K-theory and Hermitian Groups

Guoping Tang
Department of Mathematics
Graduate School of the Chinese Academy of Science
Beijing 100039 P.R. China
email: tanggp@gscas.ac.cn

Let R be a ring equipped with involution $r \mapsto \bar{r}$ and let $\lambda \in \text{center}(R)$ such that $\lambda\bar{\lambda} = 1$. If M is a right R-module and φ is a **nonsingular λ-Hermitian forma** on M. A **metabolic plane** is a Hermitian module (R^{2n}, φ) such that for some ordered basis $e_1, \ldots, e_n, f_1, \ldots, f_n$ of R^{2n}, φ corresponds to a $2n \times 2n$ matrix

$$\begin{pmatrix} A & \lambda I \\ I & 0 \end{pmatrix}$$

of $GL_{2n}(R)$ where

$$I = n \times n \text{identity matrix}$$

$$A = \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} = \begin{pmatrix} \varphi(e_1, e_1) & & \\ & \ddots & \\ & & \varphi(e_n, e_n) \end{pmatrix}.$$

By definition, the **general Hermitian group**

$$GH_{2n}(R, a_1, \cdots, a_n) = \text{Aut}(M, \varphi).$$

If $0 \leq r < n$ and $a_{r+1} = \ldots = a_n = 0$ then we shall write

$$GH_{2n}(R, a_1, \cdots, a_r) \text{in place of } GH_{2n}(R, a_1, \cdots, a_n).$$

Represent for the moment a typical element of $GH_{2n}(R, a_1, \cdots, a_r)$ by

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \text{where } \alpha, \beta, \gamma, \delta \text{denote } n \times n \text{matrices.}$$

There is a canonical embedding

$$GH_{2n}(R, a_1, \cdots, a_r) \longrightarrow GH_{2(n+1)}(R, a_1, \cdots, a_r).$$

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mapsto \begin{pmatrix} \alpha & 0 & \beta & 0 \\ 0 & 1 & 0 & 0 \\ \gamma & 0 & \delta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

One defines

$$GH(R, a_1, \cdots, a_r) = \varinjlim_{n} GH_{2n}(R, a_1, \cdots, a_r).$$

We will define an elementary subgroup $EH_{2n}(R, a_1, \cdots, a_r)$ of $GH_{2n}(R, a_1, \cdots, a_r)$ and prove that

$$EH(R, a_1, \cdots, a_r) \stackrel{\text{def.}}{=} \varinjlim_{n} EH_{2n}(R, a_1, \cdots, a_r)$$

$$= [GH(R, a_1, \cdots, a_r), GH(R, a_1, \cdots, a_r)].$$

so define

$$KH_1(R, a_1, \ldots, a_r) = GH(R, a_1, \ldots, a_r)/EH(R, a_1, \cdots, a_r).$$

Like all kind of K-theory, one could also set up a systematic K-theory for such groups.