WEAKLY REFLECTIVE ORBITS AND TANGENTIALLY DEGENERATE ORBITS OF s-REPRESENTATIONS *

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Introduction

The linear isotropy representations of Riemannian symmetric spaces are called s-representations, and their orbits are called R-spaces. They provide various examples of homogeneous submanifolds in the hypersphere of the Euclidean space and play important role in the area of differential geometry. For example, we know the following facts:

- It is a well-known fact due to Hsiang-Lawson [9] that all homogeneous hypersurfaces in the sphere can be obtained as principal orbits of s-representations of Riemannian symmetric pairs of rank two.
- Dadok [6] showed that all polar presentations are orbit equivalent to s-representations.
- All Kähler C-spaces, that are compact simply-connected homogeneous Kähler manifolds, can be obtained as adjoint orbits of semisimple Lie groups.
- Thorbergsson [22] showed that irreducible isoparametric submanifolds in the Euclidean space are R-spaces if their codimension is greater than 2 and if they are not contained in any hyperplane.
- Ferus [8] introduced the notion of symmetric submanifold in a Riemannian manifold. He proved that all symmetric submanifolds in the Euclidean space are symmetric R-spaces. Recently, Naitoh [20] and Berndt-Eschenburg-Naitoh-Tsukada [2] classified symmetric submanifolds in Riemannian symmetric spaces. In their proof, symmetric R-spaces played an essential role.

In the present paper, we shall study some differential geometric properties of orbits of s-representations as submanifolds in the sphere.

Hirohashi-Song-Takagi-Tasaki [11] proved that in each strata of the stratification of the orbit space of an s-representation there exists uniquely an orbit which is a minimal submanifold in the sphere. However, in general we can not explicitly point out which orbit among each strata is a minimal submanifold. Harvey-Lawson [9] introduced the notion of austere submanifold, which is a minimal submanifold whose second fundamental form has a certain symmetry. They showed that one can construct special Lagrangian cones, therefore absolutely area-minimizing, in the complex Euclidean space as the normal bundles of austere submanifolds in the

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sphere (see [9], [4]). In Section 2, we shall give the definition of weakly reflective submanifold, which is an austere submanifold with a certain global symmetry. As we mentioned above, the complete list of minimal orbits of s-representations in the hypersphere is unknown at the moment. Therefore we shall determine all orbits which are austere submanifolds or weakly reflective submanifolds in the hypersphere.

A submanifold is called tangentially degenerate if its Gauss mapping is degenerate. Ferus [8] obtained a remarkable result for tangentially degeneracy of submanifolds in the sphere. He showed that there exists a number, so-called the Ferus number, such that if the rank of the Gauss mapping is less than the Ferus number, then a submanifold must be a totally geodesic sphere. However, in general it is still unknown whether there exist submanifolds which satisfy the Ferus equality, that is, the equality of the Ferus inequality. In their papers [15, 16], Ishikawa, Kimura and Miyaoka studied submanifolds with degenerate Gauss mappings in the sphere via a method of isoparametric hypersurfaces. They showed that Cartan hypersurfaces and some focal submanifolds of homogeneous isoparametric hypersurfaces are tangentially degenerate. Moreover, some of them satisfy the Ferus equality. In Section 3, we shall study submanifolds with degenerate Gauss mappings via a method of symmetric spaces. We give a classification of tangentially degenerate orbits of s-representations. We note that these orbits are weakly reflective submanifolds. We observe that these orbits provide many new examples of tangentially degenerate submanifolds in the sphere which satisfy the Ferus equality.

In Section 4, we shall show some examples of austere orbits, weakly reflective orbits or tangentially degenerate orbits of s-representations. In the last section, we shall pose some open problems related to this article.

1. Orbits of s-representations

The linear isotropy representation of a Riemannian symmetric pair is called an s-representation as we mentioned in Introduction. In the following sections, we will study orbits of s-representations which are austere, weakly reflective or tangentially degenerate in the hypersphere. For this purpose, we shall provide some fundamental notions of orbits of s-representations in this section.

Let G be a compact, connected Lie group and K a closed subgroup of G. Assume that θ is an involutive automorphism of G and $G_\theta^0 \subset K \subset G_\theta$, where

$$G_\theta = \{g \in G \mid \theta(g) = g\}$$

and G_θ^0 is the identity component of G_θ. Then (G, K) is a symmetric pair with respect to θ. We denote the Lie algebras of G and K by $\mathfrak{g}$ and $\mathfrak{k}$, respectively. The involutive automorphism of $\mathfrak{g}$ induced from θ will be also denoted by θ. Then $\mathfrak{g}$ is decomposed to

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$$

where $\mathfrak{m}$ is the (-1)-eigenspace of θ. The tangent space of a compact symmetric space G/K at the origin $o = K$ can be identified with $\mathfrak{m}$ in a natural manner. The linear isotropy representation of G/K is isomorphic to the action of K on $\mathfrak{m}$ by the adjoint representation of G. Therefore we shall denote the orbit of K-action through $H \in \mathfrak{m}$ by $\mathrm{Ad}_G(K)H$, or for simplicity by $\mathrm{Ad}(K)H$. Since the action of K on $\mathfrak{m}$ is an orthogonal representation, $\mathrm{Ad}(K)H$ is a submanifold of the hypersphere S in $\mathfrak{m}$ of radius $\|H\|$.

Fix a maximal abelian subspace $\mathfrak{a}$ in $\mathfrak{m}$. For $\lambda \in \mathfrak{a}$ we set a subspace $\mathfrak{m}_\lambda$ of $\mathfrak{m}$ by

$$\mathfrak{m}_\lambda = \{X \in \mathfrak{m} \mid [H,[H,X]] = -\langle \lambda, H\rangle^2 X \quad (H \in \mathfrak{a})\}$$

and define the restricted root system R of $(\mathfrak{g}, \mathfrak{k})$ by $R = \{\lambda \in \mathfrak{a} \mid \mathfrak{m}_\lambda \neq \{0\}\}$. We take a fundamental system F of R and denote by R_+ the set of all positive roots with respect to F. Then we have the restricted root space decomposition of $\mathfrak{m}$ as

$$\mathfrak{m} = \mathfrak{a} + \sum_{\lambda \in R_+} \mathfrak{m}_\lambda.$$

We set a Weyl chamber $\mathcal{C}$ by

$$\mathcal{C} = \{H \in \mathfrak{a} \mid \langle \alpha, H\rangle > 0 \ (\alpha \in F)\}.$$

Then $\mathcal{C}$ is an open convex subset of $\mathfrak{a}$. From a fundamental result of compact symmetric spaces, we have the following proposition.

Proposition 1.1.

$$\mathrm{Ad}(K)\bar{\mathcal{C}} = \mathfrak{m},$$

where $\bar{\mathcal{C}}$ is the closure of $\mathcal{C}$. More precisely, $\bar{\mathcal{C}}$ meets all orbits perpendicularly, and each orbit $\mathrm{Ad}(K)H$ through only one point of $\bar{\mathcal{C}}$.

Hence, the orbit space of the action of K on the hypersphere S can be identified with $S \cap \bar{\mathcal{C}}$. Therefore we can assume that H is in $S \cap \bar{\mathcal{C}}$.

For a subset $\Delta \subset F$, we define

$$\mathcal{C}^\Delta = \{H \in \bar{\mathcal{C}} \mid \langle \alpha, H\rangle > 0 \ (\alpha \in \Delta),\ \langle \beta, H\rangle = 0 \ (\beta \in F - \Delta)\}.$$

Lemma 1.2. (1) *For $\Delta_1 \subset F$, the decomposition*

$$\overline{\mathcal{C}^{\Delta_1}} = \bigcup_{\Delta \subset \Delta_1} \mathcal{C}^\Delta$$

is a disjoint union. In particular, $\bar{\mathcal{C}} = \bigcup_{\Delta \subset F} \mathcal{C}^\Delta$ is a disjoint union.

(2) *For $\Delta_1, \Delta_2 \subset F$, $\Delta_1 \subset \Delta_2$ if and only if $\mathcal{C}^{\Delta_1} \subset \overline{\mathcal{C}^{\Delta_2}}$.*

For $H \in \mathfrak{m}$ we set

$$Z_K^H = \{k \in K \mid \mathrm{Ad}(k)H = H\}.$$

Then Z_K^H is a closed subgroup of K. The orbit $\mathrm{Ad}(K)H$ through H can be expressed as a homogeneous space $\mathrm{Ad}(K)H \cong K/Z_K^H$.

For $\Delta \subset F$ we set

$$\begin{aligned} N_K^\Delta &= \{k \in K \mid \mathrm{Ad}(k)\mathcal{C}^\Delta = \mathcal{C}^\Delta\}, \\ Z_K^\Delta &= \{k \in K \mid \mathrm{Ad}(k)|_{\mathcal{C}^\Delta} = 1\}. \end{aligned}$$

Then N_K^Δ and Z_K^Δ are closed subgroups of K.

Proposition 1.3 ([10])**.** *For $\Delta \subset F$ and $H \in \mathcal{C}^\Delta$ we have*

$$Z_K^\Delta = Z_K^H = N_K^\Delta.$$

From Lemma 1.2, the orbit space of the action of K on the hypersphere S is decomposed to

$$S \cap \bar{\mathcal{C}} = \bigcup_{\Delta \subset F} (S \cap \mathcal{C}^\Delta).$$

From Proposition 1.3, we have that if $\Delta \subset F$ and $H_1, H_2 \in \Delta$, then $Z_K^{H_1} = Z_K^{\Delta} = Z_K^{H_2}$, hence $\mathrm{Ad}(K)H_1$ is diffeomorphic to $\mathrm{Ad}(K)H_2$. This means that all orbits are the same isotropy type in each strata. Moreover we have that if $\Delta_1 \subset \Delta_2 \subset F$ and $H_1 \in \Delta_1, H_2 \in \Delta_2$, then $\mathcal{C}^{\Delta_1} \subset \overline{\mathcal{C}^{\Delta_2}}$, hence $Z_K^{H_1} = Z_K^{\Delta_1} \supset Z_K^{\Delta_2} = Z_K^{H_2}$. Thus an orbit which is in the interior $S \cap \mathcal{C}$ of the orbit space is a principal orbit. If $\Delta \subset F$ is a proper subset, then the orbit through $\mathcal{C}^{\Delta} \cap S$ is a singular orbit. Consequently we have the stratification of the isotropy types for the orbits of s-representations.

For orbits of s-representations which are minimal submanifolds in the hypersphere, the following theorem is known.

Theorem 1.4 ([11]). *For each subset $\Delta \subset F$, there exists a unique $H \in \mathcal{C}^{\Delta} \cap S$ such that $\mathrm{Ad}(K)H$ is a minimal submanifold in S.*

2. Classification of weakly reflective orbits

As in Theorem 1.4, there exists a unique minimal orbit in each strata of the stratification of the orbits space of an s-representation. However, it is difficult to determine minimal orbits explicitly in general. In this section, we shall introduce the notion of weakly reflective submanifold, which is an austere submanifold, hence minimal, with a certain global symmetry. We shall give the classification of all orbits of irreducible s-representations which are austere submanifolds or weakly reflective submanifolds in the hypersphere S.

Definition 2.1. Let M be a submanifold of a Riemannian manifold $\tilde{M}$. For each normal vector $\xi \in N_xM$ at each point $x \in M$, if there exists an isometry σ_ξ of $\tilde{M}$ which satisfies

$$\sigma_\xi(x) = x, \qquad (d\sigma_\xi)_x \xi = -\xi, \qquad \sigma_\xi(M) = M, \tag{2.1}$$

then we call M a *weakly reflective submanifold* and σ_ξ a *reflection* of M with respect to ξ.

A connected component M of the fixed point set of an involutive isometry σ of a Riemannian manifold is called a *reflective submanifold* ([19]). For a reflective submanifold M, the reflection σ satisfies the conditions (2.1) for all normal vectors at all points in M. Hence it is evident that a reflective submanifold is a weakly reflective submanifold.

Definition 2.2. Let M be a submanifold of a Riemannian manifold $\tilde{M}$. We denote the shape operator of M by A. M is called an *austere submanifold* if for each normal vector $\xi \in N_xM$, the set of eigenvalues with their multiplicities of A_ξ is invariant under the multiplication by -1. It is obvious that an austere submanifold is a minimal submanifold.

For these classes of submanifolds the following relation holds.

Proposition 2.3. *reflective $\subset$ weakly reflective $\subset$ austere $\subset$ minimal*

The following proposition was essentially proved by Podentà [21].

Proposition 2.4 ([21], [13]). *Any singular orbit of a cohomogeneity one action on a Riemannian manifold is a weakly reflective submanifold.*

Cohomogeneity one actions on compact symmetric spaces were completely classified by Kollross [18]. And non-compact cases were studied by Berndt-Tamaru [3].

From these results we obtain several examples of weakly reflective submanifolds in Riemannian symmetric spaces.

Now we shall give the classification of orbits of irreducible s-representations which are austere submanifolds or weakly reflective submanifolds in the hypersphere. Since these two properties of orbits are invariant under scalar multiples on the vector spaces, we do not discriminate the difference of the length of a vector. We shall follow the notations of root systems in [1].

Theorem 2.5 ([13])**.** *An orbit of an irreducible s-representation which is an austere submanifold in the hypersphere is one of the following list:*

1. *an orbit through a restricted root vector,*
2. *the orbit through the vector $2e_1 - e_2 - e_3$ and the orbit through the vector $e_1 + e_2 - 2e_3$ of the linear isotropy representation of a compact symmetric pair with the restricted root system $\{\pm(e_i - e_j)\}$ of type A_2,*
3. *the orbit through the vector $e_1+e_2-e_3-e_4$ of the linear isotropy representation of a compact symmetric pair with the restricted root system $\{\pm(e_i-e_j)\}$ of type A_3,*
4. *the orbit through the vector e_1 of the linear isotropy representation of a compact symmetric pair with the restricted root system $\{\pm(e_i \pm e_j)\}$ of type D_p,*
5. *the orbit through the vector $e_1+e_2+e_3+e_4$ and the orbit through the vector $e_1+e_2+e_3-e_4$ of the linear isotropy representation of a compact symmetric pair with the restricted root system $\{\pm(e_i \pm e_j)\}$ of type D_4,*
6. *the orbit through the vector $e_1 + \frac{e_1+e_2}{\sqrt{2}}$ of the linear isotropy representation of a compact symmetric pair with the restricted root system $\{\pm e_i, \pm e_i \pm e_j\}$ of type B_2 whose multiplicities are constant,*
7. *the orbit through the vector $\alpha_1 + \frac{\alpha_2}{\sqrt{3}}$ of the linear isotropy representation of a compact symmetric pair with the restricted root system of type G_2, where $\alpha_1 = e_1 - e_2$ and $\alpha_2 = -2e_1 + e_2 + e_3$.*

Furthermore, in the case of (1)∼(5) the orbits are weakly reflective submanifolds in S. In the case of (6) and (7) the orbits are not weakly reflective submanifolds.

Special Lagrangian normal bundles

Originally the notion of austere submanifold was introduced by Harvey-Lawson [9] in order to construct special Lagrangian cones in $\mathbb{C}^n$.

Let M be an austere submanifold in S^n. We denote by N^1M the unit normal bundle of M in S^n. We define a map

$$\begin{array}{rcl}\Phi : N^1M \times S^1 & \longrightarrow & S^{2n+1} \subset \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \\ (v_x, e^{i\theta}) & \longmapsto & (\cos\theta x,\ \sin\theta v_x).\end{array}$$

Then Harvey-Lawson proved that Φ is minimal Legendrian if and only if M is an austere submanifold in S^n. Therefore, then the cone over the image of Φ is special Lagrangian in $\mathbb{C}^n$. However, Φ fails to be an immersion in general. Later, Borrelli-Gorodski [4] defined a map Ψ modifying Φ. They showed that if for any normal vector ξ the shape operator A_ξ of M does not have 0-eigenvalue, then Ψ is an immersion. Hence, considering cones over these minimal Legendrian submanifolds we obtain special Lagrangian cones with conical singularities.

As a natural generalization of Harvey-Lawson's normal bundle construction, recently Karigiannis and Min-Oo [17] showed that the normal bundle $L = NM$ of a submanifold M in the sphere S^n is a special Lagrangian submanifold of the tangent bundle TS^n of S^n with Stenzel Calabi-Yau metric if and only if M is an austere submanifold in S^n.

3. Classification of tangentially degenerate orbits

In this section we shall give the classification of all tangentially degenerate orbits of s-representations. We will observe that all tangentially degenerate orbits are weakly reflective submanifolds in our list.

Let $f : M \longrightarrow S^n$ be an immersion of an l-dimensional manifold M into an n-dimensional sphere S^n. The Gauss mapping γ of f is defined as a mapping from M to a Grassmannian manifold $G_{l+1}(\mathbf{R}^{n+1})$ of all $(l+1)$-dimensional subspaces in $\mathbf{R}^{n+1}$ by:

$$\begin{array}{rcl}\gamma : M & \longrightarrow & G_{l+1}(\mathbf{R}^{n+1}) \\ x & \longmapsto & \mathbf{R}f(x) \oplus T_{f(x)}(f(M)).\end{array}$$

We denote by r the maximal rank of the Gauss mapping γ of an immersion f. If the Gauss mapping is degenerate, i.e. $r < l$, then an immersed submanifold $f(M) \subset S^n$ is said to be *tangentially degenerate*. We note that γ is constant, i.e. $r = 0$, if and only if $f(M)$ is a part of a totally geodesic sphere.

We denote by h and A the second fundamental form and the shape operator of f, respectively. Chern and Kuiper [5] introduced the notion of the *index of relative nullity* at $x \in M$, that is the dimension of the vector space

$$\begin{aligned}\mathcal{N}_x &= \{X \in T_x(M) \mid h(X,Y) = 0,\ {}^{\forall}Y \in T_x(M)\} \\ &= \bigcap_{\xi \in N_x M} \ker(A_\xi).\end{aligned}$$

It is easy to show $\ker(d\gamma)_x = \mathcal{N}_x$, therefore the index of relative nullity is equal to the degeneracy of the Gauss mapping at each point.

Let $f : M \longrightarrow S^n$ be an immersion of a compact, connected manifold M of dimension l. Ferus [8] showed that there exists a number $F(l)$, which only depends on the dimension l of M, such that the inequality $r < F(l)$ implies $r = 0$. Then $f(M)$ must be an l-dimensional great sphere in S^n. Here the number $F(l)$ is called the Ferus number and given by

$$F(l) = \min\{k \mid A(k) + k \geq l\},$$

where $A(k)$ is the Adams number, that is the maximal number of linearly independent vector fields at each point on the $(k-1)$-dimensional sphere S^{k-1}.

Regarding the Ferus inequality, Ishikawa, Kimura and Miyaoka posed the following problem:

Problem 3.1 ([16]). (1) *Is the inequality $r < F(l)$ best possible for the implication $r = 0$? Do there exist tangentially degenerate immersions $M^l \to S^n$ with $r = F(l)$?*
(2) *If the above problem is true, classify tangentially degenerate immersions $M^l \to S^n$ with $r = F(l)$.*

For these problems, they obtained the following results using isoparametric hypersurfaces in the sphere. It is well-known that the number g of distinct principal curvatures of an isoparametric hypersurface in the sphere is 1, 2, 3, 4 or 6. A minimal isoparametric hypersurface with $g = 3$ is called a Cartan hypersurface.

Theorem 3.2 ([15]). *A homogeneous compact hypersurface in the real projective space $\mathbb{R}P^n$ which is tangentially degenerate is projectively equivalent to a hyperplane or a Cartan hypersurface.*

Theorem 3.3 ([16]). *When M is a homogeneous isoparametric hypersurface in the sphere with $g = 6$, then both focal submanifolds of M are tangentially degenerate. Moreover, these submanifolds satisfy the Ferus equality.*

Theorem 3.4 ([16]). *When M is a homogeneous isoparametric hypersurface in the sphere with $g = 4$, then one of focal submanifolds of M is tangentially degenerate, and another one is not. Moreover, some of them satisfy the Ferus equality.*

In their work they investigated tangentially degenerate submanifolds via a theory of isoparametric hypersurfaces in the sphere. As we mentioned, all homogeneous isoparametric hypersurfaces can be obtained as orbits of s-representations of Riemannian symmetric spaces of rank 2. Therefore here we shall investigate tangentially degenerate submanifolds via a theory of symmetric spaces. As a result, we have the following theorem.

Theorem 3.5 ([14]). *An orbit of an s-representation is tangentially degenerate if and only if it is through a long root (any root when all roots have the same length), or a short root of restricted root system of type G_2. Let $\lambda \in R$ be such a root. Then the tangentially degeneracy of the orbit $\mathrm{Ad}(K)\lambda$ is $\ker(d\gamma)_\lambda = \mathfrak{m}_\lambda$.*

Remark 3.6. From Theorem 2.5, we observe that all of these tangentially degenerate orbits are weakly reflective submanifolds in S.

In Table 1, we give the list of symmetric pairs whose ranks are equal or greater than 2 such that the orbits of their s-representations have degenerate Gauss mappings. All of them are orbits through long roots except the case of type G_2. In the case of type G_2 both of orbits through a long root and a short root have degenerate Gauss mappings, and both of them have the same dimension and the same rank of Gauss mapping. In Table 1, we denote the dimension of the orbit by l and the rank of Gauss mapping by r. Then tangentially degeneracy is equal to $l - r$. In this list, we can find several orbits which satisfy the Ferus equality $r = F(l)$.

4. Examples of weakly reflective orbits and tangentially degenerate orbits

In this section we give some concrete examples of weakly reflective orbits and tangentially degenerate orbits of s-representations.

First we consider the case of symmetric spaces of rank 2. In the case of type A_2 the restricted roots are as in Figure 1. The colored area is a Weyl chamber $\mathcal{C}$. In this case the arc $S \cap \bar{\mathcal{C}}$ in $\bar{\mathcal{C}}$ can be identified with the orbit space of the action of K on S. The orbit through a restricted root locates at the mid point of the orbit space. This orbit is weakly reflective and tangentially degenerate. This orbit is called a Cartan hypersurface, that is a isoparametric hypersurface with three distinct principal curvatures. At the both of the end points of the orbit space there are two singular orbits, which are congruent with each other. These orbits are weakly reflective, but not tangentially degenerate. These orbits are projective planes called Veronese surfaces.

Figures 2 and 3 are restricted roots of types B_2 and G_2, respectively. Similarly the orbit spaces can be identified with the arcs $S \cap \bar{\mathcal{C}}$ in $\bar{\mathcal{C}}$. In these cases there

type	rank	$\mathfrak{g}$	$\mathfrak{k}$	l	r	$l-r$
A	p	$\mathfrak{su}(p+1)$	$\mathfrak{so}(p+1)$	$2p-1$	$2p-2$	1
	p	$\mathfrak{su}(p+1)^2$	$\mathfrak{su}(p+1)$	$2(2p-1)$	$2(2p-2)$	2
	p	$\mathfrak{su}(2(p+1))$	$\mathfrak{sp}(p+1)$	$4(2p-1)$	$4(2p-2)$	4
	2	$\mathfrak{e}_6$	$\mathfrak{f}_4$	24	16	8
B	p	$\mathfrak{so}(2p+1)^2$	$\mathfrak{so}(2p+1)$	$8p-10$	$8p-12$	2
	p	$\mathfrak{so}(2p+n)$	$\mathfrak{so}(p)\oplus\mathfrak{so}(p+n)$	$4p+2n-7$	$4p+2n-8$	1
C	p	$\mathfrak{sp}(p)$	$\mathfrak{u}(p)$	$2p-1$	$2p-2$	1
	p	$\mathfrak{sp}(p)^2$	$\mathfrak{sp}(p)$	$4p-2$	$4p-4$	2
	p	$\mathfrak{sp}(2p)$	$\mathfrak{sp}(p)\oplus\mathfrak{sp}(p)$	$8p-5$	$8p-8$	3
	p	$\mathfrak{su}(2p)$	$\mathfrak{su}(p)\oplus\mathfrak{su}(p)\oplus\mathbf{R}$	$4p-3$	$4p-4$	1
	p	$\mathfrak{so}(4p)$	$\mathfrak{u}(2p)$	$8p-7$	$8p-8$	1
	3	$\mathfrak{e}_7$	$\mathfrak{e}_6\oplus\mathbf{R}$	33	32	1
D	p	$\mathfrak{so}(2p)$	$\mathfrak{so}(p)\oplus\mathfrak{so}(p)$	$4p-7$	$4p-8$	1
	p	$\mathfrak{so}(2p)^2$	$\mathfrak{so}(2p)$	$2(4p-7)$	$2(4p-8)$	2
E_6	6	$\mathfrak{e}_6$	$\mathfrak{sp}(4)$	21	20	1
	6	$\mathfrak{e}_6\oplus\mathfrak{e}_6$	$\mathfrak{e}_6$	42	40	2
E_7	7	$\mathfrak{e}_7$	$\mathfrak{su}(8)$	33	32	1
	7	$\mathfrak{e}_7\oplus\mathfrak{e}_7$	$\mathfrak{e}_7$	66	64	2
E_8	8	$\mathfrak{e}_8$	$\mathfrak{so}(16)$	57	56	1
	8	$\mathfrak{e}_8\oplus\mathfrak{e}_8$	$\mathfrak{e}_8$	114	112	2
F_4	4	$\mathfrak{f}_4$	$\mathfrak{su}(2)\oplus\mathfrak{sp}(3)$	15	14	1
	4	$\mathfrak{f}_4\oplus\mathfrak{f}_4$	$\mathfrak{f}_4$	30	28	2
	4	$\mathfrak{e}_6$	$\mathfrak{su}(2)\oplus\mathfrak{su}(6)$	21	20	1
	4	$\mathfrak{e}_7$	$\mathfrak{su}(2)\oplus\mathfrak{so}(12)$	33	32	1
	4	$\mathfrak{e}_8$	$\mathfrak{su}(2)\oplus\mathfrak{e}_7$	57	56	1
G_2	2	$\mathfrak{g}_2$	$\mathfrak{so}(4)$	5	4	1
	2	$\mathfrak{g}_2\oplus\mathfrak{g}_2$	$\mathfrak{g}_2$	10	8	2
BC	p	$\mathfrak{su}(2p+n)$	$\mathfrak{su}(p)\oplus\mathfrak{su}(p+n)\oplus\mathbf{R}$	$4p+2n-3$	$4p+2n-4$	1
	p	$\mathfrak{so}(4p+2)$	$\mathfrak{u}(2p+1)$	$8p-3$	$8p-4$	1
	p	$\mathfrak{sp}(2p+n)$	$\mathfrak{sp}(p)\oplus\mathfrak{sp}(p+n)$	$8p+4n-5$	$8p+4n-8$	3
	2	$\mathfrak{e}_6$	$\mathfrak{so}(10)\oplus\mathbf{R}$	21	20	1

TABLE 1

are long roots and short roots. The orbits through restricted roots are weakly reflective, which are singular orbits. In the case of type G_2 both of two singular orbits are tangentially degenerate. On the other hand, in the case of type B_2 the orbit through a long root is tangentially degenerate, although one through a short root is not tangentially degenerate. In the cases of G_2 and B_2 with constant multiplicities, at the mid point of the orbit spaces, there exist principal orbits which are austere but not weakly reflective.

Next we give the explicit homogeneous space expressions of weakly reflective orbits in the case where (G, K) is a compact symmetric pair of classical type with the restricted root system of type A. In this case a weakly reflective orbit is one of (1) an orbit through a restricted root, (2) an orbit through $2e_1 - e_2 - e_3$ and an orbit through $e_1 + e_2 - 2e_3$ when $R = A_2$, (3) an orbit through $e_1 + e_2 - e_3 - e_4$ when $R = A_3$. As in the following list, in the case of (2) we have Veronese surfaces, and

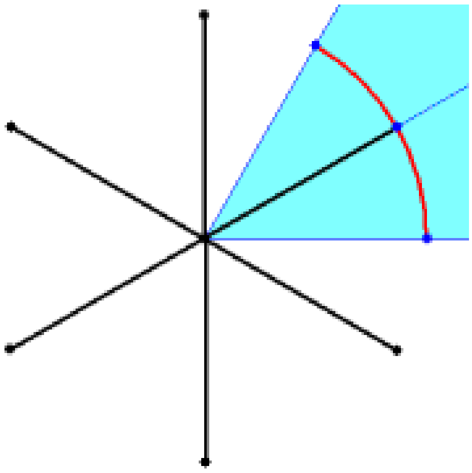

FIGURE 1. restricted roots of type A_2

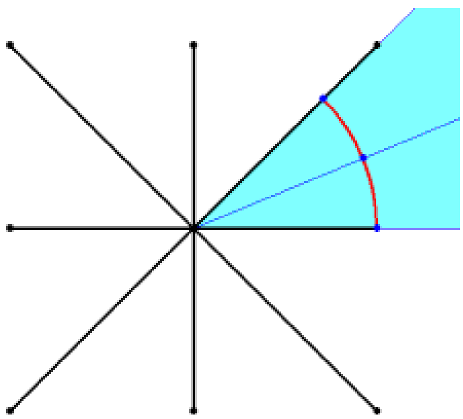

FIGURE 2. restricted roots of type B_2

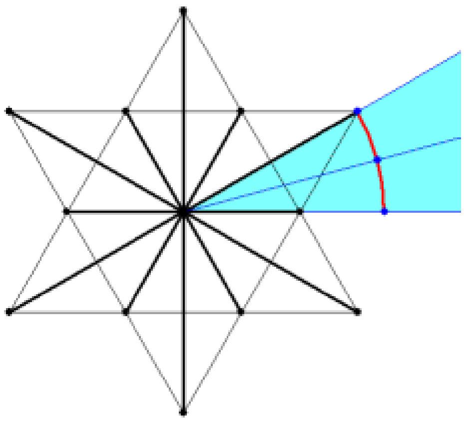

FIGURE 3. restricted roots of type G_2

in the case of (3) we have 2-plane Grassmannian manifolds. We should note that, in the cases of (2) and (3), these orbits are not only weakly reflective submanifolds but also symmetric R-spaces.

- Case of $(G, K) = (SU(p+1), SO(p+1))$
  (1) $(SU(p+1), SO(p+1))$, $\quad H = e_1 - e_2$ (restricted root)

$$\mathrm{Ad}(K)H \cong SO(p+1)/S(O(1) \times O(1) \times O(p-1)) \subset S^{\frac{1}{2}(p+1)(p+2)-2}$$

  (2) $(SU(3), SO(3))$, $\quad H = 2e_1 - e_2 - e_3$

$$\mathrm{Ad}(K)H \cong SO(3)/S(O(1) \times O(2)) \cong \mathbb{R}P^2 \subset S^4$$

  (3) $(SU(4), SO(4))$, $\quad H = e_1 + e_2 - e_3 - e_4$

$$\mathrm{Ad}(K)H \cong SO(4)/S(O(2) \times O(2)) \cong G_2(\mathbb{R}^4) \subset S^8$$

- Case of $(G, K) = (SU(p+1) \times SU(p+1), SU(p+1)^*)$

(1) $(SU(p+1)\times SU(p+1), SU(p+1)^*)$, $\quad H = e_1 - e_2$ (restricted root)

$$\mathrm{Ad}(K)H \cong SU(p+1)/S(U(1)\times U(1)\times U(p-1)) \subset S^{p^2+2p-1}$$

(2) $(SU(3)\times SU(3), SU(3)^*)$, $\quad H = 2e_1 - e_2 - e_3$

$$\mathrm{Ad}(K)H \cong SU(3)/S(U(1)\times U(2)) \cong \mathbb{C}P^2 \subset S^7$$

(3) $(SU(4)\times SU(4), SU(4)^*)$, $\quad H = e_1 + e_2 - e_3 - e_4$

$$\mathrm{Ad}(K)H \cong SU(4)/S(U(2)\times U(2)) \cong G_2(\mathbb{C}^4) \subset S^{14}$$

- Case of $(G,K) = (SU(2p+2), Sp(p+1))$

(1) $(SU(2p+2), Sp(p+1))$, $\quad H = e_1 - e_2$ (restricted root)

$$\mathrm{Ad}(K)H \cong Sp(p+1)/Sp(1)\times Sp(1)\times Sp(p-1) \subset S^{p(2p+3)-1}$$

(2) $(SU(6), Sp(3))$, $\quad H = 2e_1 - e_2 - e_3$

$$\mathrm{Ad}(K)H \cong Sp(3)/Sp(1)\times Sp(2) \cong \mathbb{H}P^2 \subset S^{13}$$

(3) $(SU(8), Sp(4))$, $\quad H = e_1 + e_2 - e_3 - e_4$

$$\mathrm{Ad}(K)H \cong Sp(4)/Sp(2)\times Sp(2)) \cong G_2(\mathbb{H}^4) \subset S^{26}$$

5. Problems

At the end of this paper we shall pose some problems related to this article.

In Section 3 we constructed several new examples of tangentially degenerate submanifolds in the sphere which satisfies the Ferus equality. But we do not know whether the Ferus inequality is best possible or not. Thus we should try Problem 3.1.

In Section 2 we gave the classification of weakly reflective orbits of s-representations. At this moment, all examples of weakly reflective submanifolds in S^n are orbits of s-representations. Are there any other weakly reflective submanifolds in S^n? Furthermore,

Problem 5.1. *Classify all weakly reflective submanifolds in Riemannian symmetric spaces.*

For this purpose probably first we should try the following problem.

Problem 5.2. *Are all weakly reflective submanifolds homogeneous?*

At the end of Section 2 we mentioned that we can construct special Lagrangian cones in $\mathbb{C}^n$ and special Lagrangian submanifolds in TS^n from austere submanifolds in S^n. Therefore, from Theorem 2.5, we can obtain several concrete examples of special Lagrangian submanifolds.

Problem 5.3. *Study the geometry of special Lagrangian submanifolds obtained as the normal bundles of these austere orbits.*

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