We show that if a holomorphic $n$ dimensional compact torus action on a compact connected complex manifold of complex dimension $n$ has a fixed point then the manifold is equivariantly biholomorphic to a smooth toric variety.

1. Introduction

We begin by recalling some notions from the theory of toric varieties.

We work in the vector space $\text{Lie}(S^1)^n \cong \mathbb{R}^n$ with the lattice $\text{Hom}(S^1, (S^1)^n) \cong \mathbb{Z}^n$. Here, we identify $\text{Lie}(S^1)$ with $\mathbb{R}$ such that the exponential map $\exp: \mathbb{R} \to S^1$ is $t \mapsto e^{2\pi i t}$.

A unimodular fan is a finite set $\Delta$ of convex polyhedral cones with the following properties.

1. A face of a cone in $\Delta$ is also a cone in $\Delta$.
2. The intersection of two cones in $\Delta$ is a common face.
3. Every cone in $\Delta$ is unimodular, i.e., it has the form $\text{pos}(\lambda_1, \ldots, \lambda_k)$ where $\lambda_1, \ldots, \lambda_k$ is part of a $\mathbb{Z}$-basis of the lattice. Here, $\text{pos}$ denotes the positive span: the set of linear combinations with non-negative coefficients.$^1$

A fan $\Delta$ is complete if the union of the cones in $\Delta$ is all of $\text{Lie}(S^1)^n$.

The theory of toric varieties associates to a unimodular fan $\Delta$ a complex manifold $M_{\Delta}$ with a holomorphic $(\mathbb{C}^*)^n$-action with the following properties.

1. The fixed points in $M_{\Delta}$ are in bijection with the $n$-dimensional cones in $\Delta$.
2. Let $p$ be a fixed point in $M_{\Delta}$. Then the isotropy weights at $p$ are a $\mathbb{Z}$-basis to the lattice $\text{Hom}((\mathbb{C}^*)^n, S^1) \subset (\text{Lie}(S^1)^n)^*$. Moreover, let $\lambda_1, \ldots, \lambda_n$ be the dual basis; then the cone in $\Delta$ that corresponds to $p$ is $\text{pos}(\lambda_1, \ldots, \lambda_n)$.
3. The manifold $M_{\Delta}$ is compact if and only if the fan $\Delta$ is complete.
For the details of the construction and the proof of these properties, we refer the reader to the book [2] by Cox, Little, and Schenck.

In fact, $M_{\Delta}$ is an algebraic variety. Moreover, every smooth complex algebraic variety that is equipped with an algebraic $(\mathbb{C}^*)^n$-action with an open dense free orbit is isomorphic to some $M_{\Delta}$. (The proof of this fact appeared in the book [6] by Kempf, Knudsen, Mumford, and Saint-Donat and in the article [9] by Miyake and Oda and relies on a lemma of Sumihiro [10]; see Corollary 3.1.8 in [2].) Our main theorem is a complex analytic variant of this result:

**Theorem 1.** Let $M$ be a connected complex manifold of complex dimension $n$, equipped with a faithful action of the torus $(S^1)^n$ by biholomorphisms. If $M$ is compact and the action has fixed points, then there exists a unimodular fan $\Delta$ and an $(S^1)^n$-equivariant biholomorphism of $M_{\Delta}$ with $M$.

**Remark 2.**

(1) Our theorem gives a negative answer to a question that was raised by Buchstaber and Panov in [11, Problem 5.23].

Let $M$ be a closed $2n$ dimensional manifold with an $(S^1)^n$-action that is locally standard: every orbit has a neighbourhood that is equivariantly diffeomorphic, up to an automorphism of $(S^1)^n$, to an invariant open subset of $\mathbb{C}^n$ with the standard $(S^1)^n$-action. Also assume that the quotient $M/(S^1)^n$ is diffeomorphic, as a manifold with corners, to a simple convex polytope $P$ in $\mathbb{R}^n$. Such manifolds, introduced in [3] and studied in the toric topology community, are called quasi-toric manifolds.

The question of Buchstaber and Panov is whether there exists a non-toric quasi-toric manifold that admits an $(S^1)^n$-invariant complex structure.

(2) Our theorem strengthens an earlier result of Ishida and Masuda, that if a closed complex manifold of complex dimension $n$ admits an $(S^1)^n$-action, and if its odd-degree cohomology groups vanish, then the Todd genus of the manifold is equal to one. See [5, Theorem 1.1 and Remark 1.2].

(3) It is necessary to assume that the action has fixed points: the complex torus $\mathbb{C}^*/(z \sim 2z)$ has a holomorphic $S^1$-action, induced from multiplication on $\mathbb{C}^*$, but it is not a toric variety.

(4) It is necessary to assume that the manifold is compact: the open unit disc in $\mathbb{C}$ with the natural circle action has a fixed point, but it is not a toric variety: the circle action does not extend to a $\mathbb{C}^*$-action.

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2 A map from $M/(S^1)^n$ to $P$ is a diffeomorphism of manifolds with corners if and only if it is a homeomorphism and, for every real valued function on $P$, the function extends to a smooth function on $\mathbb{R}^n$ if and only if its pullback to $M$ is smooth. For every $k \in \{0, \ldots, n\}$, a diffeomorphism carries the $k$ dimensional orbits in $M$ to the relative interiors of the $k$ dimensional faces of $P$.

3 Davis-Januszkiewicz [3] used the term toric manifold, but this term was already used in the literature to mean a smooth toric variety, so Buchstaber-Panov [11] introduced instead the term quasitoric manifold.
2. The complexified action

Let the torus \((S^1)^n\) act on a complex manifold \(M\) by biholomorphisms. If the manifold \(M\) is compact, then the \((S^1)^n\)-action extends to a \((\mathbb{C}^*)^n\)-action that is holomorphic not only in the sense that each element of \((\mathbb{C}^*)^n\) acts by a biholomorphism but also in the sense that the action map \((\mathbb{C}^*)^n \times M \to M\) is holomorphic. See, e.g., \([4, \text{ Theorem 4.4}]\). For the convenience of the reader, we briefly recall here some of the details of this standard construction.

Let \(\xi_1, \ldots, \xi_n\) be the fundamental vector fields of the \((S^1)^n\)-action with respect to the coordinate one-dimensional subtori. Let \(J: TM \to TM\) be the multiplication by \(\sqrt{-1}\). We claim that the vector fields \(-J\xi_1, \ldots, -J\xi_n\) are holomorphic (in the sense that their flows preserve the complex structure) and commute with each other and with the vector fields \(\xi_i\).

Because the \((S^1)^n\)-action preserves \(J\) and \(\xi_j\), it preserves \(-J\xi_j\), for each \(j\). So the vector fields \(-J\xi_j\) commute with the vector fields \(\xi_i\) that generate this action. Because \(J\) is a complex structure, its Nijenhaus tensor, \(N(Z, W) := 2(JZJW) - J[Z, JW] - J[JZ, W] - [Z, W]\), vanishes. Setting \(Z = \xi_i\) and \(W = \xi_j\), we get that \([J\xi_i, J\xi_j]\) = \(J[\xi_i, J\xi_j] + J[J\xi_i, \xi_j] + [\xi_i, \xi_j]\), and each of the three terms on the right hand side is zero. So the vector fields \(-J\xi_j\) commute with each other. A vector field \(Y\) is holomorphic if and only if \([Y, JW] = J[Y, W]\) for each vector \(W\); see \([4, \text{ Proposition 2.10 in Chapter IX}]\). Set \(Y := -J\xi_i\) and \(W\) arbitrary; because \(JY(= \xi_i)\) is holomorphic, \([JY, JW] = J[JY, W]\); by the vanishing of the Nijenhaus tensor,

\[
\]

so \(Y\) is holomorphic.

If \(M\) is compact, the vector fields \(-J\xi_1, \ldots, -J\xi_n\) are complete, and we get an \(\mathbb{R}^{2n}\)-action, \(\mathbb{R}^{2n} \times M \to M\), via

\[
\left( \sum_{i=1}^{2n} a_i \xi_i, x \right) \mapsto c_x(1),
\]

where \(c_x(r)\) is the integral curve of the vector field \(\sum_{i=1}^n -a_i J\xi_i + a_{n+i}\xi_i\) with \(c_x(0) = x\). This action descends to a \((\mathbb{C}^*)^n\)-action by biholomorphisms that extends the given \((S^1)^n\)-action. Finally, the action map \((\mathbb{C}^*)^n \times M \to M\) is holomorphic, because its differential, which at the point \((z, m)\) is the map \(\mathbb{C}^n \times T_m M \to T_{z,m} M\) that takes \((2\pi r_1 + i\theta_1, \ldots, r_n + i\theta_n, v)\) to \(\sum_{j} -r_j J\xi_j|_{z,m} + \theta_j \xi_j|_{z,m} + z_*v\), is complex linear.

Remark 3. In the next section we will see that if there exists a fixed point then the extended \((\mathbb{C}^*)^n\)-action is faithful. In general, the extended \((\mathbb{C}^*)^n\)-action might not be faithful.

Example 4. Let \((S^1)^n\) act on \(\mathbb{C}^n\) with weights \(\alpha_1, \ldots, \alpha_n\):

\[
g \cdot (z_1, \ldots, z_n) = (g^{\alpha_1}z_1, \ldots, g^{\alpha_n}z_n),
\]
where \( g^{\alpha_i} = g_1^{\alpha_i} \ldots g_n^{\alpha_i} \) for \( g = (g_1, \ldots, g_n) \in (S^1)^n \) and the isotropy weight \( \alpha_i = (\alpha_{i1}, \ldots, \alpha_{in}) \in \mathbb{Z}^n \). Then the complexified action is given by the same formula applied to \( g = (g_1, \ldots, g_n) \in (\mathbb{C}^*)^n \).

3. Structures near fixed points

Let \( M \) be a complex manifold of complex dimension \( n \). Let the torus \( (S^1)^n \) act on \( M \) faithfully by biholomorphisms. Let \( p \) be a point in \( M \) that is fixed by the \( (S^1)^n \)-action. Let \( \alpha_1, \ldots, \alpha_n \) be the isotropy weights at \( p \).

We begin with a local result:

**Lemma 5.** There exists an \( (S^1)^n \)-invariant neighbourhood \( U_p \) of \( p \) in \( M \), an \( (S^1)^n \)-invariant neighbourhood \( \tilde{U}_p \) of the origin in \( T_p M \), and an \( (S^1)^n \)-equivariant biholomorphism \( \varphi_p : U_p \to \tilde{U}_p \) whose differential at \( p \) is the identity map on \( T_p M \).

Here, \( C_{\alpha_i} \) denotes the one dimensional complex vector space \( C \) with the \( (S^1)^n \)-action that is obtained by composing the homomorphism \( (S^1)^n \to S^1 \) that is encoded by the weight \( \alpha_i \) with the standard action of \( S^1 \) on \( C \) by scalar multiplication.

**Proof.** Let \( \varphi : U \to \tilde{U} \subseteq C^n \) be a local holomorphic chart near \( p \) with \( \varphi(p) = 0 \). Identifying \( C^n \) with \( T_p M \) via the differential \( (d\varphi)_p : T_p M \to T_0 C^n \cong C^n \), we get a biholomorphism

\[
\varphi' : U \to \tilde{U} \subseteq T_p M
\]

whose differential at \( p \) is the identity map on \( T_p M \). We want to obtain such a biholomorphism that is also equivariant.

Set

\[
U' := \bigcap_{g \in (S^1)^n} g U.
\]

Clearly, \( U' \) is invariant and contains \( p \). We now show that \( U' \) is open. The complement of \( U' \) is the image of the closed subset \((S^1)^n \times (M \setminus U)\) of \((S^1)^n \times M\) under the action map \((S^1)^n \times M \to M\). Because \((S^1)^n\) is compact, the action map is proper. Being proper means that the preimage of every compact set is compact; when the target space \( M \) is a manifold\(^4\), it implies that the map is closed. Thus, the complement \( M \setminus U' \) is closed, and so \( U' \) is open.

To obtain an equivariant chart, we average \( \varphi' \): let

\[
\tilde{\varphi} := \int_{g \in (S^1)^n} (g \circ \varphi' \circ g^{-1}) \, dg : U' \to T_p M,
\]

\(^4\) In fact, it is enough to assume that the target space is Hausdorff and compactly generated. Compactly generated means that a subset is closed if and only if its intersection with every compact set \( K \) is closed in \( K \); this property holds if the space is locally compact or if the space is metrizable.
where \( dg \) is Haar measure on \((S^1)^n\). The map \( \varphi \) is holomorphic and \((S^1)^n\)-equivariant. Moreover, its differential at \( p \) is the identity map on \( T_pM \). By the implicit function theorem, \( \varphi \) restricts to a biholomorphism from some smaller open neighbourhood \( U'' \) of \( p \) in \( M \) to an open neighbourhood of the origin in \( T_pM \). The restriction of \( \varphi \) to the invariant neighbourhood \( U_p := \bigcap_{g \in (S^1)^n} g \cdot U'' \) of \( p \) in \( M \) satisfies the requirements of the lemma. \( \square \)

**Corollary 6.** There exists an \((S^1)^n\)-equivariant local holomorphic chart \[ \varphi_p : U_p \to \mathbb{D}^n \]

from an invariant open neighbourhood \( U_p \) of \( p \) to a polydisc \( \mathbb{D}^n \) in \( C_{a_1} \oplus \ldots \oplus C_{a_n} \).

**Proof.** By the definition of the isotropy weights, there exists a complex linear \((S^1)^n\)-equivariant isomorphism between the tangent space \( T_pM \) and the representation \( C_{a_1} \oplus \ldots \oplus C_{a_n} \). Corollary 6 then follows from Lemma 5 by restricting the chart to the preimage of a polydisc. \( \square \)

We would like to extend the chart of Corollary 6 to a chart whose image is all of \( C^n \). We can do this when the \((S^1)^n\) extends to a \((C^n)^n\)-action; for example, if the manifold is compact; by “sweeping” by the \((C^n)^n\)-action.

**Lemma 7.** Suppose that the \((S^1)^n\)-action extends to a \((C^n)^n\)-action. Then there exists an invariant open neighbourhood \( V_p \) of \( p \) in \( M \) and an \((S^1)^n\)-equivariant biholomorphism of \( V_p \) with \( C_{a_1} \oplus \ldots \oplus C_{a_n} \).

**Proof.** Let \( \varphi_p : U_p \to \mathbb{D}^n \) be an \((S^1)^n\)-equivariant holomorphic local chart, as in Corollary 6. Because \( \varphi_p \) is \((S^1)^n\)-equivariant and holomorphic, it intertwines the restriction to \( U_p \) of the vector fields that generate the complexified \((C^n)^n\)-action on \( M \) with the restriction to \( \mathbb{D}^n \) of the vector fields that generate the complexified \((C^n)^n\)-action on \( C^n = C_{a_1} \oplus \ldots \oplus C_{a_n} \). This, and the fact that \( \varphi_p \) is a diffeomorphism between \( U_p \) and \( \mathbb{D}^n \), implies that \( \varphi_p \) also intertwines the partial flows on \( U_p \) and on \( \mathbb{D}^n \) that are generated by these vector fields; in particular it intertwines the domains of definition of these partial flows.

For each \( t \in \mathbb{R} \), let \( g_t \) be the element of \((C^n)^n\) that acts on \( C^n \) as scalar multiplication by \( e^{-t} \), and let \( \eta \in \text{Lie}(C^n)^n \) be the generator of the one-parameter subgroup \( t \mapsto g_t \). Because \( e^{-t} \mathbb{D}^n \subset \mathbb{D}^n \) for all \( t \geq 0 \), and because \( \varphi_p \) intertwines the domains of definition of the partial flows on \( U_p \) and on \( \mathbb{D}^n \) that correspond to \( \eta \), we get that \( g_t U_p \subset U_p \) for all \( t \geq 0 \). So, for every \( t \geq 0 \), the domain of definition of the \((S^1)^n\)-equivariant biholomorphism \[ \varphi_p^{(t)} := (g_t)^{-1} \circ \varphi_p \circ g_t : g_{-t} U_p \to e^{-t} \mathbb{D}^n \]
contains \( U_p \). Here, \( g_t : g_{-t} U_p \to U_p \) and \( g_t : e^{-t} \mathbb{D}^n \to \mathbb{D}^n \) are given by the complexified actions on \( M \) and on \( C^n \). By the choice of \( g_t \), the latter map is multiplication by \( e^{-t} \).

Moreover, because \( \varphi_p \) intertwines the partial flows that correspond to \( \eta \) and these partial flows are defined for all \( t \geq 0 \), the restriction to \( U_p \) of \( \varphi_p^{(t)} \) coincides with \( \varphi_p \) for all \( t \geq 0 \). Substituting \( t - s \) instead of \( t \), we get that the maps \( \varphi_p^{(t)} \) and \( \varphi_p^{(s)} \) agree whenever they are
both defined. Thus, all these maps fit together into a map

$$\bigcup_{t \geq 0} \phi_p^{(t)} : V_p \to \mathbb{C}_{\alpha_1} \oplus \ldots \oplus \mathbb{C}_{\alpha_n},$$

where $V_p = \bigcup_{t \geq 0} g_{-t}U_p$. This map is onto, because its image is the union of the sets $e^{tD^n}$ over all $t \geq 0$. The map is one to one, because it is one to one on each $g_{-t}U_p$, and for every two points in the domain there exists a $t \geq 0$ such that the points are both in $g_{-t}U_p$. Because $V_p$ is covered by $(S^1)^n$-invariant open sets $g_{-t}U_p$ on which the map is an $(S^1)^n$-equivariant biholomorphism, we deduce that the map is itself an $(S^1)^n$-equivariant biholomorphism, as required.

$$4. \text{Obtaining a fan}$$

Let $M$ be a complex manifold of complex dimension $n$, let the torus $(S^1)^n$ act on $M$ faithfully by biholomorphisms, and assume that this action extends to a holomorphic $(\mathbb{C}^*)^n$-action. Moreover, assume that the action has at least one fixed point.

In Lemma 7 we assigned to every fixed point $p$ in $M$ an open subset $V_p$ that is biholomorphic to $\mathbb{C}^n$. By assumption, there exists at least one fixed point. So the union of the sets $V_p$ over these fixed points,

$$\bigcup_{p \in M(S^1)^n} V_p,$$

is nonempty. We fix a connected component of this union and denote it $X$.

**Remark 8.** We would like to know that if $M$ connected then the union of the sets $V_p$ is all of $M$. We do not know how to prove this directly; we do not even know if it is always true. We will eventually show that if $M$ is compact and connected then $X$ is compact; so in this case $X$ must coincides with $M$, and the union of the sets $V_p$ is indeed all of $M$.

The connected components of the fixed point sets of the circle subgroups of $(S^1)^n$ are closed complex submanifolds of $X$. If such a submanifold has complex codimension one, then, in analogy with the toric topology literature, we call it a *characteristic submanifold* of $X$ (cf. [8, p. 240]).

Because $X$ is a union of finitely many $V_p$'s and each $V_p$ has only finitely many characteristic submanifolds, there are only finitely many characteristic submanifolds in $X$. Denote them

$$X_1, \ldots, X_m.$$

Let $T_i$ be the subgroup of $T$ that fixes $X_i$. If a compact group acts faithfully on a connected manifold then at every fixed point the linear isotropy representation is faithful. Therefore, the linear isotropy representation of $T_i$ at any point $q$ of $X_i$ is faithful. Because $T_i$ acts holomorphically and fixes $X_i$, we get a faithful representation of $T_i$ on the one dimensional complex space $T_qX/T_qX_i$. This gives an injection $T_i \to S^1$, where $S^1$ acts on $T_qX/T_qX_i$ by scalar multiplication. By continuity, this injection is independent of the
choice of point \( q \) in \( X_i \). Because, by assumption, \( T_i \) contains a circle subgroup of \( T \), this injection is an isomorphism. Let
\[
\lambda_i: S^1 \to T_i \subset (S^1)^n
\]
be the inverse of this isomorphism, composed with the inclusion map into \((S^1)^n\).

We define an abstract simplicial complex:
\[
\Sigma := \left\{ I \subseteq \{1, \ldots, m\} \mid X_I := \bigcap_{i \in I} X_i \neq \emptyset \right\}.
\]
To each simplex \( I \in \Sigma \) we assign the cone
\[
C_I := \text{pos}(\lambda_i | i \in I) := \left\{ \sum_{i \in I} a_i \lambda_i \mid a_i \geq 0 \right\}
\]
in \( \text{Lie}(S^1)^n \).

**Example 9.** Take \( \mathbb{C}^n \) with coordinates \( z_1, \ldots, z_n \). Let \((S^1)^n \) act on it with weights \( \alpha_1, \ldots, \alpha_n \in \text{Hom}((S^1)^n, S^1) \subset (\text{Lie}(S^1)^n)^\ast \). Suppose that the action is faithful; then \( \alpha_1, \ldots, \alpha_n \) are a \( \mathbb{Z} \)-basis of \( \text{Hom}((S^1)^n, S^1) \). The characteristic submanifolds are the coordinate hyperplanes \( \{z_i = 0\} \) for \( i = 1, \ldots, n \). The homomorphisms \( \lambda_1, \ldots, \lambda_n \) are the basis to \( \text{Hom}(S^1, (S^1)^n) \subset \text{Lie}(S^1)^n \) that is dual to \( \alpha_1, \ldots, \alpha_n \).

Recall that a cone in \( \text{Lie}(S^1)^n \) is **unimodular** if it is generated by part of a \( \mathbb{Z} \)-basis of \( \text{Hom}(S^1, (S^1)^n) \).

Returning to our general case –

**Lemma 10.** The cones \( C_I \), for \( I \in \Sigma \), are unimodular.

**Proof.** Let \( I \in \Sigma \). By the definition of \( \Sigma \), this means that the intersection \( \bigcap_{i \in I} X_i \) is nonempty. Let \( q \) be a point in this intersection. Let \( p \) be a fixed point such that \( q \in V_p \). Because \( V_p \) is isomorphic to some \( \mathbb{C}_{\alpha_1} \oplus \ldots \oplus \mathbb{C}_{\alpha_n} \) on which the action is faithful, the lemma follows from Example 9.

Every \( V_p \) contains an open dense free \((\mathbb{C}^\ast)^n \) orbit. For any two \( V_p \)s that are in the connected component \( X \), these orbits coincide. Thus, there exists a unique free \((\mathbb{C}^\ast)^n \) orbit in \( X \), it is open and dense, and it is contained in every \( V_p \) that is contained in \( X \).

Fix a point \( q \) in the free \((\mathbb{C}^\ast)^n \) orbit in \( X \). For any \( \xi \in \text{Lie}(S^1)^n \), consider the curve
\[
c^\xi_q: \mathbb{R} \to X
\]
that is given by
\[
c^\xi_q(r) := \exp(-rJ\xi) \cdot q \quad \text{for } r \in \mathbb{R}
\]
where \( \exp: \text{Lie}(\mathbb{C}^\ast)^n \to (\mathbb{C}^\ast)^n \) is the exponential map and where \( J \) denotes multiplication by \( i \) in \( \text{Lie}(\mathbb{C}^\ast)^n \).
Denote by $C^0_I$ the relative interior of the cone $C_I$. Denote
$$X^0_I = \bigcap_{i \in I} X_i \smallsetminus \bigcap_{j \notin I} X_j.$$ 

Lemma 11. Let $\xi \in \text{Lie}(S^1)^n$ and $I \in \Sigma$. Then $\xi \in C^0_I$ if and only if the curve $c^\xi_q(r)$ converges as $r \to -\infty$ to a point $q'$ in $X^0_I$. Moreover, in this case the limit point $q'$ belongs to $V_p$ for every $p$ such that $V_p \cap X_I \neq \emptyset$.

Proof. Suppose that $\xi \in C^0_I$. By the definition of $\Sigma$, $X_I$ is nonempty. Let $p$ be such that $V_p$ meets $X_I$. Without loss of generality assume that $I = \{1, \ldots, k\}$ and that the characteristic submanifolds that meet $V_p$ are $X_1, \ldots, X_n$. Let $\alpha_1, \ldots, \alpha_n$ denote the isotropy weights at $p$. The assumption that $\xi \in C^0_I$ exactly means that $(\xi, \alpha_i)$ is positive for $i = 1, \ldots, k$ and zero for $i = k + 1, \ldots, n$. Fix an isomorphism $(z_1, \ldots, z_n) : V_p \to \mathbb{C}^n = \mathbb{C}_{\alpha_1} \oplus \cdots \oplus \mathbb{C}_{\alpha_n}$ such that $z_i(q) = 1$ for all $i$. In these coordinates, the curve $c^\xi_q(r)$ is represented as
$$(z_1, \ldots, z_n)(c^\xi_q(r)) = (e^{2\pi i (\xi, \alpha_1)}, \ldots, e^{2\pi i (\xi, \alpha_n)}).$$

As $r$ approaches $-\infty$, the curve in $\mathbb{C}^n$ approaches the point $(0, \ldots, 0, 1, \ldots, 1)$. On the other hand, the coordinates take each intersection $V_p \cap X_I$ to the coordinate hyperplane $\{(z_1, \ldots, z_n) \mid z_i = 0\}$, and they take the intersection $V_p \cap X^0_I$ to the set $\{(z_1, \ldots, z_n) \mid z_i = 0$ if $1 \leq i \leq k\}$. So the curve approaches a point in $V_p \cap X^0_I$, as required.

Now suppose that the curve $c^\xi_q(r)$ converges as $r \to -\infty$ to a point in $X^0_I$. Let $p$ be such that this limit point is contained in $V_p$. As before, without loss of generality assume that $I = \{1, \ldots, k\}$ and that the characteristic submanifolds that meet $V_p$ are exactly $X_1, \ldots, X_n$; fix an isomorphism $(z_1, \ldots, z_n) : V_p \to \mathbb{C}^n = \mathbb{C}_{\alpha_1} \oplus \cdots \oplus \mathbb{C}_{\alpha_n}$ such that $z_i(q) = 1$ for all $i$; the curve $c^\xi_q(r)$ is represented as $(z_1, \ldots, z_n)(c^\xi_q(r)) = (e^{2\pi i (\xi, \alpha_1)}, \ldots, e^{2\pi i (\xi, \alpha_n)})$. Because the curve approaches a limit as $r \to -\infty$, the pairings $(\xi, \alpha_i)$ are nonnegative for all $i = 1, \ldots, n$. Because this limit is in $X^0_I$, the pairings are positive for every $i \in I$ and they vanish for every $i \in \{1, \ldots, n\} \setminus I$. Thus, $\xi \in C^0_I$ as required. $\square$

Corollary 12. 
(1) For every $I, J \in \Sigma$, if $I \neq J$, then $C^0_I \cap C^0_J = \emptyset$.

(2) For every $I, J \in \Sigma$,
$$C_I \cap C_J = C_{I \cap J}.$$

(3) The collection of cones
$$\Delta := \{ C_I \mid I \in \Sigma \}$$
is a fan, that is, every face of every cone in $\Delta$ is itself in $\Delta$, and the intersection of every two cones in $\Delta$ is a common face.

Proof. Part (1) follows from Lemma 11 because the sets $X^0_I$ are disjoint. Part (3) follows from Part (2).

For Part (2), we only need to show the inclusion $C_I \cap C_J \subseteq C_{I \cap J}$, because the opposite inclusion is trivial. Let $\xi \in C_I \cap C_J$. Let $I' \subset I$ and $J' \subset J$ be the subsets such that $\xi \in C^0_{I'}$,
For every $I \in \Sigma$, we described an open subset $\mathbb{C}^d_J$. Then $C^0_I \cap C^0_J \neq \emptyset$. By Part (1), $I' = J'$. Let $L = I' = J'$. Then $L \subset I \cap J$, and $\xi \in C^0_L \subset C_{I \cap J}$. 

**Lemma 13.** For every $I \in \Sigma$, the set $X_I$ is an $(S^1)^n$-invariant smooth closed complex submanifold of $X$ of complex codimension $|I|$, it is connected, and it contains a fixed point.

**Proof.** Fix $I \in \Sigma$.

Because each of the sets $X_i$, for $i \in I$, is closed in $X$, so is the intersection $X_I$ of these sets.

Because $X$ is the union of open subsets $V_p$, and because every intersection $V_p \cap X_I$ is an $(S^1)^n$-invariant complex submanifold of codimension $|I|$ in $V_p$, we deduce that $X_I$ is itself an $(S^1)^n$-invariant complex submanifold of codimension $|I|$ in $X$. It remains to show that $X_I$ is connected and contains a fixed point.

Choose any $\xi \in C^d_J$ (for example, we may take $\xi = \sum_{i \in I} \lambda_i$), and choose any $q$ in the free $(\mathbb{C}^*)^n$ orbit in $X$. By Lemma 11, the curve $c^d_J(r)$ converges as $r \to -\infty$; let $q'$ be its limit. Also by Lemma 11, for every $p$ such that $V_p \cap X_I \neq \emptyset$, the limit point $q'$ belongs to $V_p$. Because $X_I$ is the union over such $p$ of the subsets $V_p \cap X_I$, and because each of these subsets is connected and contains $q'$, the union $X_I$ is connected. Also, every $p$ such that $V_p \cap X_I \neq \emptyset$ belongs to $V_p \cap X_I$; because the set of such $p$s is nonempty, $X_I$ contains a fixed point. 

**Corollary 14.** In the fan $\Delta$, every cone is contained in an $n$ dimensional cone.

**Proof.** Every cone in the fan has the form $C^d_J$ for some $I \in \Sigma$. By Lemma 3, the set $X_I$ contains a fixed point; let $p$ be such a fixed point. Since $V_p$ was chosen as in Lemma 2, by Example 9 there exist exactly $n$ characteristic submanifolds, say, $X_j$ for $j \in J \subset \{1, \ldots, m\}$ with $|J| = n$, that pass through $p$. Then $J \in \Sigma$, and $C^d_J$ is an $n$ dimensional cone in $\Delta$ that contains $C^d_I$. 

5. ISOMORPHISM OF THE SUBSET $X$ WITH A TORIC MANIFOLD

Let $M$ be a complex manifold of complex dimension $n$, let the torus $(S^1)^n$ act on $M$ faithfully by biholomorphisms, and assume that this action extends to a holomorphic $(\mathbb{C}^*)^n$-action. Moreover, assume that the action has at least one fixed point.

In Section 3 we described an open subset $X$ of $M$ and a unimodular fan $\Delta$. Let $M\Delta$ be the toric variety that is associated to the fan $\Delta$.

**Lemma 15.** There exists an $(S^1)^n$-equivariant biholomorphism between $M\Delta$ and $X$.

We recall some properties of the set $X$ and the fan $\Delta$. Let $F$ denote the fixed point set in $X$. For every fixed point $p \in F$, let $\alpha_{p,1}, \ldots, \alpha_{p,n}$ denote the isotropy weights of the torus action at $p$.

1. The set $X$ is the union over $p \in F$ of subsets $V_p$, such that each $V_p$ is an invariant open neighbourhood of $p$ that is equivariantly biholomorphic to the linear representation $\mathbb{C}_{\alpha_{p,1}}, \ldots, \mathbb{C}_{\alpha_{p,n}}$. 

(2) The \( n \)-dimensional cones in \( \Delta \) are in bijection with the fixed point sets \( p \in F \), and the cone corresponding to the fixed point \( p \) is \( \text{pos}(\lambda_{i_1}, \ldots, \lambda_{i_n}) \), where \( \lambda_{i_1}, \ldots, \lambda_{i_n} \) is a basis of \( \text{Lie}(S^1)^n \) that is dual to the basis \( \alpha_{p,1}, \ldots, \alpha_{p,n} \) of \( (\text{Lie}(S^1)^n)^* \).

The toric variety \( M_\Delta \) that is associated to the fan \( \Delta \) has similar properties: it is the union over \( p \in F \) of invariant subsets \( V'_p \), and every \( V'_p \) is equivariantly biholomorphic to \( \mathbb{C}^{\alpha_{p,1}} \oplus \ldots \oplus \mathbb{C}^{\alpha_{p,n}} \).

Lemma 15 follows immediately from these properties of \( X \) and \( M_\Delta \), by the following lemma.

**Lemma 16.** Let \( X \) and \( X' \) be complex manifolds of complex dimension \( n \), equipped with holomorphic \((\mathbb{C}^*)^n\)-actions. Suppose that there exist open dense \((\mathbb{C}^*)^n\) orbits \( O \) in \( X \) and \( O' \) in \( X' \). Suppose that there exist invariant open subsets \( V_p \) in \( X \) and \( V'_p \) in \( X' \), both indexed by \( p \in F \), such that \( X \) is the union of the sets \( V_p \) and \( X' \) is the union of the sets \( V'_p \), and that for each \( p \in F \) there exists an equivariant biholomorphism \( \varphi_p : V_p \to V'_p \). Then \( X \) is equivariantly biholomorphic to \( X' \).

**Proof.** Necessarily, \( O \) is contained in each \( V_p \) and \( O' \) is contained in each \( V'_p \). Fix a point \( q \) in \( O \) and a point \( q' \) in \( O' \). After possibly composing each \( \varphi_p \) by the action of an element of \((\mathbb{C}^*)^n\), we may assume that \( \varphi_p(q) = q' \) for each \( p \in F \). So, for each \( p \) and \( \tilde{p} \in F \), the maps \( \varphi_p \) and \( \varphi_{\tilde{p}} \) coincide at the point \( q \). By equivariance, \( \varphi_p \) and \( \varphi_{\tilde{p}} \) coincide on all of \( O \); by continuity, they coincide on the entire overlap \( V_p \cap V_{\tilde{p}} \). Thus, the \( \varphi_p \) fit together into a map

\[
\varphi = \bigcup_p \varphi_p : X \to X'.
\]

This map is holomorphic, equivariant, and onto. Similarly, the inverses \( \psi_p := \varphi_p^{-1} \) fit together into a map

\[
\psi = \bigcup_p \psi_p : X' \to X.
\]

We have that \( \psi \circ \varphi = \text{id}_X \) and \( \varphi \circ \psi = \text{id}_{X'} \); thus, \( \varphi : X \to X' \) is an equivariant biholomorphism, as required. \( \Box \)

6. The compact case

Let \( M \) be a complex manifold of complex dimension \( n \), with a faithful \((S^1)^n\)-action, with fixed points.

Suppose that \( M \) is compact. In Section \( \Box \) we extended the \((S^1)^n\)-action to a holomorphic \((\mathbb{C}^*)^n\)-action. In Section \( \Box \) we chose an open subset \( X \) of \( M \) of a particular form and we associated to it a fan \( \Delta \).

**Lemma 17.** The fan \( \Delta \) is complete.

We begin by proving a special case:
Lemma 18. Let $M'$ be a complex manifold of complex dimension one, equipped with a faithful holomorphic action of $S^1$ with at least one fixed point. Suppose that $M'$ is compact and connected. Then $M'$ is equivariantly biholomorphic to $\mathbb{C}P^1$ with a standard $\mathbb{C}^*$-action.

Proof. Consider the $S^1$-action on $M'$. Near a fixed point, it is isomorphic to the restriction of either the standard $S^1$-action on $\mathbb{C}$ or the opposite $S^1$-action on $\mathbb{C}$ to an invariant neighbourhood of the origin in $\mathbb{C}$.

Consider the flow that is generated by $-J\xi$, where $\xi$ generates the $S^1$-action. If the $S^1$-action near a fixed point is standard, then the trajectories of this flow converge to the fixed point as their parameter approaches $-\infty$. If the $S^1$-action near a fixed point is opposite from standard, then the trajectories of this flow converge to the fixed point as their parameter approaches $\infty$.

Outside the fixed point set, the action is free. The quotient $M'/S^1$ is a real one-manifold with boundary; its boundary is exactly the image of the fixed point set. Because $M'$ is compact and contains a fixed point, and by the classification of one-manifolds, the quotient $M'/S^1$ must be a closed segment.

The flow on $M'$ that is generated by $-J\xi$ descends to a flow on the interior of $M'/S^1$ that does not have fixed points. For each boundary component, the flow approaches that component either as its parameter approaches $\infty$ or as the parameter approaches $-\infty$. Necessarily, it approaches one boundary component when the parameter approaches $\infty$ and it approaches the other boundary component when the parameter approaches $-\infty$.

The corresponding fan must then be equal to the fan of $\mathbb{C}P^1$, and the manifold is equivariantly biholomorphic to $\mathbb{C}P^1$ by Lemma 16. \hfill \Box

We now return to the setup of Lemma 17: We have a complex manifold $M$ of complex dimension $n$, with a faithful $(S^1)^n$-action, with fixed points. We assume that $M$ is compact. We chose an open subset $X$ of $M$ of a particular form and we associated to it a fan $\Delta$.

Lemma 19. Every $n-1$ dimensional cone in $\Delta$ is a common face of two $n$ dimensional cones in $\Delta$.

Proof. Let $C_i$ be an $n-1$ dimensional cone in $\Delta$, corresponding to the subset $I = \{i_1, \ldots, i_{n-1}\}$ of $\{1, \ldots, m\}$.

Let $T_i$ be the codimension one subtorus of $(S^1)^n$ that is generated by the circles $T_i$ for $i \in I$. By Lemma 13, $X_i$ is a connected complex manifold of dimension one, equipped with a faithful holomorphic action of the circle $(S^1)^n/T_i$ with at least one fixed point. We will now show that $X_i$ is compact, and will deduce Lemma 19 from Lemma 13.

First note that $X_i$ is a connected component of the fixed point set of $T_i$ in $X$. This follows from the facts that $X_i$ is connected (by Lemma 13) and that, for every $V_p$ in $X$, if the intersection $V_p \cap X_i$ is nonempty then it is a connected component of the fixed point set of $T_i$ in $V_p$. Let $N$ denote the connected component of the fixed point set of $T_i$ in $M$ that contains $X_i$. As in any holomorphic torus action on a complex manifold, $N$ is a

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Here, “is” means that there exists a unique manifold-with-boundary structure on $M'/S^1$ such that a function is smooth if and only if its pullback to $M'$ is smooth.
Lemma gives an equivariant biholomorphism $V$. E. Buchstaber and T. E. Panov, \[ \text{(S^1)^n-invariant closed complex submanifold of } M. \] By examining $N$ near a point of $X_f$, we deduce that $N$ has complex dimension one. Because $N$ is closed in $M$ and $M$ is compact, $N$ is compact. By Lemma [13], $N$ is equivariantly biholomorphic to $\mathbb{C} \mathbb{P}^1$ with a standard action of the circle $(S^1)^n/T_f$. In particular, $N$ contains two fixed points; denote them $p'$ and $p''$. At least one of these fixed points is in $X_f$, by Lemma [13]. The intersection $V_{p'} \cap N$, being a $(C^*)^n$-invariant neighbourhood of $p'$ in $N$, must be all of $N \setminus \{p''\}$. Similarly, the intersection $V_{p''} \cap N$, is all of $N \setminus \{p'\}$. Thus, the intersection $V_{p'} \cap V_{p''}$ is nonempty. Because at least one of the sets $V_{p'}$ and $V_{p''}$ is contained in $X$, and because $X$ is a connected component of the union of the sets $V_p$, we deduce that $X$ contains both $V_{p'}$ and $V_{p''}$. Thus, $N$ is entirely contained in $X$, and so $N$ must be equal to $X_f$. Thus, $X$ is equivariantly biholomorphic to $\mathbb{C} \mathbb{P}^1$ with a standard action of the circle $(S^1)^n/T_f$. This implies the result of Lemma [13]. \[ \square \]

We are now ready to prove Lemma [17].

**Proof of Lemma [17].** Let $|\Delta|$ denote the union of the cones in $\Delta$, and let $|\Delta|^{n-2}$ denote the union of the cones in $\Delta$ that have codimension $\geq 2$. The complement $\text{Lie}(S^1)^n \setminus |\Delta|^{n-2}$ is connected, open, and dense in $\text{Lie}(S^1)^n$.

By Lemma [13], the union of the relative interiors of the faces of $\Delta$ of dimension $(n-1)$ and of dimension $n$ is open in $\text{Lie}(S^1)^n$. This union is $|\Delta| \setminus |\Delta|^{n-2}$. Thus, $|\Delta| \setminus |\Delta|^{n-2}$ is also open in $\text{Lie}(S^1)^n \setminus |\Delta|^{n-2}$.

But because $|\Delta|$ is closed in $\text{Lie}(S^1)^n$, we also have that $|\Delta| \setminus |\Delta|^{n-2}$ is closed in $\text{Lie}(S^1)^n \setminus |\Delta|^{n-2}$.

Because $|\Delta| \setminus |\Delta|^{n-2}$ is open and closed in $\text{Lie}(S^1)^n \setminus |\Delta|^{n-2}$ and $\text{Lie}(S^1)^n \setminus |\Delta|^{n-2}$ is connected, we deduce that $|\Delta| \setminus |\Delta|^{n-2}$ is either empty or is equal to all of $\text{Lie}(S^1)^n \setminus |\Delta|^{n-2}$.

Because, by assumption, $M$ has a fixed point, $\Delta$ has at least one $n$ dimensional cone, so $|\Delta| \setminus |\Delta|^{n-2}$ is not empty. So $|\Delta| \setminus |\Delta|^{n-2}$ is equal to all of $\text{Lie}(S^1)^n \setminus |\Delta|^{n-2}$. Taking the closures, we deduce that $|\Delta| = \text{Lie}(S^1)^n$, as required. \[ \square \]

We are now ready to prove our main theorem.

**Proof of Theorem [1].** Lemma [16] gives an equivariant biholomorphism $\varphi: M_\Delta \rightarrow X$.

By Lemma [14], the fan $\Delta$ is complete. This implies that the toric variety $M_\Delta$ is compact. So $X$ must be compact. Because $M$ is Hausdorff and connected, and $X$ is a subset that is both compact and open, $X$ is all of $M$. So $\varphi$ defines an equivariant biholomorphism from $M_\Delta$ to $M$, as required. \[ \square \]

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**References**


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