Categorifying q-Fock space via graded cyclotomic q-Schur algebras

Ben Webster

U. of Oregon

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Fock space

One of the most wonderful objects of representation theory is the **Fock space** of charge c. This is a vector space spanned by a basis s_λ where λ ranges over all partitions of all sizes.

- Thought of bosonically, this is symmetric polynomials, and s_λ is the Schur function.
- Thought of fermionically, this is a semi-infinite wedge space, and

$$s_\lambda = x_{\lambda_1+c} \wedge x_{\lambda_2-1+c} \wedge x_{\lambda_3-2+c} \wedge \cdots$$

The $\widehat{\mathfrak{sl}}_n$ action

Fix once and for all a positive integer n. Given a partition λ, we fill its boxes (i, j) with their **content mod** n which is $i - j + c \in \mathbb{Z}/n\mathbb{Z}$.

1	2	0	1
0	1	2	
2			
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$$n = 3, c = 1$$

There is a natural $\widehat{\mathfrak{sl}}_n$ action on Fock space, where we identify the Dynkin diagram of $\widehat{\mathfrak{sl}}_n$ with $\mathbb{Z}/n\mathbb{Z}$ so that i and $i + 1$ are adjacent.

- The action of F_i sums over adding a box of content i in all possible ways;
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These are obviously adjoint for the inner product where the s_λ are orthonormal.

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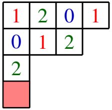

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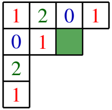

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The Heisenberg action

However, the Fock space is not irreducible as a module over $\widehat{\mathfrak{sl}}_n$. In fact, there is an infinite-dimensional algebra of endomorphisms commuting with it.

Let p_k denote the operator of multiplying by the degree k power sum function (in the bosonic realization) if $k \geq 0$, and its adjoint if $k < 0$.

Theorem

The operators p_{nk} define an action of the Heisenberg Lie algebra $\mathcal{H}$ on Fock space which commutes with $\widehat{\mathfrak{sl}}_n$ and acts irreducibly on the space of highest weight vectors.

One can think of $\widehat{\mathfrak{sl}}_n \times \mathcal{H}$ as $\widehat{\mathfrak{gl}}_n$, so this says that Fock space is irreducible over $\widehat{\mathfrak{gl}}_n$.

Generalizations

There are several directions one can take this picture:

- You can deform and obtain q-**Fock space**.
- You can categorify and obtain q-**Schur algebras**.

Oddly enough, it was known for many years how to do both of these separately before we knew how to do both at once.

$$\begin{array}{ccc} q\text{-Fock space ('90)} & \longrightarrow & \text{Fock space ('32)} \\ \uparrow & & \uparrow \\ \text{graded } q\text{-Schur algebras ('09)} & \longrightarrow & q\text{-Schur algebras ('89)} \end{array}$$

Since graded q-Schur algebras are so new, there are a lot of unresolved questions about them:

- relation to Rouquier-type categorification?
- is there a natural graded presentation? a homogenous basis?
- what about generalizations to higher level?

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Generalizations

My aim in this talk to answer (parts of) these questions.

Main idea: Develop graded Rouquier-style categorification of q-Fock space from first principles, and than check isomorphism with q-Schur algebra.

So, which principles would those be?

Categorification

It is an old observation that some numbers are really sets in disguise, and some sets are categories in disguise. Of course, this added structure is a choice, but we know of oodles of instances where it "feels right."

What has taken a little longer to develop is the linearized version of the same story.

- Sometimes, a number doesn't seem to be the size of any particular set, but is the dimension of a vector space.
- Abelian groups can be gotten as the Grothendieck group of a category with some notion of exact sequence.

Some very important and popular abelian groups, the semi-simple Lie algebras and their representations, managed to be the Grothendieck groups of categories for 100 years without anyone noticing.

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The Grothendieck group

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$$[A] + [C] = [B].$$

- The most popular choice is for $\mathcal{C}$ to be modules over some ring, with short exact sequences $0 \to A \to B \to C \to 0$.
- Another option would be projective modules over a ring; now all short exact sequences are split, so the triples are just $(P_1, P_1 \oplus P_2, P_2)$.
- More exciting would be a derived category or homotopy category of modules. In this case, triples are objects with representatives that fit in an exact sequence.

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Universal enveloping algebras

Let $\mathfrak{g}$ be your favorite symmetrizable Kac-Moody Lie algebra with Cartan matrix C (for today, you should probably choose $\widehat{\mathfrak{sl}}_n$), root lattice Y and weight lattice X.

This Lie algebra has a presentation of the form

$$[H_j, E_i] = c_{ij}E_i \qquad [H_j, F_i] = -c_{ij}F_i$$

$$[E_i, F_j] = \delta_j^i H_i \qquad \mathrm{ad}_{E_j}^{c_{ij}+1} E_i = \mathrm{ad}_{F_j}^{c_{ij}+1} F_i = 0$$

By definition, the universal enveloping algebra is the *associative* algebra generated by these symbols subject to the relations above (where $[-,-]$ means commutator).

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Universal enveloping algebras

We actually want a slightly bigger algebra $\dot{U}$, with some extra idempotents $\mathbb{1}_\lambda$ for $\lambda \in X$. These satisfy the relations

$$\mathbb{1}_\lambda \mathbb{1}_{\lambda'} = \delta_{\lambda'}^{\lambda} \mathbb{1}_\lambda \qquad H_i \mathbb{1}_\lambda = \mathbb{1}_\lambda H_i = \lambda^i \mathbb{1}_\lambda.$$

Note that

$$\mathbb{1}_\lambda E_i = E_i \mathbb{1}_{\lambda-\alpha_i} \qquad \mathbb{1}_\lambda F_i = F_i \mathbb{1}_{\lambda+\alpha_i}.$$

We can represent elements of this as pictures on a line

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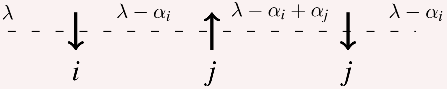
$\mathbb{1}_\lambda E_i F_j E_j \mathbb{1}_{\lambda - \alpha_i} =$ 

Categorifying UEAs

The remarkable insight of Khovanov and Lauda was that one could make these into the objects of a category $\mathcal{U}$, with morphisms given by pictures in the plane (Chuang and Rouquier had the same idea first, but never drew the pictures).

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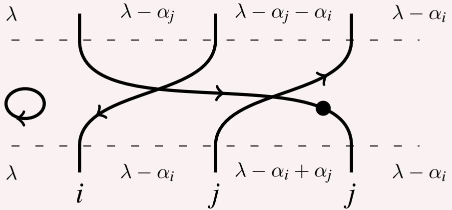
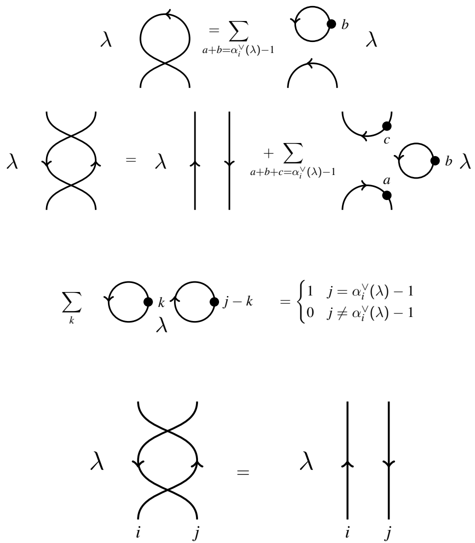
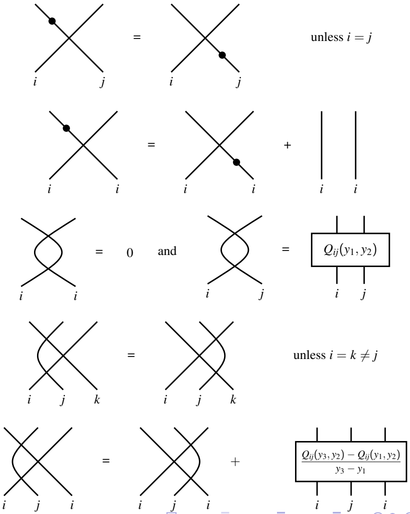


Relations in $\mathcal{U}$ (simply-laced case)



$$Q_{ij}(u,v) = Q_{ji}(v,u) = \begin{cases} 1 & i \neq j \pm 1 \\ au + bv & i = j + 1 \end{cases}$$



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These relations are all a bit overwhelming at first, but actually they match up surprisingly well with the relations of the universal enveloping algebra. Of course, this correspondence is a bit subtle.

The thing to look for is writing the identity element of any object as maps factoring through another; this is how we find direct sum decompositions.

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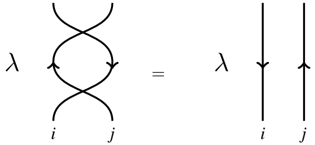

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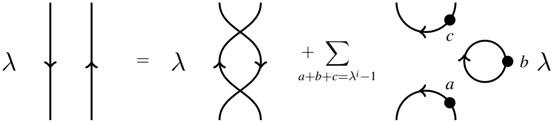
$$\mathbb{1}_\lambda F_i E_i = \mathbb{1}_\lambda E_i F_i + \mathbb{1}_\lambda \lambda^i \qquad (\lambda^i > 0)$$



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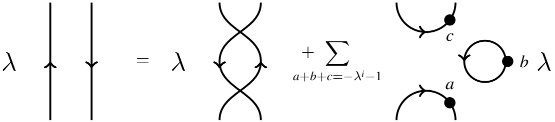

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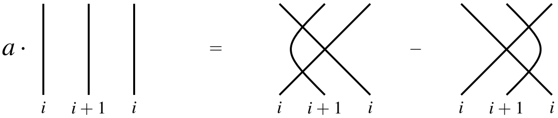

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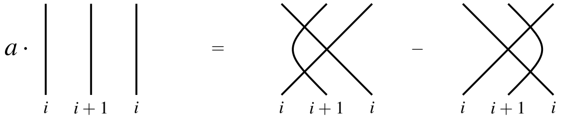

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The category $\mathcal{U}$

We let $\mathcal{U}$ be the idempotent completion of the category whose

- objects are diagrams on a line shown above and
- morphisms are the pictures in the plane, modulo the relations of the previous slide.

Idempotent completion means adding a new object for each idempotent which is the image of that idempotent as a projection.

Put another way, if we fix the weights at extreme left and right, then we get an algebra spanned by diagrams modulo these relations ${}_{\lambda}\mathfrak{U}_{\mu}$. We look at the categories of projective modules over these.

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The monoidal structure

The category $\mathcal{U}$ is monoidal; it has a tensor product. Visually, it's quite simple. You just put diagrams next to each other if the label at the edges match, and get 0 if they don't.

$$\lambda_1 \; \boxed{A} \; \lambda_2 \otimes \mu_1 \; \boxed{B} \; \mu_2 = \lambda_1 \; \boxed{A} \; \lambda_2 = \mu_1 \; \boxed{B} \; \mu_2$$

Put another way, there is a 2-category where

- objects are weights $\lambda \in X$, and
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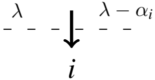
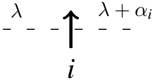
Let $\mathcal{E}_i^\lambda =$ 

and $\mathcal{F}_i^\lambda =$ 

Theorem (Khovanov-Lauda, W.)

The GG of $\mathcal{U}$ is $\dot{U}$, via the isomorphism $[\mathcal{E}_i^\lambda] \mapsto \mathbb{1}_\lambda E_i, [\mathcal{F}_i^\lambda] \mapsto \mathbb{1}_\lambda F_i$.

For example,

$\mapsto \mathbb{1}_\lambda E_i F_j E_j \mathbb{1}_{\lambda-\alpha_i}$

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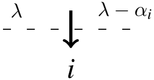
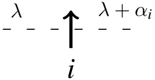
Let $\mathcal{E}_i^\lambda =$ 

and $\mathcal{F}_i^\lambda =$ 

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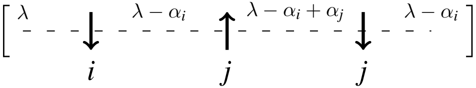

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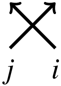
deg 

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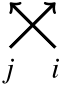
deg 

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$\widehat{\mathfrak{gl}}_n$?

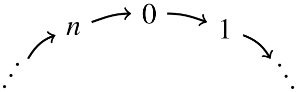
This is, of course, a beautiful story, and I'll say a lot more about it before this talk is over. But for our purposes, it's not very satisfying, since it uses extremely strongly that $\mathfrak{g}$ has a Chevalley presentation, which $\widehat{\mathfrak{gl}}_n$ doesn't.

However, this is really a sign that our principles aren't "first" enough yet.

Underlying all of this picture is geometry, which we haven't delved into yet.

Quiver varieties

Assume from now on that our Cartan matrix is symmetric, and let Γ be an orientation of the Dynkin graph of $\mathfrak{g}$. For $\widehat{\mathfrak{sl}}_n$, of course, we can choose



Choose a dimension vector $\mathbf{d}\colon V(\Gamma) \to \mathbb{Z}$. Let

$$E_{\mathbf{d}} = \bigoplus_{i \to j} \mathrm{Hom}(\mathbb{C}^{d_i}, \mathbb{C}^{d_j})$$

This is the **universal d-dimensional representation** of Γ since its points are exactly the representations of Γ on $\bigoplus \mathbb{C}^{d_i}$.

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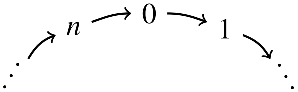

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Quiver flag varieties

Choose a sequence of dimension vectors $\mathbf{D} = (\mathbf{d}^1, \cdots, \mathbf{d}^n)$ with $\mathbf{d} = \sum \mathbf{d}^j$, and let $\mathcal{F}_{\mathbf{D}}$ denote the space of flags of subspaces

$$\cdots \subset V_i^j \subset V_i^{j+1} \subset \cdots \subset \mathbb{C}^{d_i}$$

such that $\dim V_i^{j+1}/V_i^j = d_i^j$. This is a parabolic homogeneous space for the action of $G_{\mathbf{d}} = \prod_{i \in \Gamma} GL_{d_i}$.

Definition

The **quiver flag variety** *for the sequence* $\mathcal{Q}_{\mathbf{D}}$ *is the subvariety*

$$\{(f, V_*^*) \in E_{\mathbf{d}} \times \mathcal{F}_{\mathbf{D}} | V_*^j \text{ is a subrepresentation and } V_*^j/V_*^{j-1} \text{ is a trivial representation for all } j.\}$$

This variety is $G_{\mathbf{d}}$-invariant, and thus inherits a $G_{\mathbf{d}}$-action.

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Quiver flag varieties

There are natural projection maps

$$E_{\mathbf{d}} \xleftarrow{p_{\mathbf{D}}} \mathcal{Q}_{\mathbf{D}} \xrightarrow{\pi_{\mathbf{D}}} \mathcal{F}_{\mathbf{D}}$$

- The map $\pi_{\mathbf{D}}$ is an affine bundle; its fiber is the space of quiver representations preserving the given flag.
- The map $p_{\mathbf{D}}$ is proper, and in some circumstances is a resolution of singularities of a subvariety of $E_{\mathbf{d}}$. The image this map necessarily lies in the space of nilpotent quiver representations.

In finite or affine type, the nilpotent representations have finitely many $G_{\mathbf{d}}$-orbits, and the closure each of these has a resolution of singularities of the form $\mathcal{Q}_{\mathbf{D}}$, given by choosing the dimension vectors of the radical (or socle) filtration.

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Convolution and Borel-Moore homology

Faced with a situation like this, it's quite popular to study the equivariant Borel-Moore homology of the fiber products $\mathcal{Q}_{\mathbf{D}} \times_{E_{\mathbf{d}}} \mathcal{Q}_{\mathbf{D}'}$.

Theorem

- *The vector space*

$$A_S \cong \bigoplus_{\mathbf{D},\mathbf{D}' \in S} H_*^{BM,G_{\mathbf{d}}}(\mathcal{Q}_{\mathbf{D}} \times_{E_{\mathbf{d}}} \mathcal{Q}_{\mathbf{D}'}; \mathbb{C})$$

  for any set S of sequences is an algebra under convolution.

- *This algebra is isomorphic to*

$$\mathrm{Ext}^{\bullet}_{D(E_{\mathbf{d}}/G_{\mathbf{d}})}\left(\bigoplus_{\mathbf{D} \in S} (p_{\mathbf{D}})_* \mathbb{C}_{\mathcal{Q}_{\mathbf{D}}}\right)$$

  where $D(E_{\mathbf{d}}/G_{\mathbf{d}})$ denotes the equivariant derived category.

Borel-Moore homology and categorification

Given two sequences $\mathbf{C} = (\mathbf{c}^1, \ldots, \mathbf{c}^m)$ and $\mathbf{D} = (\mathbf{d}^1, \ldots, \mathbf{d}^n)$, let $\mathbf{C} \cup \mathbf{D}$ be their concatenation.

Definition

Let $\mathcal{Q}_{\mathbf{C};\mathbf{D}} \subset \mathcal{Q}_{\mathbf{C}\cup\mathbf{D}}$ be the subvariety where the mth subspace (the last from $\mathbf{C}$) is the standard coordinate subspace.

Pull-push on the diagram

$$\mathcal{Q}_{\mathbf{C}} \times \mathcal{Q}_{\mathbf{D}} \longleftarrow \mathcal{Q}_{\mathbf{C};\mathbf{D}} \longrightarrow \mathcal{Q}_{\mathbf{C}\cup\mathbf{D}}$$

defines a map $A_S \otimes A_S \to A_S$ for any set S closed under concatenation. We call this map **horizontal multiplication**.

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Borel-Moore homology and categorification

The object $\bigoplus_{\mathbf{D}\in S}(p_{\mathbf{D}})_*\mathbb{C}_{\mathcal{Q}_{\mathbf{D}}}$ is semi-simple by the decomposition theorem, and thus its simple summands are the intersection cohomology sheaves of some set of orbits of nilpotent orbits, and these sheaves categorify some piece of the Hall algebra of nilpotent quiver representations.

Theorem (Lusztig)

- *If $S = L$ is the set of sequences where every $\mathbf{d}^j$ is a unit vector, A_L categorifies $\dot{U}(\mathfrak{b})$, with product structure induced by horizontal multiplication.*
- *If $S = M$ is all sequences, then the result is the same in finite type, but for $\widehat{\mathfrak{sl}}_n$, A_M is a categorification of $\dot{U}(\mathfrak{b}) \otimes \Lambda$, an algebra whose twisted Drinfeld double is $\dot{U}(\widehat{\mathfrak{gl}}_n)$.*

Since Fock space is cyclic for the action of $\dot{U}(\mathfrak{b}) \otimes \Lambda$, we should expect our categorification of Fock space to be "cyclic" over the monoidal category of modules over A_M.

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Algebraic and geometric categorification

Now, you'll note, I've now talked about categorifications arising from algebra (the pictorial categorification) and from geometry (Lusztig's perverse sheaves).

Question

How are these related? Does one really get two different pictures?

Algebraic and geometric categorification

If each $\mathbf{d}^j$ is a unit vector, we can think of $\mathbf{D}$ as a sequence $\mathbf{i} = (i_1, \dots, i_n)$ of nodes. Associated to this sequence, we have an element $F_{i_1} \cdots F_{i_n} \in U(\mathfrak{g})$ and its corresponding categorification $\mathcal{F}_{\mathbf{i}}$.

Theorem (Vasserot-Varagnolo/Rouquier)

We have an isomorphism $H^{BM,G_{\mathbf{d}}}(\mathcal{Q}_{\mathbf{i}} \times_{E_{\mathbf{d}}} \mathcal{Q}_{\mathbf{i}'}) \cong \mathrm{Hom}_{\mathcal{U}^-}(\mathcal{F}_{\mathbf{i}}, \mathcal{F}_{\mathbf{i}'})$ *compatible with horizontal and vertical multiplication. This isomorphism sends*

- *a dot on the jth strand to the Euler class of the line bundle* V^j/V^{j-1} *and*
- *a crossing on the kth and* $k+1$*st strands to the fundamental class of* $\mathcal{Q}_{\mathbf{i},k}$*, the space of flags with* V^{k+1}/V^{k-1} *trivial, the top of type* $\mathbf{i}$*, and the bottom of type* $(k, k+1) \cdot \mathbf{i}$ *of the form*

$$\cdots \subset V^{k-2} \subset V^{k-1} \begin{matrix} \subset V^{k} \supset \\ \supset V^{k'} \subset \end{matrix} V^{k+1} \subset V^{k+2} \subset \cdots$$

The case of $\widehat{\mathfrak{gl}}_n$

Thus $A_L(\mathbf{d})$ is actually the algebra Khovanov and Lauda call $R(\sum d_i\alpha_i)$.

So, what about affine type A? Actually, it's easier to come up with generators of A_M, since we can use any flags we want. There are two essential types: the *split* and the *join*, which we denote pictorially by

These are given by pull-push on the diagram

$$\mathcal{Q}_{(\mathbf{c},\mathbf{d})} \longleftarrow \mathcal{F}_{\mathbf{c},\mathbf{d}} \longrightarrow \mathcal{Q}_{\mathbf{c}+\mathbf{d}} = pt.$$

Horizontal and vertical compositions of these are written exactly as the name suggests.

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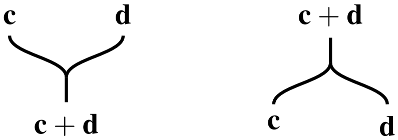

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The case of $\widehat{\mathfrak{gl}}_n$

Of course, we have to also include multiplication by classes in

$$H_*^{BM,G_{\mathbf{d}}}(\mathcal{Q}_{\mathbf{D}}) \cong H_*^{BM,G_{\mathbf{d}}}(\mathcal{F}_{\mathbf{D}}) \cong H^*_{P_{\mathbf{D}}}(pt)$$

where $P_{\mathbf{D}}$ is the stabilizer of a flag of type $\mathbf{D}$.

This ring is a polynomial ring freely generated by the Chern classes of V_i^j/V_i^{j-1} for all i, j.

Theorem (Stroppel-W.)

Under horizontal and vertical composition, the split, join and Chern classes generate A_M. In fact, we can give an explicit basis of A_M using these diagrams.

The case of $\widehat{\mathfrak{gl}}_n$

Of course, we have to also include multiplication by classes in

$$H_*^{BM,G_{\mathbf{d}}}(\mathcal{Q}_{\mathbf{D}}) \cong H_*^{BM,G_{\mathbf{d}}}(\mathcal{F}_{\mathbf{D}}) \cong H^*_{P_{\mathbf{D}}}(pt)$$

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In principle one can; we have a basis of the algebra, and an algorithm for writing a product of basis vectors in terms of this basis, using localization in equivariant cohomology.

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Representations

One of the reasons people like Kac-Moody algebras is that it has a nice representation theory. Every integrable irrep is generated by a unique line killed by all E_i, and the representation V_λ is determined by the weight λ of this line.

So, we can construct a representation $\mathcal{L}_\lambda$ of $\mathcal{U}$ by starting with a single object $\mathbb{V}$ of weight λ with boring endomorphisms, and letting $\mathcal{U}$ act by horizontal composition, subject to $\mathcal{E}_i \otimes \mathbb{V} = 0$.

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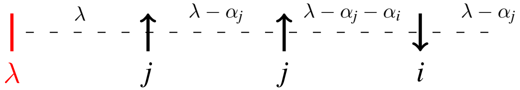

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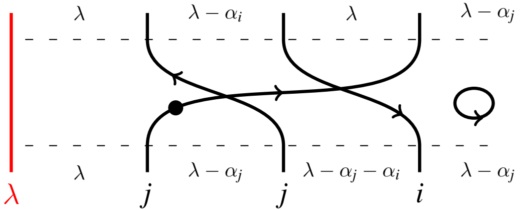


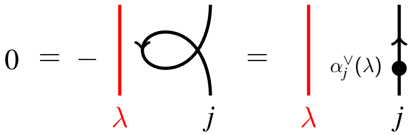
Relations

Theorem (Rouquier/Brundan-Kleshchev)

The GG of $\mathcal{L}_\lambda$ is the irreducible representation of $\dot{U}$ with highest weight λ, and $\mathcal{L}_\lambda$ is essentially the unique such module category for $\mathcal{U}$.

(You might think that this would give you the Verma module; it doesn't!).

There's just one new relation in this category, the "cyclotomic relation."



The category $\mathcal{L}_\lambda$ is just projective modules over the algebra A_L, the quotient of A_L^λ by the two-sided ideal generated under vertical composition and horizontal composition on the right by the relation above.

Representations

So, even if we don't have a categorification of the full $\widehat{\mathfrak{gl}}_n$, we can still try to define a categorification of Fock space as a representation over its lower half.

Definition

Let A_M^λ the quotient of A_M by the two-sided ideal generated by the relation above by horizontal composition on the right and vertical composition.

The category $\mathcal{M}_\lambda$ of projective A_m^λ modules still has a monoidal action by $\mathcal{U}$ given by adding black lines colored by simple roots (unit vectors).

Theorem (Stroppel-W.)

- *The graded GG of $\mathcal{M}_{\omega_i}$ is isomorphic to q-deformed Fock space as a $\widehat{\mathfrak{sl}}_n \otimes \Lambda$-module.*
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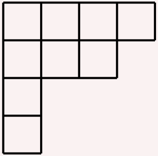
A cellular basis

At the core of the proof is the construction of a basis $C_{\mathsf{S},\mathsf{T}}$ indexed by pairs S,T of semi-standard tableaux of the same shape.

Let $\mathbf{D}_{\mathsf{T}}$ be the sequence of dimension vectors where the component of $\mathbf{d}^j_{\mathsf{T}}$ at the node i is the number of boxes with label j in T with content i.



Let $\mathbf{F}_{\mathsf{T}}$ be the associated sequence for the **ground-state tableau** on the same shape as T.

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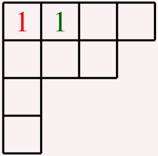

$$\mathbf{d}^{1} = (0, 1, 1)$$

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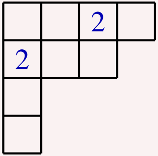

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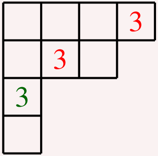

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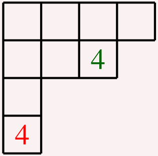

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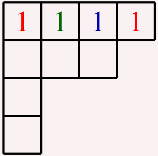

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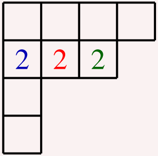

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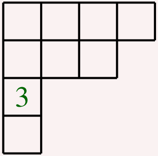

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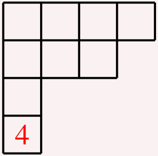

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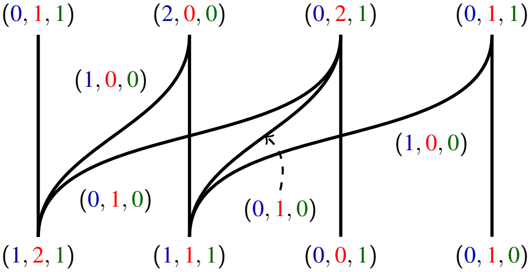
There is a natural element B_{T} attached to each tableau; its top and bottom are marked with $\mathbf{D}_{\mathsf{T}}$ and $\mathbf{F}_{\mathsf{T}}$. The element itself is given by "connect each box to its row."

$B_{\mathsf{T}} =$ 

$\mathsf{T} =$

1	1	2	3
2	3	4	
3			
4			

Of course, this involves choices; essentially the choice of a special reduced word for the permutation which puts the reading word into order. Fix a choice once and for all; this won't make any difference for our purposes.

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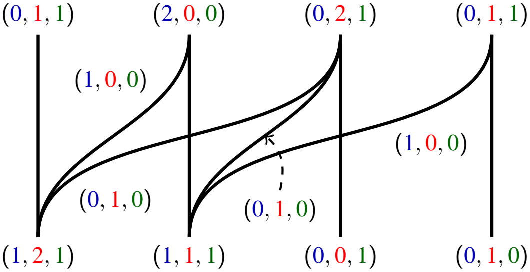
$B_{\mathsf{T}} =$ 

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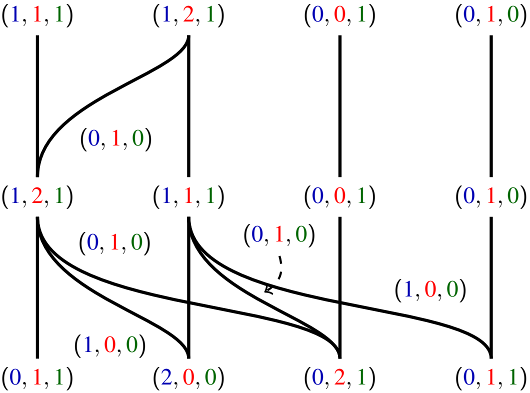
We let B_{T}^* be the reflection of this diagram, and let $C_{\mathsf{S},\mathsf{T}} = B_{\mathsf{S}} B_{\mathsf{T}}^*$. Of course, if this element is non-zero, S and T must have the same shape.



$\mathsf{S} =$

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3			
4			

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4			

A cellular basis

Theorem (Stroppel-W.)

The vectors $C_{\mathsf{S},\mathsf{T}}$ form a basis of $A_M^{\omega_i}$, which is cellular.

Under the isomorphism to the q-Schur algebra, the cellular ideals of this basis coincide with those of the DJM cellular basis, and thus its cell modules are sent to the Weyl modules.

Proof.

- Geometric considerations and a few calculations show that these vectors span.
- Reduce to the case where **D** is a unit vector (i.e. both tableaux are standard).
- This is almost the cellular basis of Hu and Mathas on the cyclotomic Khovanov-Lauda algebra (=cyclotomic Hecke algebra).

□

Graphical permutation modules

Let e_ν for $\nu = (\nu_1, \dots, \nu_n)$ a composition be the idempotent in A_M which picks out components where the flag has total type ν, that is $\sum \mathbf{d}_i^j = \nu_j$.

For example $e_{(1^d)} A_M e_{(1^d)} = A_L$. Thus, $e_\nu A_M^{\omega_i} e_{(1^d)}$ is a right module over $e_{(1^d)} A_M^{\omega_i} e_{(1^d)} \cong \mathcal{H}_d(q)$ by Brundan and Kleshchev's isomorphism.

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This right module is isomorphic to the signed permutation module over $\mathcal{H}_d(q)$ for ν. The left action of $A_M^{\omega_i}$ on $A_M^{\omega_i} e_{(1^d)}$ induces the isomorphism to the q-Schur algebra.

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What's achieved by obtaining a categorification like this? For me, a big part of the motivation was categorified knot invariants, which I'll talk about in Kyoto next week. But this can also illuminate purely representation theoretic points.

Theorem

The functor $\mathrm{Hom}_{A_M^{\omega_i}}(-, A_M^{\omega_i})$ *where the action is twisted by the vertical reflection anti-automorphism categorifies the bar involution on q-Fock space.*

This works for any module category over $\mathcal{U}$ with a nice enough equivalence to its opposite; we always get an involution which is anti-linear and commutes with F_i and E_i.

The indecomposable projectives are always invariant under this functor (in the grading so that their quotients are self-dual), and thus will be a canonical basis in the sense of Lusztig whenever we have some upper-triangularity.

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Theorem (W.)

The indecomposables of $\mathcal{L}_\lambda$ give Lusztig's canonical basis in V_λ if and only if $\mathcal{L}_\lambda$ is graded equivalent to representations of a positively graded algebra.

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Higher levels

One very interesting thing to do is consider analogues of these results for higher level q-Fock spaces. Of course, this would easier if people could agree on what “higher level q-Fock spaces” meant.

- One thing that you might mean is a highest weight representation of $U_q(\widehat{\mathfrak{gl}}_n)$; that's categorified by representations of A^λ_M for other weights λ.
- Another thing you could potentially mean is the tensor product (using the usual Hopf algebra structure on $U_q(\widehat{\mathfrak{sl}}_n)$) of level 1 Fock spaces (or $\widehat{\mathfrak{gl}}_n$-irreps as above); this is bigger than the one discussed above. This is categorified by a similar diagrammatic algebra $A^{\underline{\lambda}}_M$ which is isomorphic to the cyclotomic q-Schur algebra.

This new diagrammatic algebra now has multiple red lines, which we forbid to cross. It arises from the geometry of quiver varieties with shadow vertices.

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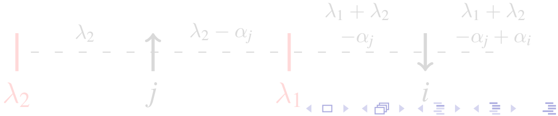

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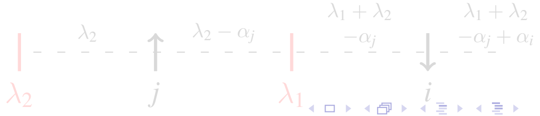
$E_i(v_{\lambda_1} \otimes F_j v_{\lambda_2}) \leftrightarrow$ 

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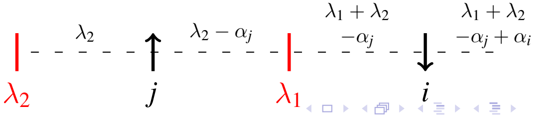
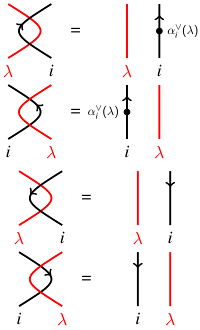
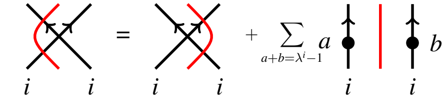
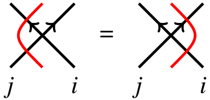
$E_i(v_{\lambda_1} \otimes F_j v_{\lambda_2}) \leftrightarrow$ 

Tensor products

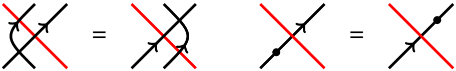
The new relations one needs to add to $A^{\boldsymbol{\lambda}}_{M}$ are







the right-most red line is killed by $\mathcal{E}_i$.



Theorem (W.)

The GG of $A^{\boldsymbol{\lambda}}_{M}$ -mod *is the tensor product of Fock spaces for the weights* λ_i*. The classes of indecomposables give the canonical basis for the bar involution discussed earlier.*

Higher levels

Yet a third thing you could have in mind is the twisted higher-level spaces in the sense of Uglov. These arise from choosing a list of integers $(c_1, \dots, c_\ell)$, and q-deform the tensor product of level 1 Fock spaces with these charges.

The key difference between Uglov's space and the tensor product is that they are associated to different partial orders on the set of ℓ-multipartitions. multi-partitions

- The tensor product quantization corresponds to the usual dominance order.
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Higher levels

The categorifical manifestation of the first order is that the coarsest partial order on the set of Weyl modules for which they form an exceptional collection is the dominance order.

Thus, we can mutate our exceptional collection with respect to the change of order from $\leq$ to $\prec_{\mathbf{c}}$, to obtain a new exceptional collection in $D^b(A^{\boldsymbol{\lambda}}_M\text{-mod})$ which we call **c-twisted Weyl modules.**

In particular, this exceptional collection and its dual define a new t-structure on $D^b(A^{\boldsymbol{\lambda}}_M\text{-mod})$, whose heart $\mathfrak{V}_{\mathbf{c}}$ is a highest weight category with standards given by the $\mathbf{c}$-twisted Weyl modules.

Conjecture

Uglov's higher level Fock spaces are the GG's of $\mathfrak{V}_{\mathbf{c}}$, with standard vectors given by the classes of $\mathbf{c}$-twisted Weyl modules and the canonical basis given by projective modules in $\mathfrak{V}_{\mathbf{c}}$.

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Thanks for listening.