Research results

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Let G be a connected semisimple group over a number field k. Writing $\mathbb{A}$ for the adele ring of k, $G(k)\backslash G(\mathbb{A})$ becomes a locally compact space, and has a $G(\mathbb{A})$-invariant measure. By this we can consider the space $L^2(G(k)\backslash G(\mathbb{A}))$ of square-integrable functions on $G(k)\backslash G(\mathbb{A})$. The representation on it induced by the right regular action of $G(\mathbb{A})$ has an orthogonal decomposition,

$$L^2(G(k)\backslash G(\mathbb{A})) = L^2_{\mathrm{disc}}(G(k)\backslash G(\mathbb{A})) \oplus L^2_{\mathrm{cont}}(G(k)\backslash G(\mathbb{A})),$$

where $L^2_{\mathrm{disc}}(G(k)\backslash G(\mathbb{A}))$ is the maximal completely reducible closed subspace and $L^2_{\mathrm{cont}}(G(k)\backslash G(\mathbb{A}))$ is its orthogonal complement. I considered the *residual spectrum* and the space of *CAP forms* for a specific G, which are two invariant subspaces of $L^2_{\mathrm{disc}}(G(k)\backslash G(\mathbb{A}))$ closely related to each other. I shall explain these spaces concretely. As G we take the unitary group of the rank 2 hyperbolic hermitian space over a quaternion division algebra R over k. It is an inner form of $Sp(2)$. We say that an element ϕ of $L^2(G(k)\backslash G(\mathbb{A}))$ is L^2-cusp form if the constant terms of ϕ along all the proper k-parabolic subgroup vanish. It is known that the space $L^2_0(G(k)\backslash G(\mathbb{A}))$ of L^2-cusp forms of G is contained in $L^2_{\mathrm{disc}}(G(k)\backslash G(\mathbb{A}))$.

The residual spectrum is the orthogonal complement of $L^2_0(G(k)\backslash G(\mathbb{A}))$ in $L^2_{\mathrm{disc}}(G(k)\backslash G(\mathbb{A}))$. I determined the irreducible decomposition of the residual spectrum of G completely when k is totally real. $GL(n)$, $Sp(2)$, $U(2,2)$ are examples of which the irreducible decomposition of the residual spectrum are completely determined. There are other examples such that the irreducible decomposition of their residual spectrum are partially determined. These examples are all quasisplit algebraic group. Since Langlands' spectral theory of Eisenstein series is applied to determine the residual spectrum we need to know an analytic behavior of Eisenstein series. In case of a quasisplit group we can know the analytic behavior of Eisenstein series from the Langlands-Shahidi theory. However G is not quasisplit group. I reduced the problem of the analytic behavior of Eisenstein series for G to that for $Sp(2)$ using the Jacquet-Langlands correspondence. Some irreducible components of the residual spectrum of G are determined by the Langlands classification, and the others are constructed by the theta lifts of the trivial representation of the unitary groups of rank 1 skew-hermitian spaces.

The space of CAP forms is a subspace of $L^2_0(G(k)\backslash G(\mathbb{A}))$. We say that an L^2-cusp form ϕ is CAP form if there exists an element ϕ' of an irreducible component of the residual spectrum such that ϕ and ϕ' share the same absolute values of Hecke eigenvalues at almost all places of k. I constructed many examples of CAP forms of G. In case of $GSp(2)$, Piatetski-Shapiro constructed the Saito-Kurokawa representations as examples of CAP forms, and Soudry determined the other CAP forms. As we know that "CAP" is the abbreviation of "Cuspidal Associated to Parabolic", a CAP representation is associated to a parabolic subgroup of G. It is not necessary that the parabolic subgroup is defined over k. I constructed two kinds of CAP representations. One of them is given by the theta correspondents from CAP representations of the unitary group $U(V)$ of rank 2 skew-hermitian space V of determinant 1 over R. These are the inner form version of the Saito-Kurokawa representations and the examples given by Howe and Piatetski-Shapiro, and the associated parabolic subgroup of G is defined over k. Another is given by the theta correspondents from automorphic representations of the unitary group of rank 1 skew-hermitian space. Many of these correspond to automorphic representations in residual spectrum of G, and the associated parabolic subgroup of G is not defined over k. Following Arthur's conjecture, all of CAP representations except for a part of Saito-Kurokawa representations should be exhausted in this manner.

There exists a CAP form which does not satisfy the multipliticity one property. This phenomenon does not occur in case of $Sp(2)$. This is caused by the failure of the Hasse principle. That is, there exist two skew-hermitian spaces which are not isometric globally but isometric locally. These produce different spaces, but which are isomorphic as representations. The multiplicity obtained in this way coincides with Arthur's multiplicity formula.