Minimization problem related with Lyapunov inequality

M. Hashizume

Department of Mathematics, Graduate School of Science, Osaka City University
3-3-138 Sugimoto Sumiyoshi-ku, Osaka-shi, Osaka 558-8585 Japan

Abstract

We consider a minimization problem on bounded smooth domain $\Omega$ in $\mathbb{R}^N$

$$S' := \inf \left\{ \left\| \nabla u \right\|_2^2 \left/ \| u \|_2^2 \right. \left| u \in H^1(\Omega) \setminus \{0\}, \int_\Omega |u|^{2^*-2} u = 0 \right. \right\}.$$  

This minimization problem plays a crucial role related with $L^p$ Lyapunov-type inequalities $(1 \leq p \leq \infty)$ for linear partial differential equations with Neumann boundary conditions (on bounded smooth domains in $\mathbb{R}^N$). In this paper, we prove that existence of the minimizer of $S'$ and $L^p$ Lyapunov-type inequalities in critical case.

Keywords: Minimization problem, Critical, Sign changing, Lyapunov inequalities, Neumann, Neumann boundary value problem

1. Introduction

Let $N \geq 3$ and $\Omega$ be a bounded domain in $\mathbb{R}^N$ with a smooth boundary. We consider the linear elliptic equation

$$\begin{cases}
-\Delta u(x) = a(x)u(x) & \text{in } \Omega \\
\frac{\partial u}{\partial n}(x) = 0 & \text{on } \partial \Omega
\end{cases} \quad (1)$$

where the function $a : \Omega \to \mathbb{R}$ belongs to the set $\Lambda$ defined as

$$\Lambda := \left\{ a \in L^{N/2}(\Omega) \setminus \{0\} \left| \int_\Omega a(x)dx \geq 0 \text{ and } (1) \text{ has nontrivial solutions} \right. \right\}.$$
We define \( \beta_p \) as

\[
\beta_p = \inf \left\{ \| a^+ \|_{L^p(\Omega)} \left| a \in \Lambda \cap L^p(\Omega) \right. \right\}.
\]

The eigenvalues of the eigenvalue problem

\[
\begin{cases}
-\Delta u(x) = \lambda u(x) & \text{in } \Omega \\
\frac{\partial u}{\partial \nu}(x) = 0 & \text{on } \partial \Omega
\end{cases}
\]

belong to \( \Lambda \). Thus \( \Lambda \) is not empty therefore \( \Lambda \) is well defined. Cañada, Montero and Villegas [4] proved that \( \beta_p \) is attained in the case \( N/2 < p \leq \infty \), \( \beta_p = 0 \) and it is not attained in the case \( 1 \leq p < N/2 \). But the case \( p = N/2 \) has not been studied so far. In this paper we prove the case \( p = N/2 \) for \( N \geq 4 \). As result, \( \beta_{N/2} \) is attained and the minimizer \( a(x) \) is represented by the form

\[
a(x) = |u(x)|^{\frac{2}{N-2}}
\]

where \( u(x) \) is solutions of some quasilinear elliptic equation. Timoshin[10] considered similar problem with Dirichlet boundary conditions, that is,

\[
\begin{cases}
-\Delta u(x) = a(x)u(x) & \text{in } \Omega \\
u(x) = 0 & \text{on } \partial \Omega
\end{cases}
\]

\( \tilde{\Lambda} := \{ a \in L^{N/2}(\Omega) \setminus \{0\} \left| (2) \right. \text{has nontrivial solutions} \} \).

\( \tilde{\beta}_p = \inf \left\{ \| a \|_{L^p(\Omega)} \left| a \in \tilde{\Lambda} \cap L^p(\Omega) \right. \right\} \).

About this problem, he proved that \( \tilde{\beta}_p \) is not attained in the case \( p = N/2 \) by using not attainability of Sobolev best constant on the bounded domains. The result is \( \tilde{\beta}_p = S \) is not attained where \( S \) is Sobolev best constant. This result is different from with Neumann boundary conditions.

2. Main Theorem

Theorem 2.1.

Let \( N \geq 4 \), \( \Omega \) be bounded with smooth boundary. Then \( \beta_{N/2} \) is attained. Furthermore \( \beta_{N/2} = S' \) where

\[
S' := \inf \left\{ \frac{\| \nabla u \|_2^2}{\| u \|_2^2} \left| u \in H^1(\Omega) \setminus \{0\}, \int_{\Omega} |u|^{2^*-2}u = 0 \right. \right\}.
\]
From this, the minimizer of $\beta_{N/2}$ is represented that
\[ a(x) = |u(x)|^{\frac{4}{N-2}} \]
where $u(x)$ is a solution of
\[
\begin{cases}
    -\Delta u(x) = |u(x)|^{\frac{4}{N-2}} u(x) & \text{in } \Omega \\
    \frac{\partial u}{\partial v}(x) = 0 & \text{on } \partial\Omega.
\end{cases}
\tag{3}
\]

3. Preliminaries

Lemma 3.1. We have
\[ S' < \frac{S}{2\pi} \]
where $S$ is Sobolev best constant.

Without loss of generality, we may assume that $0 \in \partial\Omega$, and that the mean curvature of $\partial\Omega$ at 0 is strictly positive.

For all $\varepsilon > 0$, $u_{\varepsilon}(x) \in H^1(\Omega)$ is defined by
\[ u_{\varepsilon}(x) := \frac{(N(N-2)\varepsilon^2)^{\frac{N-2}{2}}}{(\varepsilon^2 + |x|^2)^{\frac{N-2}{2}}} = \varepsilon^{\frac{2-N}{2}} U\left(\frac{x}{\varepsilon}\right) \]
where
\[ U(x) = \frac{(N(N-2))^{\frac{N-2}{2}}}{(1 + |x|^2)^{\frac{N-2}{2}}}. \]

In addition, we define $\tilde{u}_{\varepsilon}(x)$ as follows.
\[ \tilde{u}_{\varepsilon}(x) := \phi(x) u_{\varepsilon}(x) \]
where $\phi(x)$ is a suitable cut off function. Then, we have the following estimates due to Adimurthi and Mancini(see [1]) as $\varepsilon \to 0$:
\[
\frac{\| \nabla \tilde{u}_{\varepsilon} \|^2}{\| \tilde{u}_{\varepsilon} \|^2} = \begin{cases}
    \frac{S}{2\pi} (1 - c_0 \varepsilon |\log \varepsilon| + O(\varepsilon)) & N = 3 \\
    \frac{S}{2\pi} (1 - c_1 \varepsilon + O(\varepsilon^2 |\log \varepsilon|)) & N = 4 \\
    \frac{S}{2\pi} (1 - c_2 \varepsilon + O(\varepsilon^3)) & N \geq 5
\end{cases}
\]
where \(c_0, c_1, c_2\) are positive constants which depend only on \(N\).

For each \(\tilde{u}_\varepsilon\) there exist a constant \(a_\varepsilon > 0\) such that
\[
\tilde{u}_\varepsilon - a_\varepsilon \in X := \left\{ u \in H^1(\Omega) \left| \int_\Omega |u|^{2^* - 2} u = 0 \right. \right\}.
\]

**Proposition 3.2.** We obtain
\[
a_\varepsilon = O\left(\varepsilon \frac{(N-2)^2}{N+2}\right).
\]

**Proof of Proposition 3.2.** For \(s \geq 1(s \neq N/(N - 2))\) we have
\[
\|\tilde{u}_\varepsilon\|_s^* = O\left(\varepsilon \min\{s^{2-N}N, s^{N-2}\}\right).
\]

In particular,
\[
\|\tilde{u}_\varepsilon\|_1 = O\left(\varepsilon \frac{N-2}{2}\right)
\]
\[
\|\tilde{u}_\varepsilon\|_\frac{N+2}{N-2} = O\left(\varepsilon \frac{N-2}{2}\right)
\]
\[
\|\tilde{u}_\varepsilon\|_2^* = O(1)
\]
Recall that
\[
2^{p-1}(a^p + b^p) \geq (a + b)^p \quad (a, b \geq 0, p \geq 1).
\]

\(a, b\) and \(p\) are replaced by \(a = |a_\varepsilon - \tilde{u}_\varepsilon|, b = \tilde{u}_\varepsilon, p = (N + 2)/(N - 2)\) in each, we obtain
\[
2^\frac{4}{N-2}(|a_\varepsilon - \tilde{u}_\varepsilon|^{\frac{N+2}{N-2}} + \tilde{u}_\varepsilon^{\frac{N+2}{N-2}}) \geq (|a_\varepsilon - \tilde{u}_\varepsilon| + \tilde{u}_\varepsilon)^{\frac{N+2}{N-2}}
\]
\[
\geq \frac{N+2}{a_\varepsilon^{\frac{N-2}{N+2}}}. \tag{4}
\]
We integrate above inequality over \(\Omega\) and we have
\[
2^\frac{4}{N-2} \int_\Omega (|a_\varepsilon - \tilde{u}_\varepsilon|^{\frac{N+2}{N-2}} + \tilde{u}_\varepsilon^{\frac{N+2}{N-2}}) \geq \int_\Omega \frac{N+2}{a_\varepsilon^{\frac{N-2}{N+2}}}
\]
hence
\[
2^\frac{4}{N-2} \int_\Omega |a_\varepsilon - \tilde{u}_\varepsilon|^{\frac{N+2}{N-2}} \geq \int_\Omega \frac{N+2}{a_\varepsilon^{\frac{N-2}{N+2}}} - 2^\frac{4}{N-2} \int_\Omega \tilde{u}_\varepsilon^{\frac{N+2}{N-2}}. \tag{4}
\]
Since
\[ \int_{\Omega} |\tilde{u}_\varepsilon - a_\varepsilon|^{2^* - 2}(\tilde{u}_\varepsilon - a_\varepsilon) = 0 \]
we calculate using (4)

\[ 0 = \int_{\Omega} |\tilde{u}_\varepsilon - a_\varepsilon|^{2^* - 2}(\tilde{u}_\varepsilon - a_\varepsilon) \]

\[ = \int_{[\tilde{u}_\varepsilon > a_\varepsilon]} (\tilde{u}_\varepsilon - a_\varepsilon)^{\frac{N+2}{N-2}} - \int_{[a_\varepsilon > \tilde{u}_\varepsilon]} (a_\varepsilon - \tilde{u}_\varepsilon)^{\frac{N+2}{N-2}} \]

\[ = 2\frac{N+2}{N-2} \int_{[\tilde{u}_\varepsilon > a_\varepsilon]} (\tilde{u}_\varepsilon - a_\varepsilon)^{\frac{N+2}{N-2}} - 2\frac{N}{N-2} \int_{\Omega} |a_\varepsilon - u_\varepsilon|^{\frac{N+2}{N-2}} \]

\[ \leq 2\frac{N+2}{N-2} \int_{[\tilde{u}_\varepsilon > a_\varepsilon]} (\tilde{u}_\varepsilon - a_\varepsilon)^{\frac{N+2}{N-2}} - \left\{ \int_{\Omega} |a_\varepsilon - u_\varepsilon|^{\frac{N+2}{N-2}} - 2\frac{N}{N-2} \int_{\Omega} \tilde{u}_\varepsilon^{\frac{N+2}{N-2}} \right\} . \]

Thus

\[ \int_{\Omega} a_\varepsilon^{\frac{N+2}{N-2}} \leq 2\frac{N}{N-2} \left\{ \int_{[\tilde{u}_\varepsilon > a_\varepsilon]} (\tilde{u}_\varepsilon - a_\varepsilon)^{\frac{N+2}{N-2}} + \int_{\Omega} \tilde{u}_\varepsilon^{\frac{N+2}{N-2}} \right\} \]

\[ \leq 2\frac{N}{N-2} \left\{ \int_{[\tilde{u}_\varepsilon > a_\varepsilon]} \tilde{u}_\varepsilon^{\frac{N+2}{N-2}} + \int_{\Omega} \tilde{u}_\varepsilon^{\frac{N+2}{N-2}} \right\} \]

\[ \leq 2\frac{N}{N-2} \int_{\Omega} \tilde{u}_\varepsilon^{\frac{N+2}{N-2}} . \]

Therefore

\[ a_\varepsilon^{\frac{N+2}{N-2}} \leq C_0 \int_{\Omega} u_\varepsilon^{\frac{N+2}{N-2}} = C_0 \|u_\varepsilon\|^{\frac{N+2}{N-2}} = O(\varepsilon^{\frac{N-2}{2}}) \]

Hence we obtain
\[ a_\varepsilon = O\left(\varepsilon^{\frac{(N-2)^2}{2(N+2)}}\right) . \]

\[ \square \]

Proof. We estimate \( \|\tilde{u}_\varepsilon - a_\varepsilon\|_2^2 \), similarly to Girao and Weth (see [7]) using Proposition.

\[ \int_{\Omega} |\tilde{u}_\varepsilon - a_\varepsilon|^2 \geq \int_{\Omega} |\tilde{u}_\varepsilon|^2 + |\Omega|a_\varepsilon^2 - C \left( a_\varepsilon \int_{\Omega} |\tilde{u}_\varepsilon|^{\frac{N+2}{N-2}} + a_\varepsilon^{\frac{N+2}{N-2}} \int_{\Omega} |\tilde{u}_\varepsilon| \right) \]

\[ = \int_{\Omega} |\tilde{u}_\varepsilon|^2 + O\left(\varepsilon^{\frac{N(N-2)}{N+2}}\right) . \]
Consequently
\[ \| \tilde{u}_\varepsilon - a_\varepsilon \|_{2^*}^2 \geq \| \tilde{u}_\varepsilon \|_{2^*}^2 + O(\varepsilon^{\frac{N(N-2)}{N+2}}) \]
and therefore
\[ \frac{\| \nabla (\tilde{u}_\varepsilon - a_\varepsilon) \|_{2^*}^2}{\| \tilde{u}_\varepsilon - a_\varepsilon \|_{2^*}^2} \leq \frac{\| \nabla \tilde{u}_\varepsilon \|_{2^*}^2}{\| \tilde{u}_\varepsilon \|_{2^*}^2 + O(\varepsilon^{\frac{N(N-2)}{N+2}})} = \frac{\| \nabla \tilde{u}_\varepsilon \|_{2^*}^2}{\| \tilde{u}_\varepsilon \|_{2^*}^2 + O(\varepsilon^{\frac{N(N-2)}{N+2}})}.
\]
We recall that the value of Sobolev quotient of \( \tilde{u}_\varepsilon(x) \) in the case \( N = 3 \), \( N = 4 \) and \( N \geq 5 \) and taking account of the fact that \( N \geq 4 \) we obtain
\[ \frac{\| \nabla (\tilde{u}_\varepsilon - a_\varepsilon) \|_{2^*}^2}{\| \tilde{u}_\varepsilon - a_\varepsilon \|_{2^*}^2} < \frac{S}{2^*} \text{ for } \varepsilon \text{ small enough}, \]
and hence
\[ S' < \frac{S}{2^*}. \]

**Lemma 3.3.** If \( S' < S/2^{N/2} \) then \( S' \) is attained.

*Proof.* We consider a minimizing sequence \( \{ u_n \} \subset X \) for \( S' \). Then \( u_n \) is bounded in \( H^1(\Omega) \). So we can suppose, up to a subsequence,
\[
\begin{align*}
    u_n &\to u \text{ in } H^1(\Omega) \quad (n \to \infty) \\
    u_n &\to u \text{ in } L^p(\Omega) \quad (n \to \infty) \quad (1 \leq p < 2^*) \\
    u_n &\to u \text{ a.e.} \quad (n \to \infty)
\end{align*}
\]
In addition, since \( H^1(\Omega) \hookrightarrow L^{2^*} \Omega \) is a compact embedding, we have
\[ \int_{\Omega} |u_n|^{2^*-2} u_n \to \int_{\Omega} |u|^{2^*-2} u \quad (n \to \infty). \]
Furthermore, we may assume that
\[
\begin{align*}
\| u_n \|_{2^*} &= 1 \quad (n \in \mathbb{N}), \\
\| \nabla u_n \|_2^2 &= S' + o(1) \quad (n \to \infty).
\end{align*}
\]
For each $u_n$ there exist a constant $a_n$ such that
\[ u_n - u - a_n \in X. \]

We calculate similarly to the proof of Proposition 3.2. We obtain that
\[ a_n = o(1) \quad (n \to \infty). \]

Since $\|u_n\|_2^2 = 1$ for all $n \in \mathbb{N}$ by Brezis-Lieb lemma (see [2]) we have
\[ \|u_n\|_2^2 = \|u\|_2^2 + \|u_n - u\|_2^2 + o(1) \quad (n \to \infty). \]

Thus
\[ 1 = \|u_n\|_2^2 = (\|u\|_2^2 + \|u_n - u\|_2^2)^{\frac{1}{2}} + o(1) \leq \|u\|_2^2 + \|u_n - u\|_2^2 + o(1). \]

On the other hand, we have
\[ \|u\|_2^2 + (\|u_n - u - a_n\|_2^2 + \|a_n\|_2^2)^2 \leq \frac{\|
abla u\|_2^2}{S'} + \frac{\|
abla(u_n - u)\|_2^2}{S'} + o(1) \]
\[ = \frac{\|
abla u_n\|_2^2}{S'} = 1 + o(1) \]
and
\[ \|u\|_2^2 + (\|u_n - u - a_n\|_2^2 + \|a_n\|_2^2)^2 \geq \|u\|_2^2 + \|u_n - u\|_2^2. \]

Thus
\[ \|u\|_2^2 + \|u_n - u\|_2^2 \leq 1 + o(1). \]

Hence there exists a limit and we have the equality.
\[ \lim_{n \to \infty} (\|u_n - u\|_2^2 + \|u\|_2^2)^{\frac{1}{2}} = \lim_{n \to \infty} (\|u\|_2^2 + \|u_n - u\|_2^2) = 1. \]

Above equality holds if and only if $u \equiv 0$ or $u_n \to u$ in $L^2(\Omega)$. Suppose that $u \equiv 0$ a.e. By Cherrier’s inequality (see [5][6]) we obtain
\[ \frac{S}{2^\frac{N}{2}} \|u_n\|_2^2 \leq (1 + \varepsilon)\|
abla u_n\|_2^2 + C\|u_n\|_2^2 \quad (\varepsilon > 0, n \in \mathbb{N}). \]

Replacing $\varepsilon$ by $S/(S'2^{(2+N)/N}) - 1/2 > 0$ and tending $n$ to $\infty$, taking account to $u_n \to 0$ in $L^2(\Omega)$ we obtain
\[ \lim_{n \to \infty} \frac{S}{2^\frac{N}{2}} \|u_n\|_2^2 \leq \lim_{n \to \infty} \left( 1 + \frac{1}{2S'} \frac{S}{2^\frac{N}{2}} - \frac{1}{2} \right) \|
abla u_n\|_2^2. \]
Therefore
\[
\frac{S}{2^{\frac{N}{2}}} \leq \left( \frac{1}{2S'} \frac{S}{2^{\frac{N}{2}}} + \frac{1}{2} \right) \lim_{n \to \infty} \|\nabla u_n\|_2^2.
\]
Consequently
\[
\frac{S}{2^{\frac{N}{2}}} \leq S'.
\]
It is contradict \(S/2^{N/2} > S'\). Hence \(u \neq 0\) and \(u_n \to u\) in \(L^2\). Thus \(u\) is the minimizer of \(S'\).

4. Proof of Main theorem

We prove \(\beta_{N/2} = S'\) and attainability of \(\beta_{N/2}\) similar to Cañada, Montero and Villegas (see [4] the supercritical case). Since
\[
X := \{ u \in H^1(\Omega) | \phi(u) = 0 \}, \quad \phi(u) := \int_\Omega |u|^{2^*-2}u
\]
if \(u_0 \in X \setminus \{0\}\) is any minimizer of \(S'\), Lagrange multiplier theorem implies that there is \(\lambda \in \mathbb{R}\) such that
\[
F'(u_0) = \lambda \phi'(u_0)
\]
where \(F : H^1(\Omega) \to \mathbb{R}\) is defined by
\[
F(u) = \|\nabla u\|_2^2 - S'\|u\|_2^{2^*}.
\]
Also, since \(u_0 \in X\) we have \(\langle F'(u_0), 1 \rangle = 0\). Moreover, \(\langle F'(u_0), v \rangle = 0\), \(\forall v \in H^1(\Omega)\) satisfying \(\langle \phi'(u_0), v \rangle = 0\). As any \(v \in H^1(\Omega)\) may be written in the form \(v = a + w, a \in \mathbb{R}\), and \(w\) satisfying \(\langle \phi'(u_0), w \rangle = 0\), we conclude \(\langle F'(u_0), v \rangle = 0\), \(\forall v \in H^1(\Omega)\), i.e. \(F'(u_0) \equiv 0\). Hence \(u_0\) satisfies
\[
\begin{cases}
-\Delta u_0 = A(u_0)|u_0|^\frac{4}{N-2} u_0 & \text{in } \Omega \\
\frac{\partial u_0}{\partial \nu} = 0 & \text{on } \partial \Omega
\end{cases}
\]
where
\[
A(u) = S' \left( \int_\Omega |u|^{\frac{2N}{N-2}} \right)^{-\frac{2}{N}}.
\]
If $a \in \Lambda \cap L^{N/2}(\Omega)$ and $u \in H^1(\Omega)$ is a nontrivial solution in (1), then for each $k \in \mathbb{R}$ we have
\[
\|\nabla(u + k)\|_2^2 = \|\nabla u\|_2^2 = \int_\Omega a u^2 \leq \int_\Omega a u^2 + k^2 \int_\Omega a
\]
\[
= \int_\Omega a u^2 + k^2 \int_\Omega a + 2k \int_\Omega a = \int_\Omega a(u + k)^2 \leq \|a^+\|_{\frac{N}{2}} \|u + k\|_2^2.
\]
Since $u$ is a nontrivial solution of (1), $u + k$ is a nontrivial function. Consequently
\[
\|a^+\|_{\frac{N}{2}} \geq \|\nabla(u + k)\|_2^2.
\]
By choosing $k_0 \in \mathbb{R}$ such that $u + k_0 \in X$, we obtain
\[
\beta_{\frac{N}{2}} \geq S'.
\]
Conversely, if $u_0 \in X \setminus \{0\}$ is any minimizer of $S'$, then $u_0$ satisfies (5). Therefore $A(u_0)|u_0|^{\frac{4}{N-2}} \in \Lambda \cap L^{N/2}(\Omega)$ and
\[
\|A(u_0)|u_0|^{\frac{4}{N-2}}\|_{\frac{N}{2}} = S' \left(\int_\Omega |u_0|^{\frac{2N}{N-2}}\right)^{-\frac{2}{N}} \left(\int_\Omega |u_0|^{\frac{2N}{N-2}}\right)^{\frac{2}{N}} = S'.
\]
Hence $\beta_{N/2} = S'$ and $\beta_{N/2}$ is attained.

On the other hand, let $a \in \Lambda \cap L^{\frac{N}{2}}$ be any minimizer of $\beta_{N/2}$. Then
\[
\|a^+\|_{\frac{N}{2}} \|u + k_0\|_2^2 = \|\nabla(u + k_0)\|_2^2.
\]
Hence $a(x) = M|u(x) + k_0|^{\frac{N}{N-2}}$ ($M > 0$ : constant). Furthermore, since $a(x) > 0$ we have $\int_\Omega a(x) \geq 0$. In addition, since
\[
\int_\Omega a u^2 = \int_\Omega a(u + k_0)^2
\]
we obtain $k_0 \equiv 0$. Finally, we define $w(x) = M^{\frac{N-4}{N-2}} |u(x)|$ we have that
\[
|w(x)|^{\frac{N}{N-2}} = M|u(x)|^{\frac{N}{N-2}} = a(x).
\]
Moreover, since $u(x)$ is a solution of (1) and $w(x)$ is multiple of $u(x)$, then $w(x)$ is a solution of (1) and consequently a solution of (3). \qed
5. Corollary

**Corollary 5.1.** Let \( \Omega \) be a ball \( B := B(0,1) \) and \( u \) be a minimizer for \( S' \) on \( B \). Then \( u \) is foliated Schwarz symmetric, i.e. there exists a unit vector \( e \in \mathbb{R}^N \), \( |e| = 1 \) such that \( u(x) \) only depends on \( r = |x| \) and \( \theta := \arccos(x/|x| \cdot e) \), and \( u \) is nonincreasing in \( \theta \). Moreover, either \( u \) does not depend on \( \theta \)(hence it is a radial function), or \( (\partial u/\partial \theta)(r, \theta) < 0 \) for \( 0 < r \leq 1, 0 < \theta < \pi \).

**Proof.** We can prove the Corollary 5.1. similar to Girão-Weth (see [7] Proposition 4.1.) \( \Box \)

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**References**


