Summary of Research

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(1) Overview. A long knot is an embedding $f : \mathbb{R}^1 \hookrightarrow \mathbb{R}^n$ whose behaviour at infinity are fixed in advance. If a trivialization $w : V_f \xrightarrow{\approx} D^{n-1} \times \mathbb{R}^1$ of the tubular neighborhood V_f of a long knot f is specified, then we call the pair (f, w) a framed long knot. We denote by $\mathcal{K}_n$ (resp. $\widetilde{\mathcal{K}}_n$) the space of all long knots (resp. framed long knots) with C^∞-topology. I am studying the topological properties of $\mathcal{K}_n$, in particular the case $n > 3$ (my interest is in the topology of $\mathcal{K}_n$ itself in large part, though this research is related to the classical knot theory).

Since the first study of $\mathcal{K}_3$ due to V. Vassiliev, there have been appeared many researches of $\mathcal{K}_n$: In [3] and others, K. T. Chen's iterated integrals for loop spaces has been refined, and the perturbative invariants for knots in dimension three and some elements of $H^*_{DR}(\mathcal{K}_n)$, the "higher dimensional analogues" of these invariants, have been constructed as linear combinations of graphs. In [2, 8] actions of little disks operad were defined in geometric or homotopy-theoretic ways, and consequently the spaces $\mathcal{K}_n$, $\widetilde{\mathcal{K}}_n$ turn out to have homotopy types of double loop spaces. A lot of other interesting topics cross each other at $\mathcal{K}_n$, for instant, relation between the Lie algebras associated with the lower central series of pure braid groups and $\pi_*(\mathcal{K}_n) \otimes \mathbb{Q}$, rational homotopy theory of configuration spaces $\mathrm{Conf}\,(\mathbb{R}^n, k)$, and so on. I am mainly studying the Poisson algebra structure on $H_*(\mathcal{K}_n)$ induced by the double loop structure of $\mathcal{K}_n$, through associations with many perspectives mentioned above.

(2) Results. At present, for $n > 3$, explicit construction of elements of $H_*(\mathcal{K}_n)$ has not been done enough. For the construction, I made use of the Poisson bracket λ on $H_*(\mathcal{K}_n)$ and $H_*(\widetilde{\mathcal{K}}_n)$ induced by the actions of little disks operad (which is defined in at least three ways). Poisson bracket is a graded Lie bracket satisfying the Leibniz rule;

$$\lambda : H_p(X) \otimes H_q(X) \longrightarrow H_{p+q+1}(X), \quad \lambda(x, yz) = \lambda(x, y)z \pm y\lambda(x, z).$$

Here products xy etc. are induced by the connecting sum of the long knots. In general there is no way to determine $\lambda(x, y)$ is zero or not for any x, y. But I have proved that $\lambda(x, y)$ is not zero for some x, y in the following ways.

(2-1) We can define a cochain map I from certain graph complex to de Rham complex of $\widetilde{\mathcal{K}}_n$ via the iterated integrals (which was constructed in [3] for $\mathcal{K}_n$, and I generalized it to the case of $\widetilde{\mathcal{K}}_n$ in [6]). We say a graph is *trivalent* if exactly three edges emanate from each vertices of the graph. The elements in $H^*_{DR}(\widetilde{\mathcal{K}}_n)$ obtained from trivalent graph cocycles via iterated integral correspond to integral expressions of finite type invariants in dimension three. The dual elements in $H_*(\widetilde{\mathcal{K}}_n)$ to the trivalent cocycles have been given in [3] using the chord diagrams (which are trivalent graphs). In contrast, almost nothing has been known about non-trivalent graphs, because of combinatorial difficulties.

Firstly I constructed an example Γ of non-trivalent graph cocycle explicitly when $n > 3$ is odd. Then I computed $\lambda(x, y)$ determined by the operad action in the sense of [2] for some elements $x \in H_{n-3}(\widetilde{\mathcal{K}}_n)$, $y \in H_{2(n-3)}(\widetilde{\mathcal{K}}_n)$ corresponding to chord diagrams to prove the following.

Theorem 1 ([7])**.** When $n > 3$ is odd, the integration of $I(\Gamma)$ over the cycle $\lambda(x, y) \in H_{3(n-3)+1}(\widetilde{\mathcal{K}}_n)$ is not zero. Thus (co)homology classes $I(\Gamma)$, $\lambda(x, y)$ are not trivial, and moreover do not arise from trivalent graphs. □

(2-2) In [5, 6] I studied $\mathcal{K}_n$ from the viewpoint of homotopy theory. When $n > 3$, it is known that $\mathcal{K}_n$ is represented as a "totalization" of some co-simplicial space [8]. In [4] actions of operads on such totalizations were defined, hence a Poisson structure is induced on $H_*(\widetilde{\mathcal{K}}_n)$ (which does not necessarily coincide with that mentioned above). On the other hand, in [1] a spectral sequence converging to the homology group of a totalization was constructed. Its E^2-term is so-called a "Hochschild homology" which also appears in mathematical physics and so on, and on it a Poisson bracket is defined in an algebraic way [9]. In the following theorem I compare these two Poisson structures.

Theorem 2 ([5, 6])**.** The above two Poisson algebra structures are same on $H_*(\widetilde{\mathcal{K}}_n)$. In particular, by computing the spectral sequence, we can compute Poisson in the sense of [4, 8]. □

The cycle $\lambda(x, y)$ from Theorem 1 can be obtained by using the Poisson bracket in the sense of Theorem 2 [5, 6, 9].

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