{F}_q$-form $\bar{G}_{\mathbb{F}_q}$ defined by $\mathbb{F}_q[\bar{G}]$.

Setting $\tilde{\tau} = \mathrm{id}_{\bar{G}}$ when $\tau = \mathrm{id}$, we have obtained $\sigma = F\circ\tilde{\tau}$ in all cases, and Lem. 2.1 holds. As F is a morphism of algebraic groups, $F^\sharp$ is a morphism of Hopf algebras on $\mathbb{K}[\bar{G}]$. It follows that $\mathbb{F}_q[\bar{G}] = \{a\in\mathbb{K}[\bar{G}]|F^\sharp(a) = a^q\}$ forms a Hopf subalgebra of $\mathbb{K}[\bar{G}]$ over $\mathbb{F}_q$, and $\mathfrak{Sp}_{\mathbb{F}_q}(\mathbb{F}_q[\bar{G}]) = \mathbf{Alg}_{\mathbb{F}_q}(\mathbb{F}_q[\bar{G}], ?)$ gives an $\mathbb{F}_q$-form of $\mathfrak{G}$.

2.4. Let $\Bbbk = \mathbb{F}_q$. The augmentation ideal $\mathfrak{m}$ of $\mathbb{K}[\bar{G}]$ now admits a $\Bbbk$-form $\mathfrak{m}_\Bbbk$, so therefore does the algebra of distribution $\mathrm{Dist}(\bar{G})$ on $\bar{G}$ a $\Bbbk$-form $\mathrm{Dist}(\bar{G}_\Bbbk)$. In particular, the Lie algebra $\bar{\mathfrak{g}} = (\mathfrak{m}/\mathfrak{m}^2)^*$ of $\bar{G}$ admits a $\Bbbk$-form $\bar{\mathfrak{g}}_\Bbbk$. As $\mathrm{Dist}(\bar{G}) = \mathrm{Dist}(\bar{U}^-\bar{H}\bar{U}, e)$, $\mathrm{Dist}(\bar{G}_\Bbbk)$ is invariant under $\mathrm{Dist}(\tilde{\tau})$, so therefore is $\bar{\mathfrak{g}}_\Bbbk$ under $\mathrm{d}\tilde{\tau}$. Let F_* be the arithmetic Frobenius on $\mathrm{Dist}(\bar{G})$ defined by $\mathrm{Dist}(\bar{G}_\Bbbk)$

$$\begin{array}{ccc}\mathrm{Dist}(\bar{G}) & \xrightarrow{F_*} & \mathrm{Dist}(\bar{G}) \\ \| & & \| \\ \mathbb{K}\otimes_\Bbbk\mathrm{Dist}(\bar{G}_\Bbbk) & \xrightarrow[?^q\otimes_\Bbbk\mathrm{Dist}(\bar{G}_\Bbbk)]{} & \mathbb{K}\otimes_\Bbbk\mathrm{Dist}(\bar{G}_\Bbbk).\end{array}$$

As $\mathrm{d}\tilde{\tau}$ is defined over $\Bbbk$, one has $F_*\circ\mathrm{d}\tilde{\tau} = \mathrm{d}\tilde{\tau}\circ F_*$. Set $\sigma_* = F_*\circ\mathrm{d}\tilde{\tau}$, and $\mathfrak{g} = \bar{\mathfrak{g}}^{\sigma_*} = \{x\in\bar{\mathfrak{g}}|\sigma_*x = x\}$. As F_* preserves the multiplication on $\mathrm{Dist}(\bar{G}_\Bbbk)$, so does σ_*, and hence $\mathfrak{g}$ forms a p-subalgeba of $\bar{\mathfrak{g}}$ over $\Bbbk$. By definition σ_* is $\mathbb{K}$-semilinear: $\sigma_*(\xi x) = \xi^q\sigma_*x$ $\forall\xi\in\mathbb{K}$ $\forall x\in\bar{\mathfrak{g}}$.

Definition: *We call $\mathfrak{g}$ the finite Lie algebra associated to $G = \bar{G}^\sigma$.*

2.5. As we have just noted, σ_* is an automorphism of p-Lie algebra $\bar{\mathfrak{g}}$ over $\Bbbk$.

Lemma: *$\forall g\in\bar{G}$, $\mathrm{Ad}(\sigma(g))\circ\sigma_* = \sigma_*\circ\mathrm{Ad}(g)$ on $\bar{\mathfrak{g}}$. In particular, G acts on $\mathfrak{g}$ under* Ad.

Proof: Let us first imbed $\bar{G}$ in some $\mathrm{GL}_n(\mathbb{K})$ over $\Bbbk$. Consider the left regular action of $\bar{G}$ on $\mathbb{K}[\bar{G}]$: $ga = a(g^{-1}?)$ $\forall g \in \bar{G}$ $\forall a \in \mathbb{K}[\bar{G}]$. Let $\Delta : \mathbb{K}[\bar{G}] \to \mathbb{K}[\bar{G}] \otimes_{\mathbb{K}} \mathbb{K}[\bar{G}]$ denote the corresponding comodule map [J, I.2.8]:

$$\Delta = \mathrm{tp}^\sharp \circ (S \otimes_{\mathbb{K}} \mathbb{K}[\bar{G}]) \circ \Delta_{\bar{G}}, \tag{1}$$

where $\mathrm{tp} : \bar{G} \times \bar{G} \to \bar{G} \times \bar{G}$ is the transposition $(g_1, g_2) \mapsto (g_2, g_1)$ and S (resp. $\Delta_{\bar{G}}$) is the antipode (resp. comultiplication) on $\mathbb{K}[\bar{G}]$. In particular, Δ is defined over $\Bbbk$:

$$\begin{array}{ccc} \mathbb{K}[\bar{G}] & \xrightarrow{\quad\Delta\quad} & \mathbb{K}[\bar{G}] \otimes_{\mathbb{K}} \mathbb{K}[\bar{G}] \\ \| & & \wr\| \\ \mathbb{K} \otimes_{\Bbbk} \Bbbk[\bar{G}_{\Bbbk}] & \underset{\mathbb{K}\otimes_{\Bbbk}\Delta_{\Bbbk}}{\dashrightarrow} & \mathbb{K} \otimes_{\Bbbk} (\Bbbk[\bar{G}_{\Bbbk}] \otimes_{\Bbbk} \Bbbk[\bar{G}_{\Bbbk}]) \end{array} \tag{2}$$

with $\Delta_{\Bbbk} = \mathrm{tp}_{\Bbbk}^\sharp \circ (S_{\Bbbk} \otimes \Bbbk[\bar{G}_{\Bbbk}]) \circ \Delta_{\bar{G}_{\Bbbk}}$.

For $a \in \mathbb{K}[\bar{G}]$ the $\bar{G}$-submodule generated by a is $\mathrm{Dist}(\bar{G})a = \{\mu a = \{(\mathbb{K}[\bar{G}] \otimes_{\mathbb{K}} \mu) \circ \Delta\}(a) | \mu \in \mathrm{Dist}(\bar{G})\}$ [J, I.7.15]. Thus, $\Delta(\mathrm{Dist}(\bar{G})a) \subseteq \mathrm{Dist}(\bar{G})a \otimes_{\mathbb{K}} \mathbb{K}[\bar{G}]$. Write $\Delta(a) = \sum_i a_i \otimes b_i$, $a_i \in \mathrm{Dist}(\bar{G})a$, $b_i \in \mathbb{K}[\bar{G}]$, with the a_i linearly independent over $\mathbb{K}$. Then, $\forall \mu \in \mathrm{Dist}(\bar{G})$, $\mu a = \sum_i a_i \mu(b_i) \in \sum_i \mathbb{K}a_i = \coprod_i \mathbb{K}a_i$. It follows that

$$\mathrm{Dist}(\bar{G})a = \coprod_i \mathbb{K}a_i. \tag{3}$$

In case $a \in \Bbbk[\bar{G}_{\Bbbk}]$, as Δ is defined over $\Bbbk$ by (2), taking all a_i in $\mathrm{Dist}(\bar{G}_{\Bbbk})a$ yields

$$\mathrm{Dist}(\bar{G})a = (\mathbb{K} \otimes_{\Bbbk} \mathrm{Dist}(\bar{G}_{\Bbbk}))a = \mathbb{K} \otimes_{\Bbbk} \{\mathrm{Dist}(\bar{G}_{\Bbbk})a\} = \mathbb{K} \otimes_{\Bbbk} \coprod_i \Bbbk a_i. \tag{4}$$

Let now $f_1, \dots, f_r$ be a set of $\Bbbk$-algebra generators of $\Bbbk[\bar{G}_{\Bbbk}]$, and let $c_1, \dots, c_n$ be a $\mathbb{K}$-linear basis of $\sum_{i=1}^r \mathrm{Dist}(\bar{G})f_i$. By (4) one may take all c_i in $\Bbbk[\bar{G}_{\Bbbk}]$. $\forall i \in [1, n]$, write $\Delta(c_i) = \Delta_{\Bbbk}(c_i) = \sum_{j=1}^n c_j \otimes m_{ij}$, $m_{ij} \in \Bbbk[\bar{G}_{\Bbbk}]$. Let $\mathrm{GL}_n(\mathbb{K}) = \mathrm{GL}(\coprod_{i=1}^n \mathbb{K}c_i)$ with respect to the basis $c_1, \dots, c_n$. One thus obtains a group homomorphism $\bar{G} \to \mathrm{GL}_n(\mathbb{K})$ via $g \mapsto [\![m_{ij}(g)]\!]$. Write $\mathbb{K}[\mathrm{GL}_n(\mathbb{K})] = \mathbb{K}[x_{ij}, \frac{1}{\det} | i, j \in [1, n]]$ with $x_{ij}(y) = y_{ij}$ $\forall y \in \mathrm{GL}_n(\mathbb{K})$. Then $E_n = [\![m_{ij}(g)]\!][\![m_{ij}(g^{-1})]\!]$, and hence $1 = \det[\![m_{ij}(g)]\!] \det[\![m_{ij}(g^{-1})]\!]$. As $m_{ij}(?^{-1}) \in \Bbbk[\bar{G}_{\Bbbk}]$, $\det[\![m_{ij}(g)]\!] \in \Bbbk[\bar{G}_{\Bbbk}]^\times$. $\forall i$, $\forall g \in \bar{G}$, $c_i(g^{-1}?) = gc_i = \sum_j c_j m_{ij}(g)$. In particular, $c_i(1) = \sum_j c_j(g) m_{ij}(g)$, and hence

$$[\![m_{ij}(g)]\!] \begin{pmatrix} c_1(g) \\ \vdots \\ c_n(g) \end{pmatrix} = \begin{pmatrix} c_1(1) \\ \vdots \\ c_n(1) \end{pmatrix} \in \Bbbk^n \quad \text{as all } c_i \in \Bbbk[\bar{G}_{\Bbbk}].$$

Then

$$\begin{pmatrix} c_1(g) \\ \vdots \\ c_n(g) \end{pmatrix} = [\![m_{ij}(g^{-1})]\!][\![m_{ij}(g)]\!] \begin{pmatrix} c_1(g) \\ \vdots \\ c_n(g) \end{pmatrix} = [\![m_{ij}(g^{-1})]\!] \begin{pmatrix} c_1(1) \\ \vdots \\ c_n(1) \end{pmatrix},$$

and hence $c_i(g) = \sum_j m_{ij}(g^{-1}) c_j(1)$. It follows that all $c_i \in \Bbbk[m_{ij}, \frac{1}{\det[\![m_{ij}]\!]} | i, j \in [1, n]]$.

Thus,

$$\begin{array}{ccc}
x_{ij} & \longmapsto & m_{ij} \\
\mathbb{K}[\mathrm{GL}_n(\mathbb{K})] & \twoheadrightarrow & \mathbb{K}[\bar{G}] = \mathbb{K}[c_1, \dots, c_n] \\
\| & & \| \\
\mathbb{K} \otimes_{\Bbbk} \Bbbk[x_{ij}, \frac{1}{\det}|i,j]] & & \mathbb{K} \otimes_{\Bbbk} \Bbbk[\bar{G}_{\Bbbk}] \\
\uparrow & & \uparrow \\
\Bbbk[x_{ij}, \frac{1}{\det}|i,j] & \dashrightarrow\!\!\!\!\!\twoheadrightarrow & \Bbbk[\bar{G}_{\Bbbk}] = \Bbbk[c_1, \dots, c_n],
\end{array}$$

and the group hommomorphism $\bar{G} \to \mathrm{GL}_n(\mathbb{K})$ is an imbedding of algebraic groups.

If $\hat{F}$ is the geometric Frobenius on $\mathrm{GL}_n(\mathbb{K})$ defined by $\Bbbk[x_{ij}, \frac{1}{det}|i,j]$, it is compatible with the geometric Frobenius F on $\bar{G}$:

$$\begin{array}{ccc}
\mathbb{K}[\bar{G}] & \twoheadleftarrow & \mathbb{K}[\mathrm{GL}_n(\mathbb{K})] \\
\| & & \| \\
\mathbb{K} \otimes_{\Bbbk} \Bbbk[\bar{G}_{\Bbbk}] & \circlearrowright & \mathbb{K} \otimes_{\Bbbk} \Bbbk[x_{ij}, \frac{1}{\det}|i,j] \\
{\scriptstyle \hat{F}\sharp = \mathbb{K}\otimes_{\Bbbk}?^q}\downarrow & & \downarrow{\scriptstyle \mathbb{K}\otimes_{\Bbbk}?^q = \hat{F}\sharp} \\
\mathbb{K} \otimes_{\Bbbk} \Bbbk[\bar{G}_{\Bbbk}] & \twoheadleftarrow & \mathbb{K} \otimes_{\Bbbk} \Bbbk[x_{ij}, \frac{1}{\det}|i,j].
\end{array}$$

Define $\hat{F}_* : \mathfrak{gl}_n(\mathbb{K}) \to \mathfrak{gl}_n(\mathbb{K})$ to be the arithmetic Frobenius with respect to $\mathfrak{gl}_n(\Bbbk)$:

$$\begin{array}{ccc}
\mathfrak{gl}_n(\mathbb{K}) & \overset{\hat{F}_*}{\dashrightarrow} & \mathfrak{gl}_n(\mathbb{K}) \\
\| & & \| \\
\mathbb{K} \otimes_{\Bbbk} \mathfrak{gl}_n(\Bbbk) & \xrightarrow[?^q \otimes_{\Bbbk} \mathfrak{gl}_n(\Bbbk)]{} & \mathbb{K} \otimes_{\Bbbk} \mathfrak{gl}_n(\Bbbk).
\end{array}$$

Then

$$\mathrm{Ad}(\hat{F}(g)) \circ \hat{F}_* = \hat{F}_* \circ \mathrm{Ad}(g) \quad \forall g \in \mathrm{GL}_n(\mathbb{K}). \tag{5}$$

For let $g = [\![g_{ij}]\!] \in \mathrm{GL}_n(\mathbb{K})$ and $y = [\![y_{ij}]\!] = \sum_{i,j} y_{ij} e_{ij} \in \mathfrak{gl}_n(\mathbb{K})$ with $e_{ij} \in \mathfrak{gl}_n(\mathbb{K})$ such that $(e_{ij})_{ab} = \delta_{i,a}\delta_{j,b}\ \forall i,j,a,b \in [1,n]$. Then

$$\{\mathrm{Ad}(\hat{F}(g)) \circ \hat{F}_*\}(y) = [\![g_{ij}^q]\!][\![y_{ij}^q]\!][\![g_{ij}^q]\!]^{-1} = \hat{F}_*([\![g_{ij}]\!][\![y_{ij}]\!][\![g_{ij}]\!]^{-1}) = \{\hat{F}_* \circ \mathrm{Ad}(g)\}(y),$$

and hence also

$$\mathrm{Ad}(F(g)) \circ F_* = F_* \circ \mathrm{Ad}(g) \quad \forall g \in \bar{G}. \tag{6}$$

As $\mathrm{Int}(\tilde{\tau}(g)) = \tilde{\tau}(g\tilde{\tau}^{-1}(?)g^{-1}) = \tilde{\tau} \circ \mathrm{Int}(g) \circ \tilde{\tau}^{-1}$, $\mathrm{Ad}(\tilde{\tau}(g)) = \mathrm{d}\tilde{\tau} \circ \mathrm{Ad}(g) \circ (\mathrm{d}\tilde{\tau})^{-1}$. Thus,

$$\begin{aligned}
\mathrm{Ad}(\sigma(g)) \circ \sigma_* &= \mathrm{d}\tilde{\tau} \circ \mathrm{Ad}(F(g)) \circ (\mathrm{d}\tilde{\tau})^{-1} \circ F_* \circ \mathrm{d}\tilde{\tau} = \mathrm{d}\tilde{\tau} \circ \mathrm{Ad}(F(g)) \circ F_* \\
&= \mathrm{d}\tilde{\tau} \circ F_* \circ \mathrm{Ad}(g) \quad \text{by (5)} \\
&= \sigma_* \circ \mathrm{Ad}(g).
\end{aligned}$$

2.6. Let $\bar{G}\langle\sigma\rangle = \bar{G} \rtimes \langle\sigma\rangle$. Thus, in $\bar{G}\langle\sigma\rangle$, $\forall i,j \in \mathbb{Z}$, $\forall g, g' \in \bar{G}$,

$$g\sigma^i g' \sigma^j = g\sigma^i(g')\sigma^{i+j}.$$

Then

$$\begin{aligned}\mathrm{Ad}(g)\sigma_*^i\mathrm{Ad}(g')\sigma_*^j &= \mathrm{Ad}(g)\mathrm{Ad}(\sigma^i(g'))\sigma_*^{i+j} \quad \text{by 2.5}\\ &= \mathrm{Ad}(g\sigma^i(g'))\sigma_*^{i+j},\end{aligned}$$

and hence one obtains a homomorphism of abstract groups

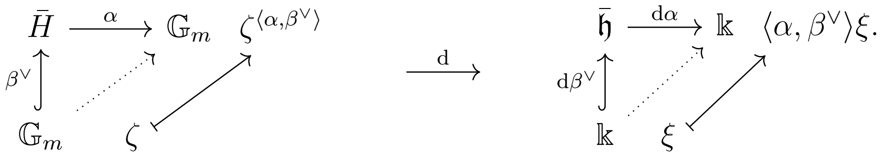
$$\bar{G}\langle\sigma\rangle \to p\mathbf{LA}_{\Bbbk}(\bar{\mathfrak{g}},\bar{\mathfrak{g}})^{\times} \quad \text{via} \quad g\sigma^i \mapsto \mathrm{Ad}(g)\sigma_*^i,$$

where $p\mathbf{LA}_{\Bbbk}(\bar{\mathfrak{g}},\bar{\mathfrak{g}})^{\times}$ denotes the set of automorphisms of p-Lie algebra $\bar{\mathfrak{g}}$ over $\Bbbk$.

2.7. Let $\bar{\mathfrak{h}} = \mathrm{Lie}(\bar{H})$ and $\bar{\mathfrak{g}}_\alpha = \mathrm{Lie}(\bar{U}_\alpha)$, $\alpha \in R$. As $\bar{U}^-\bar{H}\bar{U}$ is open in $\bar{G}$, one has a Cartan decomposition

(1) $$\bar{\mathfrak{g}} = \bar{\mathfrak{h}} \oplus \coprod_{\alpha\in R} \bar{\mathfrak{g}}_\alpha \quad \text{with} \quad \bar{\mathfrak{g}}_\alpha = \{x \in \bar{\mathfrak{g}} | \mathrm{Ad}(t)x = \alpha(t)x \ \forall t \in \bar{H}\}.$$

Thus, each $\bar{\mathfrak{g}}_\alpha$, $\alpha \in R$, is a 1-dimensional $\bar{H}$-module under Ad affording α, and hence also an $\bar{\mathfrak{h}}$-module under ad affording $\mathrm{d}\alpha$. $\forall \alpha, \beta \in R$,



As $p \geq 5$ by the standing hypothesis, $\langle\alpha,\alpha^\vee\rangle = 2 \neq 0$ in $\mathbb{K}$, and hence

(2) $$\mathrm{d}\alpha \neq 0 \quad \forall \alpha \in R.$$

Moreover, from [HLART, 13.1]

$$\det[\![\langle\alpha,\beta^\vee\rangle]\!]_{\alpha,\beta\in R^{\mathrm{s}}} = \begin{cases} l+1 & \text{if } R \text{ is of type } \mathrm{A}_l,\\ 2 & \text{if } R \text{ is of type } \mathrm{B}_l, \mathrm{C}_l, \mathrm{E}_7,\\ 4 & \text{if } R \text{ is of type } \mathrm{D}_l,\\ 3 & \text{if } R \text{ is of type } \mathrm{E}_6,\\ 1 & \text{if } R \text{ is of type } \mathrm{E}_8, \mathrm{F}_4, \mathrm{G}_2,\end{cases}$$

and hence

(3) $$\mathrm{d}\alpha \neq \mathrm{d}\beta \quad \text{if } \alpha,\beta \in R \text{ are distinct;}$$

in case $p|l+1$ in type A_l, by separate inspection. Thus,

(4) $$\bar{\mathfrak{g}}_\alpha = \bar{\mathfrak{g}}_{\mathrm{d}\alpha} \quad \forall \alpha \in R.$$

2.8. Let $\mathbb{F}$ be a field, V be a finite dimensional $\mathbb{F}$-linear space and ϕ an $\mathbb{F}$-linear endomorphism of V. For $\lambda \in \mathbb{F}$ let $V_\lambda(\phi) = \{v \in V | (\phi-\lambda)^n v = 0 \ \exists n \in \mathbb{N}\}$ the generalized λ-eigen space of f in V. Thus, $V_\lambda(\phi) = \{v \in V | (\phi-\lambda)^{\dim V} v = 0\}$. More generally, for $\Gamma \subseteq \mathbf{Lin}_{\mathbb{F}}(V,V)$ and $f : \Gamma \to \mathbb{F}$ let $V_f(\Gamma) = \cap_{x\in\Gamma} V_0(x - f(x))$, called the Fitting f-component of Γ in V.

If $\mathbb{F}$ is algebraically closed, Zassenhaus' theorem [SF, 1.4.4] asserts for a nilpotent Lie algebra $\mathfrak{n}$ that

$$V = \coprod_{f \in \mathbf{Set}(\mathfrak{n}, \mathbb{F})} V_f(\mathfrak{n}). \tag{1}$$

Let L be a finite dimensional Lie algebra. For $K \subseteq L$ under the adjoint representation $L_0(K)$ forms a subalgebra of L by Leibniz's rule; $\forall x, y, z \in L$, $\forall n \in \mathbb{N}$,

$$(\operatorname{ad} x)^n([y,z]) = \sum_{i=0}^{n} \binom{n}{i} [(\operatorname{ad} x)^i y, (\operatorname{ad} x)^{n-i} z].$$

We call $\mathrm{C}_L(K) = \{x \in L | [x,y] = 0 \ \forall y \in K\}$ the centralizer of K in L. Thus, $\mathrm{C}_L(K)$ is a subalgebra of $L_0(K)$. In case K is a subalgebra of L, We call K a Cartan subalgebra of L iff $K = L_0(K)$.

Let also $\mathrm{C}_{\bar{G}}(\bar{\mathfrak{h}}) = \{g \in \bar{G} | \mathrm{Ad}(g)(x) = x \ \forall x \in \bar{\mathfrak{h}}\}$, and $\mathrm{N}_{\bar{G}}(\bar{\mathfrak{h}}) = \{g \in \bar{G} | \mathrm{Ad}(g)(x) \in \bar{\mathfrak{h}} \ \forall x \in \bar{\mathfrak{h}}\}$.

Lemma: *(i)* $\bar{\mathfrak{h}} = \bar{\mathfrak{g}}_0(\bar{\mathfrak{h}}) = \mathrm{C}_{\bar{\mathfrak{g}}}(\bar{\mathfrak{h}})$ *is a Cartan subalgebra of* $\bar{\mathfrak{g}}$.

(ii) $\mathrm{C}_{\bar{G}}(\bar{\mathfrak{h}}) = \bar{H}$.

(iii) $\mathrm{N}_{\bar{G}}(\bar{\mathfrak{h}}) = \bar{N}$.

Proof: (i) follows from 2.7.

(ii) and (iii) Let $Z = \mathrm{C}_{\bar{G}}(\bar{\mathfrak{h}})^0$ the connected component of $\mathrm{C}_{\bar{G}}(\bar{\mathfrak{h}})$ around the unity and $\mathfrak{z} = \mathrm{Lie}(Z)$. Just suppose $Z > \bar{H}$. Then $\mathfrak{z} > \mathfrak{h}$, and hence $\bar{\mathfrak{g}}_\alpha \subseteq \mathfrak{z}$ for some $\alpha \in R$ by 2.7. Let $Z_\alpha = \mathrm{C}_Z((\ker\alpha)^0)$, which is connected [Sp, 6.4.7.(i)] with

$$\begin{aligned} \mathrm{Lie}(Z_\alpha) &= \mathrm{C}_{\mathfrak{z}}((\ker\alpha)^0) \quad \text{[Sp, 5.4.7]} \\ &\geq \mathfrak{h} \oplus \bar{\mathfrak{g}}_\alpha. \end{aligned}$$

Then a Borel subgroup of Z_α cannot be equal to $\bar{H}$ [HLAG, Prop. 21.4 B]/[Sp, 6.2.10]. Take a Borel subgroup B_α of Z_α containing $\bar{H}$, which extends to a Borel sub group B' of $\bar{G}$. As $\mathrm{R}_u(B_\alpha) \leq \mathrm{R}_u(B')$ [HLAG, 19.5], $\mathrm{R}_u(B_\alpha)$ is directly spanned by $\bar{H}$-invariant 1-dimensional unipotent groups [HLAG, 28.1], and hence $\bar{U}_\alpha \leq \mathrm{R}_u(B_\alpha)$. Let $G^\alpha = \langle \bar{U}_\alpha, \bar{U}_{-\alpha} \rangle$, and consider a morphism $\mathrm{SL}_2(\mathbb{K}) \to G^\alpha$ of algebraic groups [Sp, 8.1.4] such that

$$\begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix} \mapsto x_\alpha(\xi) \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ \xi & 1 \end{pmatrix} \mapsto x_{-\alpha}(\xi).$$

As $p \geq 5$, $\mathrm{Lie}(\mathrm{SL}_2(\mathbb{K}))$ is simple, and hence is sent isomorphically onto $\mathrm{Lie}(G^\alpha)$. On the other hand, $\forall \xi \in \mathbb{K}$,

$$\begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & -\xi \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -2\xi \\ 0 & -1 \end{pmatrix},$$

and hence $\bar{U}_\alpha \not\leq Z$, absurd. Thus, $\bar{H} = Z \trianglelefteq \mathrm{N}_{\bar{G}}(\bar{\mathfrak{h}})$. Then $\mathrm{N}_{\bar{G}}(\bar{\mathfrak{h}}) \leq \bar{N} \leq \mathrm{N}_{\bar{G}}(\bar{\mathfrak{h}})$, and (iii) follows.

Finally, if $Z < \mathrm{C}_{\bar{G}}(\bar{\mathfrak{h}})$, $\mathrm{C}_{\bar{G}}(\bar{\mathfrak{h}}) \leq \mathrm{N}_{\bar{G}}(Z) = \mathrm{N}_{\bar{G}}(\bar{H}) = \bar{N}$, and hence $\mathrm{C}_{\bar{G}}(\bar{\mathfrak{h}}) \setminus Z = \mathrm{C}_{\bar{G}}(\bar{\mathfrak{h}}) \setminus \bar{H} \subseteq \bar{N} \setminus \bar{H}$, absurd. Thus, $\mathrm{C}_{\bar{G}}(\bar{\mathfrak{h}}) = Z = \bar{H}$, and (ii) holds.

2.9. Recall from 2.4 that $\mathrm{d}\tilde{\tau}$ leaves $\bar{\mathfrak{g}}_{\mathbb{k}}$ invariant, permuting a Chevalley basis $\{\mathrm{d}x_\beta(1) | \beta \in R\}$ of $\coprod_{\beta \in R} \bar{\mathfrak{g}}_\beta$; unless $\bar{G}$ is of type A_l with $p | l+1$, $\{\mathrm{d}\alpha^\vee(1) | \alpha \in R^{\mathrm{s}}\}$ forms a basis of $\bar{\mathfrak{h}}$ [H67, 5.4] and $\mathrm{d}\tilde{\tau}$ permutes the $\mathrm{d}\alpha^\vee(1)$'s.

Lemma: *$\mathfrak{g}$ forms a $\mathbb{k}$-form of $\bar{\mathfrak{g}}$: one has a commutative diagram*

$$\begin{array}{ccc} \bar{\mathfrak{g}} & \xrightarrow{\sigma_*} & \bar{\mathfrak{g}} \\ \wr\| & & \wr\| \\ \mathbb{K} \otimes_{\mathbb{k}} \mathfrak{g} & \xrightarrow[?^q \otimes_{\mathbb{k}} \mathfrak{g}]{} & \mathbb{K} \otimes_{\mathbb{k}} \mathfrak{g}. \end{array}$$

Proof: Recall from 2.4 that $\bar{\mathfrak{g}}_{\mathbb{k}}$ is a $\mathrm{d}\tilde{\tau}$-invariant $\mathbb{k}$-form of $\bar{\mathfrak{g}}$. If $\mathrm{d}\tilde{\tau} = \mathrm{id}$, $\mathfrak{g} = \bar{\mathfrak{g}}_{\mathbb{k}}$ and the assertion holds.

Assume next that $\mathrm{ord}(\mathrm{d}\tilde{\tau}) = 2$. As $p \geq 5$, $1 \neq -1$, and $\bar{\mathfrak{g}}_{\mathbb{k}}$ admits an eigenspace decompsition with respect to $\mathrm{d}\tilde{\tau}$: $\bar{\mathfrak{g}}_{\mathbb{k}} = (\bar{\mathfrak{g}}_{\mathbb{k}})_1 \oplus (\bar{\mathfrak{g}}_{\mathbb{k}})_{-1}$ with $(\bar{\mathfrak{g}}_{\mathbb{k}})_{\pm 1} = \{y \in \bar{\mathfrak{g}}_{\mathbb{k}} | (\mathrm{d}\tilde{\tau})y = \pm y\}$, resp. Then $(\mathbb{K} \otimes_{\mathbb{k}} (\bar{\mathfrak{g}}_{\mathbb{k}})_1)^{\sigma_*} = (\bar{\mathfrak{g}}_{\mathbb{k}})_1$ is a $\mathbb{k}$-form of $\mathbb{K} \otimes_{\mathbb{k}} (\bar{\mathfrak{g}}_{\mathbb{k}})_1$. Let $v \in (\bar{\mathfrak{g}}_{\mathbb{k}})_{-1} \setminus 0$. In $\mathbb{K} \otimes_{\mathbb{k}} \mathbb{k}v = \mathbb{K}v$,

$$\begin{aligned} (\mathbb{K}v)^{\sigma_*} &= \{\lambda v | \lambda \in \mathbb{K}, -\lambda^q = \lambda\} \\ &= \mathbb{k}\lambda_0 v \quad \text{with } \lambda_0 \in \mathbb{K} \text{ such that } \lambda_0^{q-1} = -1 \end{aligned}$$

as $x^q + x = x(x^{q-1}+1) = x \prod_{\mu \in \mathbb{k}^\times}(x - \lambda_0\mu) = \prod_{\mu \in \mathbb{k}}(x - \lambda_0\mu)$ in the polynomial algebra $\mathbb{K}[x]$. Thus, $(\mathbb{K}v)^{\sigma_*} = \mathbb{k}\lambda_0 v$ is a $\mathbb{k}$-form of $\mathbb{K} \otimes_{\mathbb{k}} \mathbb{k}v$. It follows that $\mathfrak{g} = \bar{\mathfrak{g}}^{\sigma_*} = \{\mathbb{K} \otimes_{\mathbb{k}} (\bar{\mathfrak{g}}_{\mathbb{k}})_1\}^{\sigma_*} \oplus \{\mathbb{K} \otimes_{\mathbb{k}} (\bar{\mathfrak{g}}_{\mathbb{k}})_{-1}\}^{\sigma_*}$ forms a $\mathbb{k}$-form of $\bar{\mathfrak{g}}$.

Assume finally $\mathrm{ord}(\mathrm{d}\tilde{\tau}) = 3$, and let $\omega \in \mathbb{K}$ be the primitive 3rd root of 1, which exists as $p \geq 5$ again. Then $\bar{\mathfrak{g}}_{\mathbb{k}} = (\bar{\mathfrak{g}}_{\mathbb{k}})_1 \oplus (\bar{\mathfrak{g}}_{\mathbb{k}})_\omega \oplus (\bar{\mathfrak{g}}_{\mathbb{k}})_{\omega^2}$. If $v \in (\bar{\mathfrak{g}}_{\mathbb{k}})_\omega \setminus 0$. one has in $\mathbb{K} \otimes_{\mathbb{k}} \mathbb{k}v = \mathbb{K}v$

$$\begin{aligned} (\mathbb{K}v)^{\sigma_*} &= \{\lambda v | \lambda \in \mathbb{K}, \lambda^q \omega = \lambda\} = \{\lambda v | \lambda \in \mathbb{K}, \lambda^q = \lambda\omega^{-1}\} \\ &= \mathbb{k}\lambda_0 v \quad \text{with } \lambda_0 \in \mathbb{K} \text{ such that } \lambda_0^{q-1} = \omega^{-1} = \omega^2 \end{aligned}$$

as $x^q - \omega^2 x = \prod_{\mu \in \mathbb{k}}(x - \lambda_0\mu)$ in $\mathbb{K}[x]$. Thus, $(\mathbb{K}v)^{\sigma_*} = \mathbb{k}\lambda_0 v$ gives a $\mathbb{k}$-form of $\mathbb{K} \otimes_{\mathbb{k}} \mathbb{k}v$. Likewise, if $v \in (\bar{\mathfrak{g}}_{\mathbb{k}})_{\omega^2} \setminus 0$. It follows that $\mathfrak{g} = \bar{\mathfrak{g}}^{\sigma_*} = \{\mathbb{K} \otimes_{\mathbb{k}} (\bar{\mathfrak{g}}_{\mathbb{k}})_1\}^{\sigma_*} \oplus \{\mathbb{K} \otimes_{\mathbb{k}} (\bar{\mathfrak{g}}_{\mathbb{k}})_\omega\}^{\sigma_*} \oplus \{\mathbb{K} \otimes_{\mathbb{k}} (\bar{\mathfrak{g}}_{\mathbb{k}})_{\omega^2}\}^{\sigma_*}$ forms a $\mathbb{k}$-form of $\bar{\mathfrak{g}}$.

2.10. Unless $p | l+1$ in type A_l, under our restriction $p \geq 5$, $\bar{\mathfrak{g}}$ is a simple Lie algebra over $\mathbb{K}$ by [H67, 5.4], so therefore is $\mathfrak{g}$ over $\mathbb{k}$ by 2.9.

If $X_\beta = (\mathrm{d}\beta)(1)$, $\beta \in R$, $(X_\beta | \beta \in R)$ forms a $\mathbb{k}$-linear basis of $(\coprod_{\beta \in R} \bar{\mathfrak{g}}_\beta)^{F_*} = \coprod_{\beta \in R} (\bar{\mathfrak{g}}_{\mathbb{k}})_\beta$, permuted by $\mathrm{d}\tilde{\tau}$ such that $(\mathrm{d}\tilde{\tau})X_\beta = X_{\tau\beta}$ $\forall \beta \in R$. Let Ω be a τ-orbit of length r in R and $\beta \in \Omega$. Let $\sum_{i=1}^r \lambda_i \otimes (\mathrm{d}\tilde{\tau})^{i-1} X_\beta \in \sum_{i=1}^r \mathbb{K} \otimes_{\mathbb{k}} (\mathrm{d}\tilde{\tau})^{i-1} X_\beta = \coprod_{\gamma \in \Omega} \bar{\mathfrak{g}}_\gamma$, $\lambda_i \in \mathbb{K}$ $\forall i$. If $\sum_{i=1}^r \lambda_i \otimes (\mathrm{d}\tilde{\tau})^{i-1} X_\beta = \sigma_*(\sum_{i=1}^r \lambda_i \otimes (\mathrm{d}\tilde{\tau})^{i-1} X_\beta) = \sum_{i=1}^r \lambda_i^q \otimes (\mathrm{d}\tilde{\tau})^i X_\beta$,

$\lambda_2 = \lambda_1^q, \lambda_3 = \lambda_2^q = \lambda_1^{q^2}, \dots, \lambda_r = \lambda_1^{q^{r-1}}, \lambda_1 = \lambda_1^{q^r}$. Thus,

$$\begin{aligned}(\sum_{i=1}^r \mathbb{K} \otimes_{\Bbbk} (\mathrm{d}\tilde{\tau})^{i-1} X_\beta)^{\sigma_*} = \{\sum_{i=1}^r \lambda^{q^{i-1}} \otimes (\mathrm{d}\tilde{\tau})^{i-1} X_\beta = \sum_{i=0}^{r-1} \sigma_*^i(\lambda \otimes X_\beta) | \lambda \in \mathbb{F}_{q^r}\} \\ \simeq \mathbb{F}_{q^r} \quad \text{as } \Bbbk\text{-linear spaces.}\end{aligned}$$

Let $\mu \in \mathbb{F}_{q^r}^\times$ be a generator. Thus, $\deg_{\Bbbk} \mu = r$, and $1, \mu, \dots, \mu^{r-1}$ form a $\Bbbk$-linear basis of $\mathbb{F}_{q^r}$. Then $\{\sum_{i=0}^{r-1} \sigma_*^i(\mu^j \otimes X_\beta) | j \in [0, r[\}$ forms a $\Bbbk$-linear basis of $(\coprod_{\gamma \in \Omega} \bar{\mathfrak{g}}_\gamma)^{\sigma_*}$. If $h \in \bar{\mathfrak{h}}^{\sigma_*}$,

$$\begin{aligned}[h, \sum_{i=0}^{r-1} \sigma_*^i(\mu^j \otimes X_\beta)] &= \sum_{i=0}^{r-1} [\sigma_*^i h, \sigma_*^i(\mu^j \otimes X_\beta)] = \sum_{i=0}^{r-1} \sigma_*^i([h, \mu^j \otimes X_\beta]) \\ &= \sum_{i=0}^{r-1} \sigma_*^i((\mathrm{d}\beta)(h)\mu^j \otimes X_\beta) = \sum_{i=0}^{r-1} (\mathrm{d}\beta)(h)^{q^i} \sigma_*^i(\mu^j \otimes X_\beta).\end{aligned}$$

In case $h = \lambda_0 v$ with $\lambda_0 \in \mathbb{K} \setminus \Bbbk$ as in 2.9, $(\mathrm{d}\beta)(h) = \lambda_0(\mathrm{d}\beta)(v) \notin \Bbbk$ unless $(\mathrm{d}\beta)(v) = 0$.

If $R = \Omega_1 \sqcup \cdots \sqcup \Omega_m$ is the τ-orbit decomposition of R and if $\bar{\mathfrak{g}}_i = \coprod_{\alpha \in \Omega_i} \bar{\mathfrak{g}}_\alpha$, $\bar{\mathfrak{g}} = \bar{\mathfrak{h}} \oplus \coprod_{i=1}^m \bar{\mathfrak{g}}_i$ with each component σ_*-invariant, and hence from 2.9

Proposition: *$\mathfrak{g} = \mathfrak{h} \oplus \coprod_{i=1}^m (\bar{\mathfrak{g}}_i)^{\sigma_*}$ with $\mathfrak{h} = \bar{\mathfrak{h}}^{\sigma_*}$ (resp. $(\bar{\mathfrak{g}}_i)^{\sigma_*}$, $i \in [1, m]$) forming a $\Bbbk$-form of $\bar{\mathfrak{h}}$ (resp. $\bar{\mathfrak{g}}_i$).*

3° Twisting

We will often abbreviate $\mathrm{Ad}(g)$, $g \in \bar{G}$, simply as g. $\forall n \in \bar{N}$, $\sigma_* n = \sigma_* \mathrm{Ad}(n) \in p\mathbf{LA}_{\Bbbk}(\bar{\mathfrak{g}}, \bar{\mathfrak{g}})^\times$ an automorphism of p-Lie algebra over $\Bbbk$. In this section we discuss the structure of p-Lie algebra $\bar{\mathfrak{g}}^{\sigma_* n} = \{x \in \bar{\mathfrak{g}} | \sigma_* n(x) = x\}$ over $\Bbbk$.

3.1. **Lemma:** *$\forall n \in \mathbb{N}$, $\bar{\mathfrak{g}}^{\sigma_* n}$ is conjugate to $\mathfrak{g}$ under $\mathrm{Ad}(\bar{G})$. In particular, $\bar{\mathfrak{g}}^{\sigma_* n}$ gives a $\Bbbk$-form of $\bar{\mathfrak{g}}$.*

Proof: By Lang's theorem [St67, 10.10] write $n = g\sigma(g)^{-1}$ for some $g \in \bar{G}$. As $n = g\sigma g^{-1}\sigma^{-1}$ in $\bar{G}\langle\sigma\rangle$, $n\sigma_* = \mathrm{Ad}(n)\sigma_* = \mathrm{Ad}(g)\sigma_*\mathrm{Ad}(g)^{-1} = g\sigma_* g^{-1}$ in $p\mathbf{LA}_{\Bbbk}(\bar{\mathfrak{g}}, \bar{\mathfrak{g}})^\times$. Then, $\forall x \in \bar{\mathfrak{g}}$, $n\sigma_* x = x$ iff $g\sigma_* g^{-1} x = x$ iff $\sigma_* g^{-1} x = g^{-1} x$ iff $g^{-1} x \in \bar{\mathfrak{g}}^{\sigma_*} = \mathfrak{g}$ iff $x \in g\mathfrak{g}$. Thus, $\bar{\mathfrak{g}}^{\sigma_* n} = g\mathfrak{g}$.

3.2. As $n \in \bar{N}$ permutes the $\bar{H}$-root subgroups of $\bar{G}$, $\sigma_* n$ permutes $\{\bar{\mathfrak{g}}_\alpha | \alpha \in R\}$ in such a way that $\sigma_* n \bar{\mathfrak{g}}_\alpha = \bar{\mathfrak{g}}_{\tau n \alpha}$ with $n\alpha = \alpha(n^{-1}?)$. Let $\mathcal{O}_1, \dots, \mathcal{O}_r$ be the τn-orbits on R. Set $\bar{\mathfrak{g}}_i = \coprod_{\alpha \mathcal{O}_i} \bar{\mathfrak{g}}_\alpha$ $\forall i \in [1, r]$, which is invariant under $\sigma_* n$.

Proposition: *Let $n \in \bar{N}$.*

(i) $\bar{\mathfrak{g}}^{\sigma_* n} = \bar{\mathfrak{h}}^{\sigma_* n} \oplus \coprod_{i=1}^r (\bar{\mathfrak{g}}_i)^{\sigma_* n}$ forms a $\Bbbk$-form of $\bar{\mathfrak{g}}$ with all $\bar{\mathfrak{h}}^{\sigma_* n}$, $(\bar{\mathfrak{g}}_i)^{\sigma_* n}$, $i \in [1, r]$, forming $\Bbbk$-forms of $\bar{\mathfrak{h}}$, $\bar{\mathfrak{g}}_i$, resp.

(ii) The $(\bar{\mathfrak{g}}_i)^{\sigma_* n}$, $i \in [1, r]$, are all pairwise nonisomorphic irreducible $\bar{\mathfrak{h}}^{\sigma_* n}$-modules.

(iii) $\forall i,j\in[1,r]$, $[(\bar{\mathfrak{g}}_i)^{\sigma_*n},(\bar{\mathfrak{g}}_j)^{\sigma_*n}]=[\bar{\mathfrak{g}}_i,\bar{\mathfrak{g}}_j]^{\sigma_*n}$, $[\bar{\mathfrak{h}}^{\sigma_*n},(\bar{\mathfrak{g}}_j)^{\sigma_*n}]=[\bar{\mathfrak{h}},\bar{\mathfrak{g}}_j]^{\sigma_*n}$. *In particular,* $\bar{\mathfrak{h}}^{\sigma_*n}=(\bar{\mathfrak{g}}^{\sigma_*n})_0(\bar{\mathfrak{h}}^{\sigma_*n})=\{x\in\bar{\mathfrak{g}}^{\sigma_*n}|\,\mathrm{ad}(h)^{\dim\bar{\mathfrak{g}}}x=0\ \forall h\in\bar{\mathfrak{h}}^{\sigma_*n}\}=\{x\in\bar{\mathfrak{g}}^{\sigma_*n}|[h,x]=0\ \forall h\in\bar{\mathfrak{h}}^{\sigma_*n}\}$*, and hence* $\bar{\mathfrak{h}}^{\sigma_*n}$ *is a Cartan subalgebra of* $\bar{\mathfrak{g}}^{\sigma_*n}$.

(iv) Unless $p|l+1$ *in type* A_l, $\mathfrak{g}^{\sigma_*n}$ *is a simple Lie algebra over* $\Bbbk$.

Proof: (i) follows from 2.10 by 3.1. Then (iv) follows from the fact [H67, 5.4] that $\bar{\mathfrak{g}}$ is simple over $\mathbb{K}$.

(ii) and (iii) By (i)

$$\begin{aligned}[(\bar{\mathfrak{g}}_i)^{\sigma_*n},(\bar{\mathfrak{g}}_j)^{\sigma_*n}]\le[\bar{\mathfrak{g}}_i,\bar{\mathfrak{g}}_j]^{\sigma_*n}&=[\mathbb{K}\otimes_\Bbbk(\bar{\mathfrak{g}}_i)^{\sigma_*n},\mathbb{K}\otimes_\Bbbk(\bar{\mathfrak{g}}_j)^{\sigma_*n}]^{\sigma_*n}\\&=\{\mathbb{K}\otimes_\Bbbk[(\bar{\mathfrak{g}}_i)^{\sigma_*n},(\bar{\mathfrak{g}}_j)^{\sigma_*n}]\}^{\sigma_*n}\le[(\bar{\mathfrak{g}}_i)^{\sigma_*n},(\bar{\mathfrak{g}}_j)^{\sigma_*n}],\end{aligned}$$

and hence $[(\bar{\mathfrak{g}}_i)^{\sigma_*n},(\bar{\mathfrak{g}}_j)^{\sigma_*n}]=[\bar{\mathfrak{g}}_i,\bar{\mathfrak{g}}_j]^{\sigma_*n}$. Likewise, $[\bar{\mathfrak{h}}^{\sigma_*n},(\bar{\mathfrak{g}}_i)^{\sigma_*n}]=[\bar{\mathfrak{h}},\bar{\mathfrak{g}}_i]^{\sigma_*n}$. As $[\bar{\mathfrak{h}},\bar{\mathfrak{g}}_i]=\bar{\mathfrak{g}}_i$ under the standing hypothesis $p\ge 5$, $\bar{\mathfrak{h}}^{\sigma_*n}=(\bar{\mathfrak{g}}^{\sigma_*n})_0(\bar{\mathfrak{h}}^{\sigma_*n})$.

Let M be a nonzero $\bar{\mathfrak{h}}^{\sigma_*n}$-submodule of $(\bar{\mathfrak{g}}_i)^{\sigma_*n}$. Then $\mathbb{K}\otimes_\Bbbk M$ is a nonzero $\bar{\mathfrak{g}}$-submodule of $\bar{\mathfrak{g}}_i$-module by (i) again, and hence $\bar{\mathfrak{g}}_\beta\subseteq\mathbb{K}\otimes_\Bbbk M$ for some $\beta\in\mathcal{O}_i$. As $\mathbb{K}\otimes_\Bbbk M$ is invariant under $\sigma_*n={?}^q\otimes_\Bbbk \mathrm{d}\tilde{\tau}\mathrm{Ad}(n)$, one must have $\mathbb{K}\otimes_\Bbbk M=\bar{\mathfrak{g}}_i$. Then $M=(\bar{\mathfrak{g}}_i)^{\sigma_*n}$, and hence $(\bar{\mathfrak{g}}_i)^{\sigma_*n}$ is irreducible over $\bar{\mathfrak{h}}^{\sigma_*n}$. Likewise, an isomorphism $f:(\bar{\mathfrak{g}}_i)^{\sigma_*n}\to(\bar{\mathfrak{g}}_j)^{\sigma_*n}$ of $\bar{\mathfrak{h}}^{\sigma_*n}$-modules yields an isomorphism $\mathbb{K}\otimes_\Bbbk f:\bar{\mathfrak{g}}_i\to\bar{\mathfrak{g}}_j$ of $\bar{\mathfrak{h}}$-modules, and hence $i=j$.

4° **Maximal tori**

A maximal torus of G is by definition $\bar{T}^\sigma=\{t\in\bar{T}|\sigma t=t\}$ for a σ-invariant maximal torus $\bar{T}$ of $\bar{G}$. We will first recall the definition of a toral subalgebra of a p-Lie algebra, abbreviated as a TSA, or simply called a torus. We then determine the maximal TSA's of $\mathfrak{g}$, and show that they are all $\bar{G}$-conjugate to some $\bar{\mathfrak{h}}^{\sigma_*n}$, $n\in\bar{N}$, in $\bar{\mathfrak{g}}^{\sigma_*n}$, and exhaust the Cartan subalgebras of $\mathfrak{g}$. We thus obtain a Cartan decomposition of $\mathfrak{g}$ with respect to any Cartan subalgebra of $\mathfrak{g}$.

4.1. Let $\mathfrak{x}$ be a p-Lie algebra over a field $\mathbb{F}$ of characteristic $p>0$ with a p-power map $?^{[p]}$. We say $x\in\mathfrak{x}$ is p-nilpotent (resp. p-semisimple) iff $x^{[p]^r}=0$ for some $r\in\mathbb{N}$ (resp. $x\in\sum_{i\in\mathbb{N}^+}\mathbb{F}x^{[p]^i}$) [SF, 2.1.5 (resp. 2.3.3)]. Let $f:\mathfrak{x}\to\mathfrak{gl}(V)$ is a p-representation, i.e., a homomorphism of p-Lie algebras, of $\mathfrak{x}$ in a finite dimensional $\mathbb{F}$-linear space V. From [SF, 2.3.3.vi] one has that

(1) $f(x)$ is semisimple (resp. nilpotent) in the classical sense
for x p-semisimple (resp. p-nilpotent), and conversely if f is an imbedding.

A toral subalgebra, abbreviated as TSA, or simply a torus, of $\mathfrak{x}$ is an abelian p-subalgebra Y of X consisting entirely of p-semisimple elements [SF, 2.4.1], equivalently, iff $\bar{\mathbb{F}}\otimes_\mathbb{F} Y$ contains no nonzero p-nilpotents, $\bar{\mathbb{F}}$ denoting the algebraic closure of $\mathbb{F}$ [W, 4.5.2]. Recall from [W, 4.5.18] that

(2) a maximal TSA remains so under base field extension.

In case $\mathfrak{x}$ is the Lie algebra of a connected algebraic group X over $\mathbb{K}$ one has from [BS, 1.3] that

(3) $x \in \mathfrak{x}$ is p-semisimple (resp. p-nilpotent)
iff x is contained in the Lie algebra of a torus (resp. a unipotent subgroup) of X.

For if T is a torus of X, $y^{[p]} = y\ \forall y \in \mathrm{Lie}(T)$ [J, I.7.8]. If V is a unipotent subgroup of X, $\mathrm{Lie}(V)$ is nilpotent, and hence "if" holds. The converse is more subtle; the functor Lie is in general not compatible with taking an intersection [Mi, 10.14]. In [BS] they call $x \in X$ semisimple (resp. nilpotent) iff x is contained in the Lie algebra of a torus (resp. a unipotent subgroup) of X and show in [BS, 1.3] that

(4) each $x \in \mathfrak{x}$ admits a unique decomposition $x = x_s + x_n$ with x_s semisimple, x_n nilpotent and $[x_s, x_n] = 0$, and that if $\phi : X \to \mathrm{GL}_n(\mathbb{K})$ is a morphism of algebraic groups, $\mathrm{d}\phi(x) = \mathrm{d}\phi(x_s) + \mathrm{d}\phi(x_n)$ gives the classical Jordan decomposition of $\mathrm{d}\phi(x)$.

As the 2nd part of (4) follows from the existence of such a decomposition in the 1st part, one obtains the unicity of the decomposition in the 1st part by taking ϕ to be an imbedding. Likewise, "only if" of (3) follows from the existence of the decomposition in the 1st part by (1). Now, to see the existence, if M ia a Borel subgroup of X, $X = \cup_{g \in X} \mathrm{Ad}(g)\mathrm{Lie}(M)$ by [SGA3, XIV, Th. 4.11], and hence we may assume that X is solvable. Let $x \in \mathfrak{x}$. We argue by induction on $\dim X$. Write $X = T\mathrm{R}_u(X)$ with T a maximal torus and $\mathrm{R}_u(X)$ the unipotent radical of X [Bo, 10.6]/[HLAG, 19.3]. There is a 1-dimensional central subgroup Y of $R_u(X)$ normalized by X [Bo, 10.4]. Let $\pi : X \to X/Y$ be the quotient, and let $\bar{\mathfrak{x}} = \mathrm{Lie}(X/Y) = \mathfrak{x}/\mathfrak{y}$ with $\mathfrak{y} = \mathrm{Lie}(Y)$. By induction one has $\mathrm{d}\pi(x) = \bar{a} + \bar{b}$ with $\bar{a}$ semisimple, $\bar{b}$ nilpotent in $\bar{\mathfrak{x}}$ and $[\bar{a}, \bar{b}] = 0$. As $\mathrm{R}_u(X/Y) = \pi(\mathrm{R}_u(X))$ [Bo, 10.6] and as a torus of X/Y lifts to a torus of X [Bo, 11.14, 8.3]/[HLAG, 21.3.C, 16.2], one can write $x = a + b$ with a semisimple, b nilpotent in $\mathfrak{x}$, $b \in \mathrm{Lie}(\mathrm{R}_u(X))$, $\mathrm{d}\pi(a) = \bar{a}$ and $\mathrm{d}\pi(b) = \bar{b}$. Let T' be a maximal torus of X with $a \in T'$. Under $\mathrm{Ad}(T')$ write $\mathrm{Lie}(\mathrm{R}_u(X)) = \mathfrak{r} \oplus \mathfrak{y}$, and let $b = b_1 + b_2$ with $b_1 \in \mathfrak{r}$ and $b_2 \in \mathfrak{y}$. Then

$$
\begin{aligned}
\mathfrak{y} \ni [a, b] &\quad \text{as } Y \trianglelefteq X \\
= [a, b_1] + [a, b_2] &\quad \text{with } [a, b_1] \in \mathfrak{r} \text{ while } [a, b_2] \in \mathfrak{y},
\end{aligned}
$$

and hence $[a, b_1] = 0$. If $[a, b_2] = 0$, we are done. Thus, assume that $[a, b_2] \neq 0$. One has only to check show that $a + b_2$ is semisimple. Write Y as a T'-root group: $Y = \{y_\gamma(\xi) | \xi \in \mathbb{K}\}$ for some isomorphism $y_\gamma : \mathbb{G}_a \to Y$ of algebraic groups and character $\gamma : T' \to \mathrm{GL}_1(\mathbb{K})$ such that $t y_\gamma(\xi) t^{-1} = y_\gamma(\gamma(t)\xi)\ \forall t \in T',\ \forall \xi \in \mathbb{K}$. If $y = (\mathrm{d}y_\gamma)(1)$,

$$
\begin{aligned}
\mathrm{Ad}(y_\gamma(\xi))a - a &= \sum_{i \in \mathbb{N}} \xi^i \frac{y^i}{i!} \bullet a - a \quad \text{[J, II.1.19.6]} \\
&= (a + \xi[y, a] + \xi^2 \frac{y^2}{2!} \bullet a + \dots) - a \quad \text{[HLAG, 10.4]} \\
&= \xi[y, a] \quad \text{as } \frac{y^i}{i!} a \in \mathrm{Lie}(T'Y)_{i\gamma} \text{ [J, II.1.19.5]},
\end{aligned}
$$

and hence $a + [a, -\xi y] = \mathrm{Ad}(y_\gamma(\xi))a$ is semisimple. As Y is 1-dimensional, there is $\xi \in \mathbb{K}$ such that $b_2 = -\xi y$.

Back to the general $\mathfrak{x}$ we let $\mathcal{T}(\mathfrak{x})$ denote the set of maximal TSA's of $\mathfrak{x}$. In particular, we let $\mathcal{T}(\bar{\mathfrak{g}})^{\sigma_*}$ dentote set of maximal TSA's of $\bar{\mathfrak{g}}$ stabilized by σ_*. For groups we let $\mathcal{T}(\bar{G})$ denote the set of maximal tori of $\bar{G}$, and $\mathcal{T}(\bar{G})^\sigma$ the set of σ-invariant maximal tori of $\bar{G}$. A maximal tous of G is by definition $\bar{T}^\sigma$ for some $\bar{T} \in \mathcal{T}(\bar{G})^\sigma$, the collection of all maximal tori is denoted $\mathcal{T}(G)$. By (3) one has from [H67, 13,6] a bijection

$$\mathcal{T}(\bar{G}) \to \mathcal{T}(\bar{\mathfrak{g}}) \quad \text{via} \quad T \mapsto \mathrm{Lie}(T), \tag{5}$$

which is $\bar{G}$-equivariant with respect to the conjugation on $\bar{G}$ and Ad on $\bar{\mathfrak{g}}$. As $\bar{\mathfrak{h}} \in \mathcal{T}(\bar{\mathfrak{g}})$ by 2.8,

$$\mathcal{T}(\bar{\mathfrak{g}}) = \{\mathrm{Ad}(g)\bar{\mathfrak{h}} | g \in \bar{G}\}. \tag{6}$$

4.2. We will show

Theorem: *(i) There is a G-equivariant bijection between $\mathcal{T}(G)$ and $\mathcal{T}(\mathfrak{g})$.*

(ii) $\forall \mathfrak{t} \in \mathcal{T}(\mathfrak{g})$, there is $n \in \bar{N}$ such that $(\mathfrak{g}, \mathfrak{t})$ is $\bar{G}$-conjugate to $(\bar{\mathfrak{g}}^{\sigma_* n}, \bar{\mathfrak{h}}^{\sigma_* n})$.

(iii) The G-conjugacy classes of $\mathcal{T}(\mathfrak{g})$ are in bijective correspondence with the W-conjugacy classes in $W\sigma$.

(iv) $\mathcal{T}(\mathfrak{g})$ coincides with the set of Cartan subalgebras of $\mathfrak{g}$.

4.3. In 4.2.(iii) by the G-conjugacy classes we mean those under the $\mathrm{Ad}(G)$-action on $\mathfrak{g}$ defined in 2.5, and by the W-conjugacy classes in $W\sigma$ we mean those such that $w\sigma$ and $w'\sigma$, $w, w' \in W$, are W-conjugate iff $w\sigma = yw'\sigma y^{-1}$ for some $y \in W$ in $W\langle\sigma\rangle$. The proof of 4.2 will rely on the lemmas to follow.

Lemma: *(i) $\forall \mathfrak{t} \in \mathcal{T}(\mathfrak{g})$, $\mathbb{K} \otimes_{\Bbbk} \mathfrak{t} \in \mathcal{T}(\bar{\mathfrak{g}})^{\sigma_*}$.*

(ii) $\forall \bar{\mathfrak{t}} \in \mathcal{T}(\bar{\mathfrak{g}})^{\sigma_*}$, $(\mathfrak{g}, \bar{\mathfrak{t}}^{\sigma_*})$ is $\bar{G}$-conjugate to some $(\bar{\mathfrak{g}}^{\sigma_* n}, \bar{\mathfrak{h}}^{\sigma_* n})$, $n \in \bar{N}$.

(iii) There is a bijection $\mathcal{T}(\bar{\mathfrak{g}})^{\sigma_*} \to \mathcal{T}(\mathfrak{g})$ via $\bar{\mathfrak{t}} \mapsto \bar{\mathfrak{t}}^{\sigma_*}$ with inverse $\mathfrak{t} \mapsto \mathbb{K}\mathfrak{t}$ the $\mathbb{K}$-span of $\mathfrak{t}$.

Proof: (i) is immediate from 4.1.2 and 2.9.

(ii) By 4.1.6 write $\bar{\mathfrak{t}} = g\bar{\mathfrak{h}}$. Then

$$\begin{aligned} g\bar{\mathfrak{h}} = \bar{\mathfrak{t}} = \sigma_*\bar{\mathfrak{t}} &= \sigma_* g\bar{\mathfrak{h}} \\ &= \sigma(g)\sigma_*\bar{\mathfrak{h}} \quad \text{by 2.5} \\ &= \sigma(g)\bar{\mathfrak{h}}, \end{aligned}$$

and hence 2.8 yields that

$$g^{-1}\sigma(g) \in \bar{N}. \tag{1}$$

Now let $y \in \bar{\mathfrak{t}}$ and write $y = gx$, $x \in \bar{\mathfrak{h}}$. As $\sigma\bar{H} = \bar{H}$, $\sigma^{-1}(g^{-1}\sigma(g)) \in \bar{N}$. Letting $n = \sigma^{-1}(g^{-1}\sigma(g))$, $\sigma_* gx = gx$ iff $\sigma(g)\sigma_* x = gx$ by 2.5 again iff $\sigma(n)\sigma_* x = x$ iff $\sigma_* nx = x$

iff $x \in \bar{\mathfrak{h}}^{\sigma_* n}$, and hence $\bar{\mathfrak{t}}^{\sigma_*} = g\bar{\mathfrak{h}}^{\sigma_* n}$. As $\bar{\mathfrak{g}} = g\bar{\mathfrak{g}}$, the same argument yields that $\mathfrak{g} = \bar{\mathfrak{g}}^{\sigma_*} = g\bar{\mathfrak{g}}^{\sigma_* n}$, and hence

(2)
$$\begin{array}{ccc} \mathfrak{g} & \xleftarrow[\sim]{\mathrm{Ad}(g)} & \bar{\mathfrak{g}}^{\sigma_* n} \\ \uparrow & & \uparrow \\ \bar{\mathfrak{t}}^{\sigma_*} & \overset{\cdots\cdots}{\underset{\sim}{\leftarrow}} & \bar{\mathfrak{h}}^{\sigma_* n}. \end{array}$$

(iii) now follows from (i) and (ii) as $\bar{\mathfrak{h}}^{\sigma_* n}$ is a CSA, and hence also a maximal TSA of $\bar{\mathfrak{g}}^{\sigma_* n}$ by 3.2.

4.4. Recall next from [S, 2.6] that, $\forall T \in \mathcal{T}(G)$,

(1)
$$\mathrm{C}_{\bar{G}}(T)^0 \in \mathcal{T}(\bar{G})^\sigma.$$

For write $T = \bar{T}^\sigma$ for some $\bar{T} \in \mathcal{T}(\bar{G})^\sigma$. Then

$$\begin{aligned} \bar{T} &\leq \mathrm{C}_{\bar{G}}(T)^0 \leq \mathrm{C}_{\bar{G}}(T \cap O^{p'}(G)) \\ &\leq \bar{T} \quad \text{by [S, loc. cit.].} \end{aligned}$$

This is key to all that follows.

Lemma: *(i) There is a bijection $\mathcal{T}(G) \to \mathcal{T}(\bar{G})^\sigma$ via $T \mapsto \mathrm{C}_{\bar{G}}(T)^0$ with inverse $\bar{T} \mapsto \bar{T}^\sigma$.*

(ii) The $\bar{G}$-equivariant bijection 4.1.5 restricts to a G-equivariant bijection $\mathcal{T}(\bar{G})^\sigma \to \mathcal{T}(\bar{\mathfrak{g}})^{\sigma_*}$.

Proof: (i) is immediate from (1).

(ii) Let $\bar{T} \in \mathcal{T}(\bar{G})^\sigma$, and write $\bar{T} = g\bar{H}g^{-1}$, $g \in \bar{G}$. Then $g\bar{H}g^{-1} = \bar{T} = \sigma(\bar{T}) = \sigma(g\bar{H}g^{-1}) = \sigma(g)\bar{H}\sigma(g)^{-1}$, and hence $g^{-1}\sigma(g) \in \bar{N}$. Putting $n = g^{-1}\sigma(g)$ and $\bar{\mathfrak{t}} = \mathrm{Lie}(\bar{T})$ yields

$$\begin{aligned} \sigma_*\bar{\mathfrak{t}} &= \sigma_* g\bar{\mathfrak{h}} = \sigma(g)\sigma_*\bar{\mathfrak{h}} \quad \text{by 2.5} \\ &= \sigma(g)\bar{\mathfrak{h}} = gn\bar{\mathfrak{h}} = g\bar{\mathfrak{h}} = \bar{\mathfrak{t}}, \end{aligned}$$

and hence $\bar{\mathfrak{t}} \in \mathcal{T}(\bar{\mathfrak{g}})^{\sigma_*}$.

On the other hand, let $\bar{\mathfrak{t}}' \in \mathcal{T}(\bar{\mathfrak{g}})^{\sigma_*}$. Write $\bar{\mathfrak{t}}' = y\bar{\mathfrak{h}}$ for some $y \in \bar{G}$ by 4.1.6. Then $y^{-1}\sigma(y) \in \bar{N}$ by 4.3.1. Thus, letting $z = y^{-1}\sigma(y)$ yields that $\sigma(y\bar{H}y^{-1}) = \sigma(y)\sigma(\bar{H})\sigma(y)^{-1} = yz\bar{H}z^{-1}y^{-1} = y\bar{H}y^{-1}$, and hence $y\bar{H}y^{-1} \in \mathcal{T}(\bar{G})^\sigma$ with $\mathrm{Lie}(y\bar{H}y^{-1}) = \mathrm{Ad}(y)\bar{\mathfrak{h}} = \bar{\mathfrak{t}}'$.

4.5. We are now ready to show 4.2.

(i) now follows from composing the bijections 4.4.(i), (ii) and 4.3.(iii)

(1)
$$\begin{array}{ccccccc} \mathcal{T}(G) & \xrightarrow{\sim} & \mathcal{T}(\bar{G})^\sigma & \xrightarrow{\sim} & \mathcal{T}(\bar{\mathfrak{g}})^{\sigma_*} & \xrightarrow{\sim} & \mathcal{T}(\mathfrak{g}) \\ T & \longmapsto & \mathrm{C}_{\bar{G}}(T)^0 & \longmapsto & \mathrm{Lie}(\mathrm{C}_{\bar{G}}(T)^0) & \longmapsto & \mathrm{Lie}(\mathrm{C}_{\bar{G}}(T)^0)^{\sigma_*}, \end{array}$$

which are all G-equivariant.

(ii) Given $\mathfrak{t} \in \mathcal{T}(\mathfrak{g})$, write $\mathfrak{t} = \bar{\mathfrak{t}}^{\sigma_*}$ for some $\bar{\mathfrak{t}} \in \mathcal{T}(\bar{\mathfrak{g}})^{\sigma_*}$ by 4.3.(iii), and $\bar{\mathfrak{t}} = g\bar{\mathfrak{h}}$ for some $g \in \bar{G}$ as in 4.3.(ii). With $n = \sigma^{-1}(g^{-1}\sigma(g)) \in \bar{N}$ one has

$$\begin{array}{ccc} \mathfrak{g} & \xleftarrow[\sim]{\mathrm{Ad}(g)} & \bar{\mathfrak{g}}^{\sigma_* n} \\ \uparrow & & \uparrow \\ \bar{\mathfrak{t}}^{\sigma_*} & \xleftarrow[\sim]{\quad\cdots\quad} & \bar{\mathfrak{h}}^{\sigma_* n}. \end{array}$$

(iv) If $\mathfrak{t} \in \mathcal{T}(\mathfrak{g})$, $(\mathfrak{g}, \mathfrak{t})$ is $\bar{G}$-conjugate to some $(\bar{\mathfrak{g}}^{\sigma_* n}, \bar{\mathfrak{h}}^{\sigma_* n})$, $n \in \bar{N}$, by (ii), and $\bar{\mathfrak{h}}^{\sigma_* n}$ is a CSA of $\bar{\mathfrak{g}}^{\sigma_* n}$ by 3.2, so therefore is $\mathfrak{t}$.

On the other hand, if X is a CSA of $\mathfrak{g}$,

$$\begin{aligned} X &= \mathrm{C}_{\mathfrak{g}}(\mathfrak{t}') \quad \text{for some } \mathfrak{t}' \in \mathcal{T}(\mathfrak{g}) \text{ by [W, 4.5.17]} \\ &\subseteq \mathfrak{g}_0(\mathfrak{t}') = \mathfrak{t}' \quad \text{as } \mathfrak{t}' \text{ is a CSA of } \mathfrak{g} \text{ by above} \\ &\subseteq \mathrm{C}_{\mathfrak{g}}(\mathfrak{t}'), \end{aligned}$$

and hence $X = \mathfrak{t}' \in \mathcal{T}(\mathfrak{g})$.

(iii) Let $M = \bar{G}/\bar{N}$. Thus, $M \simeq \mathcal{T}(\bar{G})$ via $g\bar{N} \mapsto g\bar{H}g^{-1}$. As $\bar{H}$ is σ-stable, so is $\bar{N}$, and hence σ acts on M to yield bijections

$$(2) \qquad \begin{array}{ccccc} M^\sigma & \xrightarrow{\sim} & \mathcal{T}(\bar{G})^\sigma & \xrightarrow[4.4.(i)]{\sim} & \mathcal{T}(G) \\ g\bar{N} & \longmapsto & g\bar{H}g^{-1} & \longmapsto & (g\bar{H}g^{-1})^\sigma. \end{array}$$

Consider the left action of $\bar{G}$ on M, which is σ-equivariant: $\sigma(gm) = \sigma(g)\sigma(m)\ \forall g \in \bar{G}$ $\forall m \in M$. As the stabilizer of $\bar{N} \in M$ under the left multiplication by $\bar{G}$ is $\bar{N}$ itself and as $\bar{N}/\bar{N}^0 = \bar{N}/\bar{H} \simeq W$, an application of [SS, I.2.7, the end of I.2.5] yields a bijection

$$(3) \qquad G\backslash M^\sigma \simeq W/\sim \quad \text{via} \quad Gg\bar{N} \mapsto [g^{-1}\sigma(g)],$$

where the equivalence $\sim$ on W is defined such that $w \sim w'$ iff $w = yw'\sigma(y)^{-1}$ for some $y \in W$, equivalently, $w\sigma(y) = yw'$ for some $y \in W$ iff $w\sigma y\sigma^{-1} = yw'$ in $W\langle\sigma\rangle$ for some $y \in W$ iff $y^{-1}w\sigma y = w'\sigma$ in $W\langle\sigma\rangle$ for some $y \in W$. As (2) is G-equivariant, the G-conjugacy classes of $\mathcal{T}(G)$ corresponds bijectively to the W-conjugacy classes in $W\sigma$. Altogether, one has a commutative diagram

$$(4) \qquad \begin{array}{cccccc} \mathrm{Lie}(\mathrm{C}_{\bar{G}}((g\bar{H}g^{-1})^\sigma)^0)^{\sigma_*} & \mathcal{T}(\mathfrak{g}) & \longrightarrow & G\backslash\mathcal{T}(\mathfrak{g}) & & \\ \uparrow & \uparrow{\scriptstyle\sim} & & \uparrow{\scriptstyle\sim} & & \\ (g\bar{H}g^{-1})^\sigma & \mathcal{T}(G) & \longrightarrow & G\backslash\mathcal{T}(G) & & \\ \uparrow & \uparrow{\scriptstyle\sim} & & \uparrow{\scriptstyle\sim} & & \\ g\bar{H}g^{-1} & \mathcal{T}(\bar{G})^\sigma & \longrightarrow & G\backslash\mathcal{T}(\bar{G})^\sigma & & \\ \uparrow & \uparrow{\scriptstyle\sim} & & \uparrow{\scriptstyle\sim} & & \\ g\bar{N} & (\bar{G}/\bar{N})^\sigma & \longrightarrow & G\backslash(\bar{G}/\bar{N})^\sigma & & \\ \Big\downarrow & \downarrow & & \downarrow{\scriptstyle\sim} & & \\ & \bar{N}/\bar{H} = W & \longrightarrow & W/\sim & \xrightarrow{\qquad\sim\qquad} & W\backslash W\sigma \\ \downarrow & & & & & \\ g^{-1}\sigma(g)\bar{H} & & \longmapsto & [g^{-1}\sigma(g)] & \longmapsto & [g^{-1}\sigma(g)\sigma] = [g^{-1}\sigma g\sigma^{-1}\sigma] = [g^{-1}\sigma g]. \end{array}$$

5° Subgroups and subalgebras containing maximal tori

In this section we show how, under additional field restrictions, to assign a subalgebra of $\mathfrak{g}$ to each subgroup of G containing a maximal torus. We extend the bijection 4.2.(i) to a bijection from the collection $\mathcal{S}(G)$ of subgroups of G generated by maximal tori onto the collection $\mathcal{S}(\mathfrak{g})$ of subalgebras of $\mathfrak{g}$ containing maximal tori, i.e., maximal TSA's. Throughout the section we make a further restriction that $q \geq 13$, so that the results of [S] apply.

5.1. Let X be a subgroup of G containing a maximal torus T of G. Let $X^0 = \langle T^X \rangle$ be the normal closure of T in X. Recall from [S, Th. 12.2] that X^0 is independent of the choice of the maximal torus contained in X, the proof of which requires the classification of finite simple groups. Thus, $\mathcal{S}(G)$ consists precisely of those X containing a maximal torus with $X = X^0$.

Let also $\mathcal{S}(\bar{G})$ be the collection of closed connected σ-invariant subgroups of $\bar{G}$ containing a maximal torus of $\bar{G}$. From [S, Th. 12.1] one has a bijection $\mathcal{S}(\bar{G}) \to \mathcal{S}(G)$ via $\bar{Y} \mapsto \bar{Y}^\sigma$. Thus, for $X \in \mathcal{S}(G)$ there is a unique $\bar{X} \in \mathcal{S}(\bar{G})$ such that $\bar{X}^\sigma = X$. It turns out that $\mathrm{Lie}(\bar{X})$ is σ_*-invariant. We set $\mathcal{L}(X) = \mathrm{Lie}(\bar{X})^{\sigma_*}$, and call it the Lie algebra associated to X. We will show

Theorem: $X \mapsto \mathcal{L}(X)$ *gives a bijection from* $\mathcal{S}(G)$ *onto* $\mathcal{S}(\mathfrak{g})$.

5.2. We first need

Lemma: *Let $\bar{Y}$ be a closed connected subgroup of $\bar{G}$ containing $\bar{H}$, and let $R(\bar{Y}) = \{\alpha \in R | \bar{U}_\alpha \leq \bar{Y}\}$. Let $\bar{\mathfrak{y}}$ be a subalgebra of $\bar{\mathfrak{g}}$ containing $\bar{\mathfrak{h}}$, and let $R(\bar{\mathfrak{y}}) = \{\alpha \in R | \bar{\mathfrak{g}}_\alpha \leq \bar{\mathfrak{y}}\}$.*

(i) $R(\bar{Y}) = R(\mathrm{Lie}(\bar{Y}))$.

(ii) $R(\bar{\mathfrak{y}}) = R(\mathrm{Lie}(\langle \bar{H}, \bar{U}_\alpha | \alpha \in R(\bar{\mathfrak{y}})\rangle))$.

Proof: Recall from [S, 2.5] that $\bar{Y} = \mathrm{R}_u(\bar{Y}) \rtimes \bar{L}$ with $\bar{L}$ reductive containing $\bar{H}$ and $\mathrm{R}_u(\bar{Y}) = \prod_{\alpha \in R_1} \bar{U}_\alpha$ for some $R_1 \subseteq R$. Let $R_2 = R(\bar{L})$, so $R_1 \sqcup R_2 \subseteq R(\bar{Y})$. On the other hand, $\forall \alpha \in R(\bar{Y})$, $\bar{\mathfrak{g}}_\alpha \leq \mathrm{Lie}(\bar{Y}) = \mathrm{Lie}(\mathrm{R}_u(\bar{Y})) \oplus \mathrm{Lie}(\bar{L}) = (\coprod_{\alpha \in R_1} \bar{\mathfrak{g}}_\alpha) \oplus \bar{\mathfrak{h}} \oplus (\coprod_{\alpha \in R_2} \bar{\mathfrak{g}}_\alpha)$, and hence $\alpha \in R(\mathrm{Lie}(\bar{Y})) = R_1 \sqcup R_2$. Thus,

$$(1) \quad R(\bar{Y}) = R_1 \sqcup R_2 = R(\mathrm{Lie}(\bar{Y})), \quad \bar{Y} = \langle \bar{H}, \bar{U}_\alpha | \alpha \in R(\bar{Y})\rangle, \quad \mathrm{Lie}(\bar{Y}) = \bar{\mathfrak{h}} \oplus \coprod_{\alpha \in R(\bar{Y})} \bar{\mathfrak{g}}_\alpha,$$

and (i) holds.

In (ii), by definition LHS $\subseteq$ RHS. Let $\alpha, \beta \in R$ with $\alpha + \beta \in R$. If $(\mathbb{Z}\alpha + \beta) \cap R = \{-a\alpha + \beta, \ldots, \beta, \ldots, b\alpha + \beta\}$, $a, b \in \mathbb{N}$, is the α-string through β [HLART, 8.4], $[X_\alpha, X_\beta] = \pm(a+1)X_{\alpha+\beta}$ [HLART, Th. 25.2], where $X_\gamma = \mathrm{d}x_\gamma(1)$ $\forall \gamma \in R$. Under the standing characteristic restriction, $X_{\alpha+\beta} \in [X_\alpha, X_\beta]$ as $(\mathbb{Z}\alpha + \mathbb{Z}\beta) \cap R$ forms a root system of rank 2. Thus, $R(\bar{\mathfrak{y}})$ is closed. Write $R(\bar{\mathfrak{y}}) = R_1' \sqcup R_2'$ such that $R_1' \cap -R_1' = \emptyset$ and $R_2' = -R_2'$. Then R_2' forms a closed subsystem of R. Let W_2 be the Weyl group of R_2', and let $\bar{G}_1 = \langle \bar{U}_\alpha | \alpha \in R_1' \rangle$, $\bar{G}_2 = \langle \bar{H}, \bar{U}_\alpha | \alpha \in R_2' \rangle$. Then

$$\bar{G}_1 = \prod_{\alpha \in R_1'} \bar{U}_\alpha \text{ is unipotent.} \tag{2}$$

For let $\alpha, \beta \in R_1'$ with $\alpha \neq \beta$. By the Chevalley commutator relation [Sp, 8.2.3] one has

$$[\bar{U}_\alpha, \bar{U}_\beta] \subseteq \prod_{i,j>0} \bar{U}_{i\alpha + j\beta}.$$

If $\alpha + \beta \notin R$, $\bar{U}_\alpha$ and $\bar{U}_\beta$ commute. Suppose now that $\alpha + \beta \in R$. Then $(\mathbb{Z}\alpha + \mathbb{Z}\beta) \cap R$ forms a root system of type A_2, B_2, or G_2. By inspectioon one has

$$(\mathbb{N}^+\alpha + \mathbb{N}^+\beta) \cap R = \begin{cases} \{\alpha+\beta\} & \text{in type } \mathrm{A}_2, \\ \{\alpha+\beta, \alpha+2\beta\} & \text{in type } \mathrm{B}_2 \text{ if } \alpha \text{ is long}, \\ \{\alpha+\beta\} & \text{in type } \mathrm{B}_2 \text{ if } \alpha \text{ is short}, \\ \{\alpha+\beta, \alpha+2\beta, \alpha+3\beta, 2\alpha+3\beta\} & \text{in type } \mathrm{G}_2 \text{ if } \alpha \text{ is long}, \\ \{\alpha+\beta, 2\alpha+\beta, 3\alpha+\beta, 3\alpha+2\beta\} & \text{in type } \mathrm{B}_2 \text{ if } \alpha \text{ is short}. \end{cases}$$

In all cases $(\mathbb{N}^+\alpha + \mathbb{N}^+\beta) \cap R \subseteq R_1'$ as $R(\bar{\mathfrak{h}})$ is closed, and hence $\bar{G}_1 = \prod_{\alpha \in R_1'} \bar{U}_\alpha$ by the commutator formula. As R_1' extends to a positive system of R [BLA, Prop. VI.1.7.22], $\bar{G}_1$ is unipotent. On the other hand, $\bar{G}_2$ is reductive. As $R(\bar{\mathfrak{h}})$ is closed again, $(\mathbb{N}R_1' + \mathbb{N}R_2') \cap R(\mathfrak{h}) \subseteq R_1'$, and hence $\bar{G}_2$ normalize $\bar{G}_1$. Thus,

$$\begin{aligned} \langle \bar{H}, \bar{U}_\alpha | \alpha \in R(\bar{\mathfrak{h}}) \rangle &= \bar{G}_1 \bar{G}_2 \\ &= \bar{G}_1 \rtimes \bar{G}_2 \quad \text{as } R_1' \cap R_2' = \emptyset. \end{aligned}$$

If $\bar{U}_\beta \leq \langle \bar{H}, \bar{U}_\alpha | \alpha \in R(\bar{\mathfrak{h}}) \rangle$,

$$\begin{aligned} \beta &\in R(\mathrm{Lie}(G_1 \rtimes G_2)) \quad \text{by (1)} \\ &= R_1' \cup R_2' = R(\bar{\mathfrak{h}}), \end{aligned}$$

and (ii) holds.

5.3. We now check

Lemma: $\forall \bar{X} \in \mathcal{S}(\bar{G})$, $\mathrm{Lie}(\bar{X})$ *is* σ_**-invariant.*

Proof: As $\bar{X}$ is connected, all maximal tori are conjugate under $\bar{X}$ [HLAG, Cor. 21.3A], and hence $\bar{X}$ contains a σ-invariant maximal torus of $\bar{X}$ [St67, 10.10]. Thus, let $\bar{T} \in \mathcal{T}(\bar{G})^\sigma$ with $\bar{T} \leq \bar{X}$. Write $\bar{T} = g\bar{H}g^{-1}$ for some $g \in \bar{G}$. As in 4.4 one has $g^{-1}\sigma(g) \in \bar{N}$. Let $n = g^{-1}\sigma(g)$. Then $\sigma(g^{-1}\bar{X}g) = \sigma(g^{-1})\sigma(\bar{X})\sigma(g) = n^{-1}g^{-1}\bar{X}gn$, and hence

$$n\sigma(g^{-1}\bar{X}g)n^{-1} = g^{-1}\bar{X}g. \tag{1}$$

Now let $\bar{Y} = g^{-1}\bar{X}g$, and write $\bar{Y} = \mathrm{R}_u(\bar{Y}) \rtimes \bar{L}$ as in 5.2 with $\mathrm{R}_u(\bar{Y}) = \prod_{\alpha \in R_1} \bar{U}_\alpha$ and $\bar{L} = \langle \bar{H}, \bar{U}_\alpha | \alpha \in R_2 \rangle$. Thus, $R_1 \cap (-R_1) = \emptyset$, $R_2 = -R_2$, and $\bar{Y} = \langle \bar{H}, \bar{U}_\alpha | \alpha \in R_1 \sqcup R_2 \rangle$.

Then $\mathrm{Lie}(\bar{Y}) = \bar{\mathfrak{h}} \oplus \coprod_{\alpha\in R_1\sqcup R_2} \bar{\mathfrak{g}}_\alpha$ by 5.2.1, and hence

$$\begin{aligned} n\sigma_*\mathrm{Lie}(\bar{Y}) &= \bar{\mathfrak{h}} \oplus \coprod_{\alpha\in R_1\sqcup R_2} \bar{\mathfrak{g}}_{n\tau\alpha} \\ &= \mathrm{Lie}(\bar{Y}) \quad \text{as } \bar{U}_{n\tau\alpha} = n\sigma(\bar{U}_\alpha)n^{-1} \subseteq \bar{Y}\ \forall\alpha\in R_1\cup R_2 \text{ by (1).} \end{aligned}$$

Then

$$\begin{aligned} \sigma_*\mathrm{Lie}(\bar{X}) &= \sigma_* g\mathrm{Lie}(\bar{Y}) = \sigma(g)\sigma_*\mathrm{Lie}(\bar{Y}) \quad \text{by 2.5} \\ &= gn\sigma_*\mathrm{Lie}(\bar{Y}) = g\mathrm{Lie}(\bar{Y}) = \mathrm{Lie}(\bar{X}). \end{aligned}$$

5.4. Let $\mathcal{S}(\bar{\mathfrak{g}})$ be the collection of σ_*-invariant subalgebras of $\bar{\mathfrak{g}}$ containing a σ_*-invariant maximal TSA of $\bar{\mathfrak{g}}$.

Let now $\bar{\mathfrak{x}} \in \mathcal{S}(\bar{\mathfrak{g}})$ and $\bar{\mathfrak{t}} \in \mathcal{T}(\bar{\mathfrak{g}})^{\sigma_*}$ with $\bar{\mathfrak{x}} \geq \bar{\mathfrak{t}}$. As in 4.3 write $\bar{\mathfrak{t}} = g\bar{\mathfrak{h}}$ for some $g \in \bar{G}$, and let $n = \sigma^{-1}(g^{-1})g \in \bar{N}$. Then

$$\begin{aligned} g^{-1}\bar{\mathfrak{x}} &= g^{-1}\sigma_*\bar{\mathfrak{x}} = \sigma_*\sigma^{-1}(g^{-1})\bar{\mathfrak{x}} \quad \text{by 2.5} \\ &= \sigma_* n g^{-1}\bar{\mathfrak{x}}, \end{aligned}$$

and hence $g^{-1}\bar{\mathfrak{x}}$ is σ_*n-invariant. Write $g^{-1}\bar{\mathfrak{x}} = \bar{\mathfrak{h}} \oplus \coprod_{\alpha\in R'} \bar{\mathfrak{g}}_\alpha$ with $R' = R(g^{-1}\bar{\mathfrak{x}})$. As $\bar{\mathfrak{h}}$ is σ_*n-invariant, $\bar{\mathfrak{h}} \oplus \coprod_{\alpha\in R'} \bar{\mathfrak{g}}_\alpha = g^{-1}\bar{\mathfrak{x}} = \sigma_*n(g^{-1}\bar{\mathfrak{x}}) = \bar{\mathfrak{h}} \oplus \coprod_{\alpha\in R'} \bar{\mathfrak{g}}_{\tau n\alpha}$, and hence

(1) $\qquad R'$ is τn-invariant.

We now let $\bar{Y} = \langle \bar{H}, \bar{U}_\alpha | \alpha \in R'\rangle$ and $\bar{X} = g\bar{Y}g^{-1}$. As $\sigma(g\bar{H}g^{-1}) = \sigma(g)\bar{H}\sigma(g)^{-1} = g\sigma(n)\sigma(\bar{H})\sigma(n)^{-1}g^{-1} = g\bar{H}g^{-1}$,

(2) $\qquad g\bar{H}g^{-1} \in \mathcal{T}(\bar{G})^\sigma$.

Also, $\forall\alpha\in R'$,

$$\sigma(g\bar{U}_\alpha g^{-1}) = \sigma(g)\sigma(\bar{U}_\alpha)\sigma(g)^{-1} = g\sigma(n)\sigma(\bar{U}_\alpha)\sigma(n)^{-1}g^{-1} = g\sigma(\bar{U}_{n\alpha})g^{-1} = g\bar{U}_{\tau n\alpha}g^{-1}.$$

As R' is τn-invariant, $\bar{X} \in \mathcal{S}(\bar{G})$.

We check next that $\bar{X}$ is independent of the choice of $\bar{\mathfrak{t}} \in \mathcal{T}(\bar{\mathfrak{g}})^{\sigma_*}$ and $g \in \bar{G}$. Let $\bar{\mathfrak{t}}' \in \mathcal{T}(\bar{\mathfrak{g}})^{\sigma_*}$ with $\bar{\mathfrak{t}}' \leq \bar{\mathfrak{x}}$, and write $\bar{\mathfrak{t}}' = a\bar{\mathfrak{h}}$, $a \in \bar{G}$. Then $\bar{\mathfrak{h}} = g^{-1}a\bar{\mathfrak{h}}$, and hence $g^{-1}a \in \bar{N}$ by 2.8. Let $z = g^{-1}a$ and $n' = \sigma^{-1}(a^{-1})a \in \bar{N}$. Then $a^{-1}\bar{\mathfrak{x}} = \bar{\mathfrak{h}} \oplus \coprod_{\alpha\in R''} \bar{\mathfrak{g}}_\alpha$ with $R'' = R(a^{-1}\bar{\mathfrak{x}})$ σ_*n'-invariant as in (1). One has that $a\bar{H}a^{-1} = gg^{-1}a\bar{H}(g^{-1}a)^{-1}g^{-1} = g\bar{H}g^{-1}$ as $g^{-1}a \in \bar{N}$. Also, $g^{-1}\bar{\mathfrak{x}} = g^{-1}aa^{-1}\bar{\mathfrak{x}} = g^{-1}a(\bar{\mathfrak{h}} \oplus \coprod_{\alpha\in R''} \bar{\mathfrak{g}}_\alpha) = z(\bar{\mathfrak{h}} \oplus \coprod_{\alpha\in R''} \bar{\mathfrak{g}}_\alpha) = \bar{\mathfrak{h}} \oplus \coprod_{\alpha\in R''} \bar{\mathfrak{g}}_{z\alpha}$, and hence $zR'' = R'$. Thus, $\forall\alpha\in R''$,

$$a\bar{U}_\alpha a^{-1} = gg^{-1}a\bar{U}_\alpha(g^{-1}a)^{-1}g^{-1} = gz\bar{U}_\alpha z^{-1}g^{-1} = g\bar{U}_{z\alpha}g^{-1} \leq \bar{X},$$

and hence $a\langle \bar{H}, \bar{U}_\alpha | \alpha \in R''\rangle a^{-1} = \bar{X}$, as desired.

We now set $\bar{X} = G(\bar{\mathfrak{x}})$.

Proposition: *The map $\mathcal{S}(\bar{G}) \to \mathcal{S}(\bar{\mathfrak{g}})$ via $\bar{X} \mapsto \mathrm{Lie}(\bar{X})$ is a bijection with inverse $\bar{\mathfrak{x}} \mapsto G(\bar{\mathfrak{x}})$.*

Proof: Given $\bar{X} \in \mathcal{S}(\bar{G})$, let $\bar{T} \in \mathcal{T}(\bar{G})^\sigma$ with $\bar{T} \leq \bar{X}$, $g \in \bar{G}$ with $\bar{T} = g\bar{H}g^{-1}$, and let $\bar{Y} = g^{-1}\bar{X}g$. Then $\bar{Y} = \langle \bar{H}, \bar{U}_\alpha | \alpha \in R(\bar{Y})\rangle = \langle \bar{H}, \bar{U}_\alpha | \alpha \in R(\mathrm{Lie}(\bar{Y}))\rangle$ by 5.2.1. As $\mathrm{Lie}(\bar{Y}) = g^{-1}(\mathrm{Lie}(\bar{X}))$, $G(\mathrm{Lie}(\bar{X})) = g\bar{Y}g^{-1} = \bar{X}$ by definition.

Given $\bar{\mathfrak{x}} \in \mathcal{S}(\bar{\mathfrak{g}})$, let $\bar{\mathfrak{t}} \in \mathcal{T}(\bar{\mathfrak{g}})^{\sigma_*}$ with $\bar{\mathfrak{t}} \leq \bar{\mathfrak{x}}$, and $a \in \bar{G}$ with $\bar{\mathfrak{t}} = a\bar{\mathfrak{h}}$. Then $G(\bar{\mathfrak{x}}) = a\langle \bar{H}, \bar{U}_\alpha | \alpha \in R(a^{-1}\bar{\mathfrak{x}})\rangle a^{-1}$ by definition. Then

$$\begin{aligned}
\mathrm{Lie}(G(\bar{\mathfrak{x}})) &= a\{\mathrm{Lie}(\langle \bar{H}, \bar{U}_\alpha | \alpha \in R(a^{-1}\bar{\mathfrak{x}})\rangle)\} \\
&= a(\bar{\mathfrak{h}} \oplus \coprod_{\alpha \in R(a^{-1}\bar{\mathfrak{x}})} \bar{\mathfrak{g}}_\alpha) \quad \text{by 5.2.1 again} \\
&= a(a^{-1}\bar{\mathfrak{x}}) = \bar{\mathfrak{x}}.
\end{aligned}$$

5.5. Finally, one has from 4.3 a commutative diagram

$$\begin{array}{ccc}
\mathcal{S}(\bar{\mathfrak{g}}) & \xrightarrow{?^{\sigma_*}} & \mathcal{S}(\mathfrak{g}) \\
\uparrow & & \uparrow \\
\mathcal{T}(\bar{\mathfrak{g}})^{\sigma_*} & \xrightarrow[\sim]{} & \mathcal{T}(\mathfrak{g}).
\end{array}$$

Proposition: $?^{\sigma_*} : \mathcal{S}(\bar{\mathfrak{g}}) \to \mathcal{S}(\mathfrak{g})$ *is a bijection with inverse* $\mathfrak{x} \mapsto \mathbb{K}\mathfrak{x}$ *the* $\mathbb{K}$*-span of* $\mathfrak{x}$*.*

Proof: Let $\mathfrak{x} \in \mathcal{S}(\mathfrak{g})$. By 4.3 one has $\mathcal{S}(\bar{\mathfrak{g}}) \ni \mathbb{K}\mathfrak{x} \simeq \mathbb{K} \otimes_{\Bbbk} \mathfrak{x}$, and hence $(\mathbb{K}\mathfrak{x})^{\sigma_*} = \mathfrak{x}$.

On the other hand, let $\bar{\mathfrak{x}} \in \mathcal{S}(\bar{\mathfrak{g}})$ and let $\bar{\mathfrak{t}} \in \mathcal{T}(\bar{\mathfrak{g}})^{\sigma_*}$ with $\bar{\mathfrak{x}} \geq \bar{\mathfrak{t}}$. As in 5.4 write $\bar{\mathfrak{t}} = g\bar{\mathfrak{h}}$ for some $g \in \bar{G}$, and let $n = \sigma^{-1}(g^{-1})g \in \bar{N}$. Then $g^{-1}\bar{\mathfrak{x}} = \bar{\mathfrak{h}} \oplus \coprod_{\alpha \in R(g^{-1}\bar{\mathfrak{x}})} \bar{\mathfrak{g}}_\alpha$ with $R(g^{-1}\bar{\mathfrak{x}})$ τn-invariant as in 5.4.1. $\forall x \in \bar{\mathfrak{x}}$, $x \in \bar{\mathfrak{x}}^{\sigma_*}$ iff $g^{-1}x = g^{-1}\sigma_* x = \sigma_*\sigma^{-1}(g^{-1})x = \sigma_* n g^{-1}x$ by 2.5 iff $g^{-1}x \in (g^{-1}\bar{\mathfrak{x}})^{\sigma_* n}$. Thus, $\bar{\mathfrak{x}}^{\sigma_*} = g(g^{-1}\bar{\mathfrak{x}})^{\sigma_* n} = g\{\bar{\mathfrak{h}}^{\sigma_* n} \oplus (\coprod_{\alpha \in R(g^{-1}\bar{\mathfrak{x}})} \bar{\mathfrak{g}}_\alpha)^{\sigma_* n}\}$. Then

$$\begin{aligned}
\mathbb{K}(\bar{\mathfrak{x}}^{\sigma_*}) &= g(\bar{\mathfrak{h}} \oplus \coprod_{\alpha \in R(g^{-1}\bar{\mathfrak{x}})} \bar{\mathfrak{g}}_\alpha) \quad \text{by 3.2} \\
&= \bar{\mathfrak{x}}.
\end{aligned}$$

5.6. The bijectivity of $\mathcal{L}$ now follows from a commutative diagram

$$\begin{array}{ccc}
\mathcal{S}(\bar{G}) & \xrightarrow[\sim]{?^{\sigma}} & \mathcal{S}(G) \\
{\scriptstyle \mathrm{Lie}}\downarrow{\scriptstyle \sim} & & \downarrow{\scriptstyle \mathcal{L}} \\
\mathcal{S}(\bar{\mathfrak{g}}) & \xrightarrow[?^{\sigma_*}]{\sim} & \mathcal{S}(\mathfrak{g})
\end{array}$$

with bijections from [S, Th. 12.2], 5.4, and 5.5.

5.7. As a consequence of 5.1 certain questions concerning generation by subsets of $\mathcal{S}(G)$ can be settled by passing to $\mathcal{S}(\mathfrak{g})$.

Corollary: *Let* $X_1, \ldots, X_r \in \mathcal{S}(G)$*.*

(i) $\mathcal{L}(\langle X_1,\dots,X_r\rangle) = \langle \mathcal{L}(X_1),\dots,\mathcal{L}(X_r)\rangle$.

(ii) $\langle X_1,\dots,X_r\rangle = G(\langle \mathcal{L}(X_1),\dots,\mathcal{L}(X_r)\rangle)$.

(iii) $\mathcal{L}(X_1\cap\cdots\cap X_r) = \mathcal{L}(X_1)\cap\cdots\cap\mathcal{L}(X_r)$ *if* $X_1\cap\cdots\cap X_r\in\mathcal{S}(G)$.

(iv) $X_1\cap\cdots\cap X_r = G(\mathcal{L}(X_1)\cap\cdots\cap\mathcal{L}(X_r))$ *if* $X_1\cap\cdots\cap X_r\in\mathcal{S}(G)$.

Proof: (ii) (resp. (iv)) follows from (i) (resp. (iii)) by 5.1.

(i) As $\mathcal{L}:\mathcal{S}(G)\to\mathcal{S}(\mathfrak{g})$ and $G:\mathcal{S}(\mathfrak{g})\to\mathcal{S}(G)$ both preserve inclusions,

$$\mathcal{L}(\langle X_1,\dots,X_r\rangle) \geq \langle \mathcal{L}(X_1),\dots,\mathcal{L}(X_r)\rangle, \tag{1}$$
$$\mathcal{L}(X_1\cap\cdots\cap X_r) \leq \mathcal{L}(X_1)\cap\cdots\cap\mathcal{L}(X_r) \quad \text{if } X_1\cap\cdots\cap X_r\in\mathcal{S}(G). \tag{2}$$

(i) As $X_i = G(\mathcal{L}(X_i)) \leq G(\langle\mathcal{L}(X_1),\dots,\mathcal{L}(X_r)\rangle)$ $\forall i\in[1,r]$, $\langle X_1,\dots,X_r\rangle \leq G(\langle\mathcal{L}(X_1),\dots,\mathcal{L}(X_r)\rangle)$, and hence

$$\mathcal{L}(\langle X_1,\dots,X_r\rangle) \leq \mathcal{L}(G(\langle\mathcal{L}(X_1),\dots,\mathcal{L}(X_r)\rangle)) = \langle\mathcal{L}(X_1),\dots,\mathcal{L}(X_r)\rangle.$$

Together with (1), (i) follows.

(iii) As $\bigcap_i X_i\in\mathcal{S}(G)$ by the hypothesis,

$$\mathcal{S}(\mathfrak{g}) \ni \mathcal{L}(\bigcap_i X_i) \leq \bigcap_i \mathcal{L}(X_i), \tag{3}$$

and hence $\bigcap_i \mathcal{L}(X_i)\in\mathcal{S}(\mathfrak{g})$. Then $G(\bigcap_i\mathcal{L}(X_i)) \leq \bigcap_j G(\mathcal{L}(X_j)) = \bigcap_j X_j$, and

$$\begin{aligned}\bigcap_i\mathcal{L}(X_i) &= \mathcal{L}(G(\bigcap_i\mathcal{L}(X_i))) \leq \mathcal{L}(\bigcap_i X_i)\\ &\leq \bigcap_i\mathcal{L}(X_i) \quad \text{by (3)},\end{aligned}$$

and (iii) holds.

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