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In a study of low-dimensional topology, it is fundamental and important to characterize manifolds by using algebraic or combinatorial methods since such methods are (easily) computable in many cases and hence effective in classifying manifolds. In my future research, I am going to study five objects as follows.

Object 1: Smooth structures of 4-manifolds

One of my projects is to study smooth strucutures of 4-manifolds by using invariants of knots and links. In Period 7, we obtained methods to study smooth structure of a non-compact 4-manifold by using concordance invariants of knots and links. To accomplish new results, it is necessary to use or discover new, effective invariant. We cannot apply any algebraically defined invariant to the research since we must classify homeomorphic 4-manifolds. In this research, we consider the following problems.

Problem 1. Does every Casson handle have more than one smooth structure?
Problem 2. Does every non-compact, smooth 4-manifold have an uncountable number of smoothings?
Problem 3. Do all closed, smooth 4-manifolds have more than one smooth structure?

Problem 3 contains smooth Poincaré conjecture.
In the 1990's, Lee Rudolph discovered an invariant of a knot via contact geometry. Though the invariant is not concordance invariant, he found that it could be used to give an exotic Casson handles. Later, Z. Bizaca showed a concrete example of an exotic Casson handle by using Rudolph's invariant. (Here, an exotic Casson handle is a non-compact 4-manifold which is homeomorphic but not diffeomorphic to the open 2-handle.) We studied this invariant for positive knots, alternating knots and doubles of knots and obtained formulas of the invariant for them in Periods 1 and 7. By studying this invariant more deeply and producing formulas, we want to apply our method to more large class of Casson handles to solve Problem 1.
On the other hand, in the last five years, J. Rasmussen discovered new invariant by using Lee's variant of the Khovanov homology. He showed that the invariant is a concordance invariant and that it provides a lower bound for the slice genus of a knot. As a result, he gave a purely combinatorial proof of the Milnor conjecture. Since Rasmussen's invariant is concordance invariant and has many good properties, we expect to give many new results for studying 4-manifolds by using our methods. We want to produce new formulas of Rasmussen's invariant to apply it for investigating smooth structures of manifolds.
In fact, we can show the existence of an exotic structure of 4-dimensional space by using Rasmussen's invariant. This fact shows that the existence of an exotic structure of a 4-dimensional space is shown by using topological methods and combinatorial methods. It is surprising because the existence of such a structure was only shown via gauge theory so far.
I have shown the existence of exotic smooth structures of a certain infinite family of Casson handles using Rasmussen's invariant. I also have shown that every non-compact, connected smooth 4-submanifold of the 4-dimensional space has an exotic smooth structure by using the invariant.

Therefore we admit the usefulness of this research. On the other hand, we have obtained similar result for any non-compact, connected smooth 4-submanifold of a connected sum of any number of CP^2 by using Donaldson's theory. It is our assignment to consider if such 4-submanifolds admit infinitely many exotic smooth structures. We expect to be able to show that all Casson handles and some other 4-manifolds admit exotic smooth structures for giving answers to Problem 1 and 2 by advancing this method. We want to advance our reseach on Problem 2 to investigate how to apply our method to compact 4-manifolds. Moreover, we want to find out what causes the richness of smooth structures peculiar to 4-manifolds combinatorially.

Object 2: **Geometric structures of 3-manifolds reflected in quantum invariants**

The research related with the volume conjecture which has suggested the relation of the simplicial volume (Gromov invariant) of the complement of a knot and the colored Jones polynomial shows that a quntum invariant is deeply related to a geometric structure of a manifold. For example, the colored Jones polynomial of a figure 8 knot gives the hyperbolic volume of its complement as a certain limit. I want to investigate a relation between the Jones polynomial of a knot and a geometric structure of a 3-manifold by considering this conjecture. I have already obtained a simple formula for the colored Jones polynomial of a double of a knot in Period 4. Therefore I want to consider if the volume conjecture holds for doubles of knots. If the answer is true, then we have already confirmed that the colored Jones polynomial can detect the unknot.

Object 3: Slice knots

A slice knot is a knot which bounds a smooth disk in the 4-ball. A ribbon knot is a knot which bounds a singular disk in 3-space with only ribbon singularities. Every ribbon knot is a slice knot. But the following problem is still open.

Problem 4. (R. H. Fox (1962)) Is every slice knot a ribbon knot?

To consider this problem, we want to characterize ribbon knots more explicitly. S. Kinoshita and H. Terasaka studied a way of connecting a knot and its mirror image which is called a symmetric union in the 1950's. Recently, C. Lamm generalized the operation and the definition of symmetric union. He showed that any symmetric union is a ribbon knot and gave the following problem.

Problem 5. (C. Lamm (2000)) Are all ribbon knots symmetric unions?

It is known that every (ribbon) 2-knot has a ribbon knot as an equatorial cross section. Hence, we consider the following problem instead of Problem 5.

Problem 6. Are all ribbon 2-knots have symmetric unions as their equatorial cross sections?

We confirmed this problem for a ribbon 2-knot of 1-fusion. We want to consider if Problem 5 holds for more general class of ribbon 2-knots.

Object 4: Khovanov homology, link Floer homology and the Gordian distance for knots and links

For two knots K and K', the Gordian distance between K and K' is the minimum number of crossing changes needed to transform a diagram of K into that of K', where the minimum is taken over all diagrams of K and K'. The Grodian distance is a generalization of the unknotting number.

We want to make a complete table of the Gordian distance at least for knots with up to 10 crossings by using the signauture, polynomial invariants, Rasmussen's invariant, double branched covering spaces of 3-sphere, Khovanov homology and link Floer homology. We also want to apply the same methods to estimate the ribbon number of a ribbon knot to study Problem 5.