Results of research

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I have been studying on compactifications of infinite, connected graphs and complete, noncompact, connected Riemannian manifolds relative to the space of bounded functions with finite p-Dirichlet sums and properties invariant under quasi isometries. In what follows, a graph (resp. a Riemannian manifold) is assumed to be infinite and connected (resp. complete, noncompact and connected).

Although a quasi isometry do not need to be continuous and to preserve topological properties, it is known that it preserves some properties related to points at infinity. For example, the volume growth, the isoperimetric inequality and the existence of non constant harmonic functions with finite p-Dirichlet sums are invariant under a quasi isometry between graphs of bounded degrees. The space of all bounded functions with finite p-Dirichlet sums on a graph becomes the Banach algebra with unit element. Thus there exists a compact Hausdorff space, unique up to homeomorphisms, satisfying the following conditions: (1) the set of vertices is embedded in it as an open and dense subset; (2) every p-Dirichlet finite function extends to a continuous function on it; (3) the extended function separates two points of its boundary. We call it the Royden p-compactification and the boundary obtained by removing the set of vertices from the Royden p-compactification is called the Royden p-boundary. For Riemannian manifolds, we can define Royden p-compactifications in the same way.

For two graphs of bounded degrees which are quasi isometric, a quasi isometry between them extends to a continuous mapping between the Royden p-compactifications of them; the restriction of the extended continuous mapping to the Royden p-boundary induces a homeomorphism such that the image of the Royden p-harmonic boundary coincides with the Royden p-harmonic boundary of the target space. Moreover, we have a bijective correspondence between the spaces of bounded p-harmonic functions with finite p-Dirichlet sums. For Riemannian manifolds with bounded geometry, we can define a graph with bounded degree which is quasi isometric to the manifold, called a net of the manifold. By using this, we can prove similar results to the above one.

To show that the Royden p-boundaries of two quasi isometric graphs are homeomorphic, we use a particular property of graphs with bounded degree. This does not hold for Riemannian manifolds. In fact, we can not expect the same results for Riemannian manifolds without any restriction of exponents p. It is verified that the Royden p-boundaries of two quasi isometric Riemannian manifolds with bounded geometry whose dimensions are less than p are homeomorphic. In the case when the dimension of a Riemannian manifold is not less than p, any set consisting of a single point of the Royden p-boundary is not a G_δ-set. On the other hand, the Royden p-harmonic boundary of a graph is possible to be a G_δ-set. We made some observations obtained by varying the exponents p.

In addition, we construct a Riemannian manifold that (1) it is quasi isometric to the hyperbolic space form of dimension 3; (2) it admits non constant bounded harmonic functions of finite Dirichlet energies; (3) it is not of bounded geometry. This example illustrates a role of the condition of bounded geometry.