REFINED KIRBY CALCULUS
FOR RATIONAL HOMOLOGY SPHERES OF PRIME ORDERS

KENICHI FUJIWARA

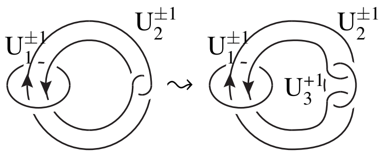
We study a theory of integral framed links in $\mathbf{S}^3$. They provide a way to present all connected, smooth, orientable, closed 3-manifolds. Kirby calculus acts to relate all equivalent presentations. In order to visualize a 3-manifold and to study its invariants, it is sometimes useful to regard 3-manifolds as a set of link-surgery presentations with equivalence.

$$\therefore \{\text{oriented closed 3-manifolds}\} = \{\text{framed links}\}/\text{Kirby calculus}.$$

Such theory fortunately can be refined for integral homology 3-spheres. Habiro proposed refined Kirby calculus and framed links without linking to present all of them. Then he showed that it suffices to relate all equivalent presentations.

$$\therefore \{\text{integral homology 3-spheres}\} = \{\text{framed links with } lk = 0\}/\text{refined Kirby calculus}.$$

In this talk, we extend his theory direct to rational homology 3-spheres of prime order p if $|p| \in 4\mathbf{Z} - 1$. Moreover, we see that it happens to extend otherwise.



Examples of framed links in Habiro's theory:
stabilize the left hand side by U_3^{+1} and then apply a band slide of U_2 over U_3^{+1}.

For rational homology 3-spheres of each prime order p, we use framed links to present them with linking matrix $\mathrm{diag}(\begin{smallmatrix}-1 & 0\\ 0 & 1\end{smallmatrix}, \ldots, \begin{smallmatrix}-1 & 0\\ 0 & 1\end{smallmatrix}, p)$, that is, one surgery coefficient is p, others are ± 1 and linking number are all 0. We aim to prove that Habiro's calculus suffices to relate them. We review his method to present a sequence of (oriented and ordered) Kirby calculus $s \in \mathscr{S}$ by $\varphi : \mathscr{S} \to \mathrm{GL}(n; \mathbf{Z})$. The method $(0,0)$-*coupling* re-normalizes linking matrices of our links according to the unimodular transformation $\mathrm{diag}(\begin{smallmatrix}-1 & 0\\ 0 & 1\end{smallmatrix}, \ldots, \begin{smallmatrix}-1 & 0\\ 0 & 1\end{smallmatrix}, p) \to \mathrm{diag}(\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}, \ldots, \begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}, p) =: \mathrm{A}$. Now, we can assume $\varphi(s) \in \mathrm{O}(\mathrm{A}; \mathbf{Z})$. We claim that we can modify s to $\overline{s}$ with $\varphi(\overline{s}) = I_n$, which *Habiro's lemma* applies to obtain the conclusion. To verify the claim, we study generators of $\mathrm{O}(\mathrm{A}; \mathbf{Z})$ to realize them into $\mathscr{S}$.

Do such presentations give all homology 3-spheres? An affirmative answer will be showed if $|p| \in 4\mathbf{Z} - 1$. Otherwise, a delicate observation is required.