Research Plan, Hiro-aki Narita

I would like to study the following two subjects in future.

(1) Arithmetic of automorphic forms on $Sp(1,q)$ generating quaternionic discrete series (mainly when $q=1$),
(2) Dimension formula for holomorphic automorphic forms on classical tube domains.

As one problem in the direction of (1), I would like to study the arithmetic meanings of the Fourier coefficients of the automorphic forms. In the theory of automorphic forms the arithmetic of Fourier coefficients is one of the central problems. For instance, there is a well-known work by Siegel on the representation numbers of quadratic forms by means of holomorphic theta series. As I explained in the achievement of the research I have calculated the Fourier coefficients of the forms lifted from elliptic Poincare series. In addition I have provided an explicit formula of Fourier coefficients of the Eisenstein series. But I think that one needs more examples to understand the arithmetic meanings of the Fourier coefficients for the forms. For that purpose I think that we have to provides more explicit constructions of our automorphic forms to have more data on the Fourier coefficients. As another approach to this problem, I am also considering to give a trace formula of Hecke operators. The traces of Hecke operators can be regarded as Fourier coefficents of Hecke eigenforms and are expexted to be written in terms of class numbers of algebras. I hope that more data of the Fourier coefficients and a trace formula of Hecke operators will make the arithmetic meanings of the Fourier coefficiens clearer.

On the other hand, we should note that there are still many remaining problems on the study of the theta lifting. For instance, there are several impotant problems such as the non-vanishing of the theta lifting and how to characterize the images of the lifiting in the total space of the automorphic forms. The methods for the former problem would be to study the Fourier coefficients of the lifted forms or an inner product formula of them. As for the latter, I was expecting that the images would be charecterized by the location of poles of their automorphic L-functions. However, judging from an explicit form of the (Spinor) L-functions for our lifting which Murase and I have obtained recently, I have realized that such characterization is not useful and we need another way of the characterization.

Next let me explain my plan of (2). In general, given an integrable representation π of a semi-simple Lie group G having discrete series representations and a fixed arithmetic subgroup Γ of G, the dimension of the space $S_\pi(\Gamma)$ of Γ-invariant bounded automorphic forms on G generating π can be expressed as

$$\dim_{\mathbb{C}} S_\pi(\Gamma) = \int_{\Gamma\backslash G} \sum_{\gamma\in\Gamma} f_\pi(g^{-1}\gamma g)dg,$$

which should be called the "Godement's formula". Here f_π is a spherical trace function for π. What I have proved for the theme (2) is that, exchanging the sums and the integrals formally, the unipotent contribution

$$\int_{\Gamma\backslash G} \sum_{\substack{\gamma\in\Gamma \\ :\text{unipotent}}} f_\pi(g^{-1}\gamma g)dg$$

to the dimension formula can be written as a finite sum of (analogous) zeta integrals for prehomogeneous vector spaces. This research is motivated by the conjecture on the vanishing of the contribution of non-central unipotent elements to the dimension formula, where non-central unipotent elements mean unipotent elements not conjugate to any elements in the center of the unipotent radical of a maximal parabolic subgroup. For this conjecture we proved that the test functions appearing in the zeta integrals corresponding to the non-central unipotent contribution have integral expressions which are expected to vanish, when G is a Lie group whose corresponding symmetric domain is of tube type (the case of "type II" domain is not considered) and π is a holomorphic discrete series. As I said, this result is given under allowing formal exchanging of the sums and the integrals. Thus it does not provide a complete proof of the conjecture but can be regarded as evidence of it. I would like to find out how to overcome such difficulty in order to prove the conjecture.