Previous achievements

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I distinguish my previous achievements into two parts mainly.

(i) One way I have been studied is to research Sobolev type inequalities in various directions. I strongly believe that this research surely reaches to the development of the real analysis such as embeddings between functional spaces, and its applications to the nonlinear partial differential equations. Especially, I focus on the following Gagliardo-Nirenberg type interpolation inequality obtained by T.Ozawa [∗∗], which implies the embedding between the Lebesgue space and the critical Sobolev space :

$$\|u\|_{L^q} \le C\, q^{1-\frac{1}{p}} \|u\|_{L^p}^{\frac{p}{q}} \|(-\Delta)^{\frac{n}{2p}} u\|_{L^p}^{1-\frac{p}{q}} \tag{1}$$

holds for all $u \in H^{\frac{n}{p},p}(\mathbb{R}^n)$, where $n \in \mathbb{N}$, $1 < p \le q < \infty$ and C depends only on n and p. It is well-known that $H^{s,p}(\mathbb{R}^n)$ cannot be embedded into $L^\infty(\mathbb{R}^n)$ if $s = \frac{n}{p}$, unlike the higher order case $s > \frac{n}{p}$, and we see that the growth order $q^{1-\frac{1}{p}}$ as $q \to \infty$ in (1) is optimal. In the paper [1], we consider the best constant of C in (1), and as a result, we succeed in giving the lower bound of it. In particular case $p = 2$, our lower bound is actually the best constant.

As another way to investigate (1) further, we try to obtain similar inequalities in various functional spaces systematically. Indeed, in the paper [2], we prove (1) in terms of the function space BMO, which is a class of functions having bounded mean oscillation with the optimal growth order as $q \to \infty$. As a corollary of this inequality, we easily get the so-called John-Nirenberg inequality which guarantees the exponential decay of the distribution function belonging to BMO. Moreover, in the papers [3, 10], we reconstruct (1) in terms of the Besov and the Triebel-Lizorkin spaces, which are defined through the real interpolation between Sobolev spaces based on Littlewood-Paley functions. The papers [4, 11] are also concerned with a generalization of (1). In the papers [4, 11], we prove weighted Gagliardo-Nirenberg type inequalities including (1) and give the optimal singular weight so that (1) remains to hold.

(ii) The second purpose in my research is to apply the inequalities stated in **(i)** into the nonlinear partial differential equations. I mainly discuss the existence or the nonexistence of positive solutions to some nonlinear elliptic equations. In particular, we investigate the elliptic equation including the critical nonlinear term in the sense of the Sobolev-Hardy inequality. In order to get the positive solution to this equation, we consider the minimizing problem of the best constant of the Sobolev-Hardy inequality. In the case of the usual Sobolev inequality, it is well-known that the corresponding best constant never attains its minimizer unless the domain is the whole space. However, by virtue of the singular weight at the origin of the Sobolev-Hardy inequality, this problem strictly depends on the geometric assumption of the domain. Indeed, N.Ghoussoub-F.Robert [∗] proves the existence of a minimizer corresponding to the best constant of the Sobolev-Hardy inequality provided that the origin of the smooth bounded domain lies on the boundary, and the mean curvature at this point is strictly negative. They apply the blow-up analysis and Pohozaev's identity for the proof in [∗], which are very detailed. However, in the papers [5, 8], we give comparatively an easier proof of the existence of a minimizer to the similar Sobolev-Hardy type variational by combining the blow-up analysis with the method of Brézis-Nirenberg. As a result, we succeed in constructing a positive solution to the following elliptic equation :

$$\begin{cases} \Delta u + \mu\, u^{\frac{2n}{n-2}-1} + \frac{u^{2^*(s)-1}}{|x|^s} = 0 \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \quad 2^*(s) := \frac{2(n-s)}{n-2}, \end{cases} \tag{2}$$

where $n \ge 3$, $0 < s < 2$, $\mu > 0$ is some positive constant, Ω is smooth bounded domain with $0 \in \partial\Omega$, and the mean curvature of $\partial\Omega$ at 0 is negative. The hypothesis for negativity of the mean curvature at 0 is one sufficient condition so that the equation (2) admits a positive solution. Though we do not know this condition is also necessary or not, but seems to be rather essential. Indeed, if we assume the domain to be star-shaped with respect to $0 \in \partial\Omega$, we see that the equation (2) has no positive solution through Pohozaev's identity. Note that star-shaped domains with respect to $0 \in \partial\Omega$ have nonnegative mean curvatures at this point.

• Related references

[∗] N.Ghoussoub, F.Robert, *The effect of curvature on the best constant in the Hardy-Sobolev inequalities*, Geom. Funct. Anal. **16**, 1201-1245 (2006).

[∗∗] T.Ozawa, *On critical cases of Sobolev's inequalities*, J. Funct. Anal. **127**, 259-269 (1995).