Asymptotic uniqueness for a biharmonic equation with nearly critical growth on symmetric convex domains

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Abstract

We consider a biharmonic equation with the nearly critical Sobolev exponent under the Navier boundary condition on a smooth bounded, strictly convex domain of dimension $N \geq 5$, which is symmetric with respect to the coordinate hyperplanes.

We prove that the number of positive solutions of the above problem is exactly one when the nonlinear exponent is subcritical and

sufficiently near to the critical exponent. Furthermore, this unique solution is nondegenerate in the sense that the associated linearized problem admits only the trivial solution.

1 Introduction

We consider the problem (P_ε) with the Navier boundary condition:

$$(P_\varepsilon)\begin{cases} \Delta^2 u = c_0 u^{p_\varepsilon} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N (N \geq 5)$ be a smooth bounded domain, $c_0 = (N-4)(N-2)N(N+2)$, $\varepsilon > 0$ is a small positive parameter, $p_\varepsilon = p - \varepsilon$ and $p = (N+4)/(N-4)$ is the critical Sobolev exponent from the view point of the Sobolev embedding $H^2 \cap H_0^1(\Omega) \hookrightarrow L^{p+1}(\Omega)$. The existence of at least one solution is easy to obtain for $\varepsilon > 0$ small. In this paper, we prove a uniqueness and a qualitative property of solution for the problem (P_ε).

We impose some geometric assumptions on the domain.

(H1) Ω is symmetric with respect to the hyperplane $\{x_i = 0\}, (i = 1, \cdots, N)$.

(H2) Ω is strictly convex.

Note that (P_ε) is equivalent to the cooperative system

$$\begin{cases} -\Delta u = \overline{u} & \text{in } \Omega, \\ -\Delta \overline{u} = c_0 u^{p_\varepsilon} & \text{in } \Omega, \\ u > 0, \overline{u} > 0 & \text{in } \Omega, \\ u = \overline{u} = 0 & \text{on } \partial\Omega. \end{cases}$$

Therefore under the assumptions (H1) and

(H2') Ω is convex in the x_i-direction, $(i = 1, \cdots, N)$,

any solution u_ε to (P_ε) is symmetric with respect to the hyperplane $\{x_i = 0\}$:

$$u_\varepsilon(x_1, \cdots, x_i, \cdots, x_N) = u_\varepsilon(x_1, \cdots, -x_i, \cdots, x_N),\ (i = 1, \cdots, N),$$

and monotone with respect to the positive x_i-direction:

$$\frac{\partial u}{\partial x_i} < 0 \quad \text{for } x_i > 0,\ (i = 1, \cdots, N),$$

see [16] Lemma 4.3. From these and since (P_0) has no solution, we easily see

$$\|u_\varepsilon\|_{L^\infty(\Omega)} = u_\varepsilon(0) \to +\infty \quad \text{as } \varepsilon \to 0.$$

In this note, we prove

Theorem 1 *(Asymptotic uniqueness) Assume $\Omega \subset \mathbb{R}^N, N \geq 5$ satisfies (H1) and (H2). Let u_ε and v_ε be two solutions to (P_ε). Then there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, we have $u_\varepsilon \equiv v_\varepsilon$ on Ω.*

Theorem 2 *The unique solution u_ε to (P_ε) in Theorem 1 is nondegenerate in the sense that $\lambda = 0$ is not an eigenvalue for the linearized eigenvalue problem*

$$\begin{cases} \Delta^2 w = c_0 p_\varepsilon u_\varepsilon^{p_\varepsilon - 1} w + \lambda w & \text{in } \Omega, \\ w = \Delta w = 0 & \text{on } \partial\Omega. \end{cases}$$

As far as we know, the uniqueness of solutions to the subcritical problem

$$\begin{cases} \Delta^2 u = u^p & \text{in } \Omega, \quad 1 < p < (N+4)/(N-4), \\ u > 0 & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega \end{cases}$$

where Ω satisfies (H1) and (H2) (or (H2')), seems widely open, except for the case Ω is a ball. We note that the blow up phenomenon does induce the uniqueness result in Theorem 1.

Our argument goes along the line of Grossi [11]; see also [3]. Grossi obtained the same uniqueness and the nondegeneracy results for the problem

$$\begin{cases} -\Delta u = N(N-2) u^{p_\varepsilon} & \text{in } \Omega \subset \mathbb{R}^N \ N \geq 3, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

under the assumptions (H1) and (H2').

In the proof, Grossi used a fine blow up analysis by YanYan Li [18] to show that the results of Han [12] hold true for general solutions of the above problem, under (H1) and (H2'). In the Laplacian case, the uniform supremum estimate near the boundary for general solutions is obtained by the method of moving planes of Gidas, Ni and Nirenberg [10], and the additional use of the Kelvin transformation if the domain is not strictly convex.

The method of moving planes also assures that the uniform boundedness near the boundary for general solutions of (P_ε) if the domain is strictly convex. However, in our biharmonic case, the Kelvin transformation does not work well because the Navier boundary condition is not preserved under the transformation; see [5]. This is the reason why we assume (H2).

Once we confirm that a blow up point is isolated and not on the boundary, then we can employ the local blow up analysis and the theory of isolated simple blow up points, recently obtained by Djadli, Malchiodi and Ahmedou [7] for biharmonic equations. See also Felli [9]. Their works assure that the origin is an isolated simple blow up point for any solution sequence of (P_ε), and the results of Chou and Geng [5], known to be valid for solutions minimizing the Sobolev quotient, hold true also for general solutions under (H1) and (H2).

In the proof of Theorem 2, we extend a lemma of Damascelli, Grossi and Pacella [6] to a polyharmonic problem. We hope this is itself interesting, see Lemma 13.

2 Preliminaries

In this section, we collect some useful facts in the sequel. Let $G = G(x, y)$ denote the Green function of Δ^2 under the Navier boundary condition:

$$\begin{cases} \Delta^2 G(\cdot, y) = \delta_y & \text{in } \Omega, \\ G(\cdot, y) = \Delta G(\cdot, y) = 0 & \text{on } \partial\Omega. \end{cases}$$

The Robin function is defined as

$$R(x) = \lim_{y \to x} \left[\Gamma(x, y) - G(x, y) \right],$$

where

$$\Gamma(x, y) = \begin{cases} \frac{1}{(N-4)(N-2)\sigma_N} |x - y|^{4-N}, & N \geq 5, \\ \frac{1}{\sigma_4} \log |x - y|^{-1}, & N = 4 \end{cases}$$

and σ_N is the volume of the $(N-1)$ dimensional unit sphere in $\mathbb{R}^N$. We see that $R > 0$ on Ω and $R(x) \to +\infty$ as x tends to the boundary of Ω.

Lemma 3 *(Pohozaev identity for the Green function) The identity*

$$\int_{\partial\Omega} ((x - y) \cdot \nabla G) \frac{\partial}{\partial \nu} (-\Delta G) ds_x = (N - 4) R(y)$$

holds true for any $y \in \Omega$.

Proof: See [5]. Note that there is a mistake in the claimed formula in [5]. □

Next lemma concerns a classical elliptic regularity for a solution to the biharmonic equation, recently obtained by Caristi and Mitidieri ([1] Theorem 4.9).

Lemma 4 *Let $u \in H^2_{loc}(\Omega)$ be a weak solution of*

$$\Delta^2 u = a(x)u \quad \text{in } \Omega$$

where $a \in L^\alpha(\Omega)$ with $\alpha > N/4$. Then for any $q \in (0, +\infty)$, there exist $C = C(q) > 0$ and $R > 0$ such that for any $0 < r < R$ and $y \in \mathbb{R}^N$, we have

$$\sup_{B(y,r)\cap\Omega} |u| \le C \left[\frac{1}{r^N} \int_{B(y,2r)\cap\Omega} |u|^{q+1} dx \right]^{1/(q+1)}.$$

Next lemma claims that the origin is an *isolated blow up point* for any solution sequence u_ε of (P_ε). Proof will be done by a standard blow up analysis just as in [11] Lemma A.1., because we know u_ε is uniformly bounded in sup-norm near the boundary thanks to our assumption (H2), see [5] p.925.

Lemma 5 *Assume (H1) and (H2). Let u_ε be any solution to (P_ε). Then there exists $C > 0$ independent of ε such that*

$$|x|^{4/(p_\varepsilon - 1)} u_\varepsilon(x) \le C$$

for any $x \in \Omega$.

Under more general situation, an isolated blow up point has to be an *isolated simple blow up point*: see [7] Proposition 2.19 and [9]. We refer [7], [9] to the definitions of isolated, or isolated simple blow up point for our biharmonic case. Then by using the estimates for isolated simple blow up points ([7] Lemma 2.11 and Lemma 2.17), we have the followings:

Lemma 6 *Assume (H1) and (H2). Let u_ε be any solution to (P_ε). Then we have*

$$\lim_{\varepsilon \to 0} \|u_\varepsilon\|^{\varepsilon}_{L^\infty(\Omega)} = 1 \tag{2.1}$$

and

$$\frac{\int_\Omega |\Delta u_\varepsilon|^2 dx}{\left(\int_\Omega |u_\varepsilon|^{p_\varepsilon+1}\right)^{2/(p_\varepsilon+1)}} \to S_N \tag{2.2}$$

where S_N is the best Sobolev constant of the embedding $H^2 \cap H^1_0(\Omega) \hookrightarrow L^{p+1}(\Omega)$.

Next theorem is the main result of Chou and Geng [5].

Theorem 7 *([5]) Assume $\Omega \subset \mathbb{R}^N, N \geq 5$ is strictly convex. Let u_ε be a solution to (P_ε) satisfying (2.2). Let $x_\varepsilon \in \Omega$ be a point such that $u_\varepsilon(x_\varepsilon) = \|u_\varepsilon\|_{L^\infty(\Omega)}$. Then after passing to a subsequence, we have*

(1) $\lim_{\varepsilon\to 0} x_\varepsilon = x_0$ *for some interior point $x_0 \in \Omega$.*

(2) For any open neighborhood ω of $\partial\Omega$ not containing x_0, there holds

$$\|u_\varepsilon\|_{L^\infty(\Omega)} u_\varepsilon \to 2(N-4)(N-2)\sigma_N G(\cdot, x_0) \quad \text{as } \varepsilon \to 0 \tag{2.3}$$

in $C^{3,\alpha}(\omega)$ for some $\alpha \in (0,1)$.

(3) There exists a constant $C > 0$ independent of ε and solution u_ε such that

$$u_\varepsilon(x) \leq C \frac{\|u_\varepsilon\|_{L^\infty(\Omega)}}{\left(1 + \|u_\varepsilon\|_{L^\infty(\Omega)}^{4/(N-4)} |x - x_\varepsilon|^2\right)^{(N-4)/2}} \tag{2.4}$$

holds for any $x \in \Omega$.

(4)

$$\lim_{\varepsilon\to 0} \varepsilon \|u_\varepsilon\|_{L^\infty(\Omega)}^2 = \frac{16\sigma_N^2 \Gamma(N) c_0}{\pi^{N/2} N(N-4)(N+2)^2 \Gamma(N/2)} R(x_0). \tag{2.5}$$

By Lemma 6 (2.2), we see that the results of Theorem 7 hold for any solution sequence u_ε to (P_ε) with $x_\varepsilon = x_0 = 0$ under (H1) and (H2).

In what follows, we use a symbol $\|\cdot\|$ to denote the L^∞ norm of functions. Now, let us consider the scaled function

$$\tilde{u}_\varepsilon(y) := \frac{1}{\|u_\varepsilon\|} u_\varepsilon \left(\frac{y}{\|u_\varepsilon\|^{(p_\varepsilon - 1)/4}} \right), \quad y \in \Omega_\varepsilon := \|u_\varepsilon\|^{(p_\varepsilon-1)/4} \Omega.$$

We see $0 < \tilde{u}_\varepsilon \leq 1, \tilde{u}_\varepsilon(0) = 1$, and $\tilde{u}_\varepsilon$ satisfies

$$\begin{cases} \Delta^2 \tilde{u}_\varepsilon = c_0 \tilde{u}_\varepsilon^{p_\varepsilon} & \text{in } \Omega_\varepsilon, \\ \tilde{u}_\varepsilon = \Delta \tilde{u}_\varepsilon = 0 & \text{on } \partial\Omega_\varepsilon, \end{cases}$$

Since $\|u_\varepsilon\| \to \infty$ as $\varepsilon \to 0$, we see $\Omega_\varepsilon \to \mathbb{R}^N$ and by standard elliptic estimates, we have a subsequence denoted also by $\tilde{u}_\varepsilon$ that

$$\tilde{u}_\varepsilon \to U \quad \text{compact uniformly in } \mathbb{R}^N \tag{2.6}$$

as $\varepsilon \to 0$ for some function U. Passing to the limit, we obtain that U is a solution of

$$\begin{cases} \Delta^2 U = c_0 U^p \quad \text{in } \mathbb{R}^N, \\ 0 < U \leq 1, \ U(0) = 1, \\ \lim_{|y|\to\infty} U(y) = 0. \end{cases}$$

Thus according to the uniqueness theorem by Chang Shou Lin [4], we obtain

$$U(y) = \left(\frac{1}{1+|y|^2}\right)^{(N-4)/2}. \tag{2.7}$$

3 A uniqueness result

In this section, we will prove Theorem 1. Assume the contrary that there exist solutions u_ε and v_ε to (P_ε), $u_\varepsilon \not\equiv v_\varepsilon$ for some $\{\varepsilon\} \downarrow 0$. Consider the function

$$w_\varepsilon(y) = \frac{1}{\|u_\varepsilon - v_\varepsilon\|}\left(u_\varepsilon\left(\frac{y}{\|u_\varepsilon\|^{(p_\varepsilon-1)/4}}\right) - v_\varepsilon\left(\frac{y}{\|u_\varepsilon\|^{(p_\varepsilon-1)/4}}\right)\right)$$

for $y \in \Omega_\varepsilon = \|u_\varepsilon\|^{(p_\varepsilon-1)/4}\Omega$. It is easy to check that w_ε solves

$$\begin{cases} \Delta^2 w_\varepsilon = c_\varepsilon(y) w_\varepsilon & \text{in } \Omega_\varepsilon, \\ w_\varepsilon = \Delta w_\varepsilon = 0 & \text{on } \partial\Omega_\varepsilon, \\ \|w_\varepsilon\| = 1 \end{cases} \tag{3.1}$$

where

$$c_\varepsilon(y) = c_0 p_\varepsilon \int_0^1 [t\tilde{u}_\varepsilon(y) + (1-t)\tilde{v}_\varepsilon(y)]^{p_\varepsilon-1}\, dt, \tag{3.2}$$

here we set

$$\tilde{u}_\varepsilon(y) = \frac{1}{\|u_\varepsilon\|} u_\varepsilon\left(\frac{y}{\|u_\varepsilon\|^{(p_\varepsilon-1)/4}}\right) \quad \text{and} \quad \tilde{v}_\varepsilon(y) = \frac{1}{\|u_\varepsilon\|} v_\varepsilon\left(\frac{y}{\|u_\varepsilon\|^{(p_\varepsilon-1)/4}}\right)$$

for $y \in \Omega_\varepsilon$.

By Theorem 7 (2.5), we see $\lim_{\varepsilon\to 0}\|u_\varepsilon\| = \lim_{\varepsilon\to 0}\|v_\varepsilon\| = +\infty$ and

$$\lim_{\varepsilon\to 0}\frac{\|u_\varepsilon\|}{\|v_\varepsilon\|} = 1.$$

By this, we have $\|\tilde{v}_\varepsilon\| = \tilde{v}_\varepsilon(0) = v_\varepsilon(0)/\|u_\varepsilon\| \to 1$, so as in (2.6), we see

$$\tilde{u}_\varepsilon \to U \quad \text{and} \quad \tilde{v}_\varepsilon \to U \quad \text{uniformly on compact sets of } \mathbb{R}^N$$

where U is as in (2.7). Thus

$$c_\varepsilon \to c_0 p \int_0^1 [tU + (1-t)U]^{p-1} dt = c_0 p U^{p-1} \tag{3.3}$$

uniformly on compact sets of $\mathbb{R}^N$. Since $\|w_\varepsilon\| = 1$, standard elliptic regularity allows us to pass to the limit in the equation (3.1). Then we get

$$w_\varepsilon \to w \quad \text{uniformly on compact sets of } \mathbb{R}^N \tag{3.4}$$

for some function w, and the limit function w satisfies

$$\Delta^2 w = c_0 p U^{p-1} w \quad \text{in } \mathbb{R}^N, \quad \|w\| \le 1. \tag{3.5}$$

Since w_ε is symmetric with respect to the hyperplanes $\{x_i = 0\}$ $(i = 1, \cdots, N)$, we see by (3.4) that w is a symmetric function.

Furthermore, arguing as in [11], we check that

$$\int_{\Omega_\varepsilon} |\Delta w_\varepsilon|^2 dy \le C \tag{3.6}$$

where C is independent of ε. Thus by Fatou's lemma, we also have

$$\int_{\mathbb{R}^N} |\Delta w|^2 dy \le C. \tag{3.7}$$

Now, we recall the classification theorem by Bartsch, Weth and Willem ([2]).

Lemma 8 *Let w be a solution to (3.5) with the property (3.7). Then there exist a_j $(j = 1, 2, \cdots, N), b \in \mathbb{R}$ such that w can be written as*

$$w = \sum_{j=1}^{N} a_j \frac{y_j}{(1+|y|^2)^{(N-2)/2}} + b \frac{1-|y|^2}{(1+|y|^2)^{(N-2)/2}}.$$

In the following, we divide the proof into several steps.

Step 1. $a_j = 0, j = 1, \cdots, N$.

This is a simple consequence of the fact that w is a symmetric function with respect to the hyperplanes $\{x_j = 0\}, j = 1, \cdots, N$.

Step 2. $b = 0$.

By step 1, we have

$$w = b \frac{1-|y|^2}{(1+|y|^2)^{(N-2)/2}}.$$

Now, we need the following lemma: In the proof, we argue as in [11] Lemma A.5 with the crucial use of Lemma 4.

Lemma 9 *Let w_ε be a solution of (3.1). Then we have the estimate*

$$|w_\varepsilon(y)| \le C\frac{1}{|y|^{N-4}} \quad \text{for } y \in \Omega_\varepsilon \cap \{|y| \ge \delta\} \tag{3.8}$$

for some $C > 0$ and $\delta > 0$.

Proof: Consider the Kelvin transformation of w_ε:

$$w_\varepsilon^*(z) = |z|^{4-N} w_\varepsilon(\frac{z}{|z|^2}), \quad z \in \Omega_\varepsilon^* := \{\frac{y}{|y|^2} : y \in \Omega_\varepsilon\}.$$

To prove (3.8), it will be enough to show that $|w_\varepsilon^*|$ is bounded in $B(0,R)\cap\Omega_\varepsilon^*$ for some $R > 0$. Direct calculation shows that

$$\begin{aligned}
&\Delta_z^2 w_\varepsilon^*(z) = |z|^{-4-N}\Delta_y^2 w_\varepsilon(y), \quad z \in \Omega_\varepsilon^*,\\
&\Delta_z w_\varepsilon^*(z) = 4|z|^{-2}(z\cdot\nu)\frac{\partial w_\varepsilon^*}{\partial\nu}, \quad z \in \partial\Omega_\varepsilon^*,\\
&\int_{\Omega_\varepsilon^*} |w_\varepsilon^*|^{p+1}dz = \int_{\Omega_\varepsilon} |w_\varepsilon|^{p+1}dy.
\end{aligned}$$

Thus, w_ε^* satisfies the equation

$$\begin{cases}
\Delta^2 w_\varepsilon^* &= |z|^{-8}c_\varepsilon(\frac{z}{|z|^2})w_\varepsilon^* & \text{in } \Omega_\varepsilon^*,\\
w_\varepsilon^* &= 0 & \text{on } \partial\Omega_\varepsilon^*,\\
\Delta w_\varepsilon^* &= 4|z|^{-2}(z\cdot\nu)\frac{\partial w_\varepsilon^*}{\partial\nu}, & \text{on } \partial\Omega_\varepsilon^*.
\end{cases}$$

Now, we claim that there exists a constant $C > 0$ such that

$$\|a_\varepsilon\|_{L^\infty(\Omega_\varepsilon^*)} \le C \tag{3.9}$$

where

$$a_\varepsilon(z) := |z|^{-8}c_\varepsilon(\frac{z}{|z|^2}).$$

Indeed, since $\Omega_\varepsilon \subset B(0, \gamma\|u_\varepsilon\|^{(p_\varepsilon-1)/4})$ for some $\gamma > 0$, we see that $\Omega_\varepsilon^* \subset \mathbb{R}^N \setminus B(0, 1/(\gamma\|u_\varepsilon\|^{(p_\varepsilon-1)/4}))$. By (2.4), we know that

$$\tilde{u}_\varepsilon(y) \le CU(y) \quad \text{and} \quad \tilde{v}_\varepsilon \le CU(y),$$

thus

$$|c_\varepsilon(y)| \le CU^{p_\varepsilon-1}(y) \quad \text{for } y \in \Omega_\varepsilon. \tag{3.10}$$

Therefore, we have

$$\begin{aligned}|z|^{-8}c_\varepsilon(\frac{z}{|z|^2}) &\le C|z|^{-8}\left(\frac{|z|^2}{1+|z|^2}\right)^{((N-4)/2)(p_\varepsilon-1)}\\ &= C|z|^{-8+(N-4)(p_\varepsilon-1)}\frac{1}{(1+|z|^2)^{4-\varepsilon((N-4)/2)}}\\ &\le C|z|^{-8+(N-4)(p_\varepsilon-1)} = C|z|^{-\varepsilon(N-4)}\end{aligned}$$

Since $|z| \ge 1/(\gamma\|u_\varepsilon\|^{(p_\varepsilon-1)/4})$ for $z \in \Omega_\varepsilon^*$, we have

$$|z|^{-\varepsilon(N-4)} \le \gamma^{\varepsilon(N-4)}\|u_\varepsilon\|^{\varepsilon(N-4)(p_\varepsilon-1)/4} \longrightarrow 1$$

as $\varepsilon \to 0$. Here we have used (2.1). From these, we confirm that the claim (3.9).

Now, for any $R > 0$, we have

$$\begin{aligned}\int_{\Omega_\varepsilon^*\cap B(0,2R)}|w_\varepsilon^*|^{p+1}dz &\le \int_{\Omega_\varepsilon^*}|w_\varepsilon^*|^{p+1}dz = \int_{\Omega_\varepsilon}|w_\varepsilon|^{p+1}dy\\ &\le \left(\frac{1}{S_N}\int_{\Omega_\varepsilon}|\Delta w_\varepsilon|^2dy\right)^{(p+1)/2} \le C,\end{aligned}$$

here we have used the Sobolev inequality for $H^2 \cap H_0^1$ functions and (3.6). Let us take $q = p, y = 0, \Omega = \Omega_\varepsilon^*$ in Lemma 4. Thus for $R > 0$ in Lemma 4, we obtain

$$\sup_{B(0,R)\cap\Omega_\varepsilon^*}|w_\varepsilon^*| \le C\left[\frac{1}{R^N}\int_{B(0,2R)\cap\Omega_\varepsilon^*}|w_\varepsilon^*|^{p+1}dz\right]^{1/(p+1)} \le C.$$

□

By Lemma 9 and Theorem 7 (2.4), we have the following convergence result.

Lemma 10 *Let $\omega \subset \Omega$ be any neighborhood of $\partial\Omega$ not containing 0. Then we have*

$$\|u_\varepsilon\|^2\frac{(u_\varepsilon - v_\varepsilon)}{\|u_\varepsilon - v_\varepsilon\|} \to -2(N-2)(N-4)\sigma_N bG(\cdot,0) \quad in\ C^3(\omega).$$

Proof: We see

$$\Delta^2\left(\|u_\varepsilon\|^2\frac{(u_\varepsilon - v_\varepsilon)}{\|u_\varepsilon - v_\varepsilon\|}\right) = \frac{\|u_\varepsilon\|^2}{\|u_\varepsilon - v_\varepsilon\|}d_\varepsilon(x)(u_\varepsilon - v_\varepsilon) =: f_\varepsilon(x) \qquad (3.11)$$

for $x \in \Omega$ with the boundary condition

$$\|u_\varepsilon\|^2 \frac{(u_\varepsilon - v_\varepsilon)}{\|u_\varepsilon - v_\varepsilon\|} = \Delta \left(\|u_\varepsilon\|^2 \frac{(u_\varepsilon - v_\varepsilon)}{\|u_\varepsilon - v_\varepsilon\|} \right) = 0$$

on $\partial\Omega$, where

$$d_\varepsilon(x) = c_0 p_\varepsilon \int_0^1 [t u_\varepsilon(x) + (1-t) v_\varepsilon(x)]^{p_\varepsilon - 1} dt, \quad x \in \Omega.$$

Note that

$$\frac{1}{\|u_\varepsilon\|^{p_\varepsilon - 1}} d_\varepsilon\left(\frac{y}{\|u_\varepsilon\|^{(p_\varepsilon - 1)/4}}\right) = c_\varepsilon(y), \quad y \in \Omega_\varepsilon,$$

see (3.2). Thus

$$|d_\varepsilon(x)| \le C \frac{\|u_\varepsilon\|^{-(p_\varepsilon - 1)}}{|x|^{(N-4)(p_\varepsilon - 1)}} \tag{3.12}$$

for any $x \in \Omega, x \neq 0$ by (2.1) and (3.10).

We have by (3.12) and (3.8),

$$\begin{aligned}
f_\varepsilon(x) &= \|u_\varepsilon\|^2 d_\varepsilon(x) w_\varepsilon(\|u_\varepsilon\|^{(p_\varepsilon - 1)/4} x) \\
&\le C \|u_\varepsilon\|^2 \frac{\|u_\varepsilon\|^{-(p_\varepsilon - 1)}}{|x|^{(N-4)(p_\varepsilon - 1)}} \frac{1}{(\|u_\varepsilon\|^{(p_\varepsilon - 1)/4} |x|)^{N-4}} \\
&\le C \frac{\|u_\varepsilon\|^{3 - p_\varepsilon - (p_\varepsilon - 1)(N-4)/4}}{|x|^{(N-4)p_\varepsilon}} \to 0
\end{aligned}$$

for any $x \neq 0$, since $3 - p_\varepsilon - (p_\varepsilon - 1)(N-4)/4 = -8/(N-4) + \varepsilon(N/4) < 0$ for $\varepsilon > 0$ small.

Also by using (3.10), (3.8), (3.3), (2.1) and the dominated convergence theorem, we obtain

$$\begin{aligned}
\int_\Omega f_\varepsilon(x) dx &= \|u_\varepsilon\|^{2 - (p_\varepsilon - 1)N/4} \int_{\Omega_\varepsilon} d_\varepsilon\left(\frac{y}{\|u_\varepsilon\|^{(p_\varepsilon - 1)/4}}\right) w_\varepsilon(y) dy \\
&= \|u_\varepsilon\|^{p_\varepsilon + 1 - (p_\varepsilon - 1)N/4} \int_{\Omega_\varepsilon} c_\varepsilon(y) w_\varepsilon(y) dy \\
&\to c_0 p \int_{\mathbb{R}^N} U^{p-1} w dy = c_0 p b \int_{\mathbb{R}^N} \frac{1 - |y|^2}{(1 + |y|^2)^{N/2+3}} dy \\
&= c_0 p b \sigma_N \left(\int_0^\infty \frac{r^{N-1}}{(1 + r^2)^{N/2+3}} dr - \int_0^\infty \frac{r^{N+1}}{(1 + r^2)^{N/2+3}} dr \right) \\
&= -2(N-2)(N-4) b \sigma_N.
\end{aligned}$$

Last integrals are computed by the formula

$$\int_0^\infty \frac{r^\alpha}{(1+r^2)^\beta} dr = \frac{\Gamma((\alpha+1)/2)\Gamma(\beta-(\alpha+1)/2)}{2\Gamma(\beta)}.$$

From these, we confirm that

$$f_\varepsilon \to -2(N-2)(N-4)\sigma_N b\delta_0 \tag{3.13}$$

in the sense of distributions. On the other hand, we can apply the L^p-theory of [8] to the equation (3.11) with the Navier boundary condition to get

$$\|\Delta\left(\|u_\varepsilon\|^2 \frac{(u_\varepsilon - v_\varepsilon)}{\|u_\varepsilon - v_\varepsilon\|}\right)\|_{C^{1,\alpha}(\omega)} \le C(\omega)\left(\|f_\varepsilon\|_{L^1(\Omega)} + \|f_\varepsilon\|_{L^\infty(\omega')}\right)$$

for $\omega \subset\subset \omega'$ is a neighborhood of $\partial\Omega$ not containing 0. Since we have seen that RHS of the above estimate is bounded by a constant independent of ε, Ascoli-Arzelá theorem implies that the function $\Delta\left(\|u_\varepsilon\|^2 \frac{(u_\varepsilon - v_\varepsilon)}{\|u_\varepsilon - v_\varepsilon\|}\right)$ converges to some function in $C^{1,\alpha}$-topology. Finally, (3.13) implies that this limit function is $-2(N-2)(N-4)\sigma_N bG(x,0)$. □

In the following, we will use Theorem 7 with $x_\varepsilon = x_0 = 0$. Recall the Pohozaev identity for u_ε and v_ε ([14] or [17]):

$$A_\varepsilon \varepsilon \int_\Omega u_\varepsilon^{p_\varepsilon+1} dx = \int_{\partial\Omega} (x\cdot\nabla u_\varepsilon)\frac{\partial \overline{u}_\varepsilon}{\partial\nu} ds_x, \tag{3.14}$$

$$A_\varepsilon \varepsilon \int_\Omega v_\varepsilon^{p_\varepsilon+1} dx = \int_{\partial\Omega} (x\cdot\nabla v_\varepsilon)\frac{\partial \overline{v}_\varepsilon}{\partial\nu} ds_x, \tag{3.15}$$

where

$$-\Delta u_\varepsilon = \overline{u}_\varepsilon, \quad -\Delta v_\varepsilon = \overline{v}_\varepsilon$$

and

$$A_\varepsilon = \frac{c_0(N-4)^2}{2(2N-(N-4)\varepsilon)}.$$

Subtracting (3.15) from (3.14), and writing as $u_\varepsilon^{p_\varepsilon+1} - v_\varepsilon^{p_\varepsilon+1} = h_\varepsilon(x)(u_\varepsilon - v_\varepsilon)$ where

$$h_\varepsilon(x) = (p_\varepsilon+1)\int_0^1 [tu_\varepsilon(x) + (1-t)v_\varepsilon(x)]^{p_\varepsilon} dt,$$

we have

$$\begin{aligned} &A_\varepsilon \varepsilon \int_\Omega h_\varepsilon(x)(u_\varepsilon - v_\varepsilon)dx \\ &= \int_{\partial\Omega}(x\cdot\nabla v_\varepsilon)\frac{\partial}{\partial\nu}(\overline{u}_\varepsilon - \overline{v}_\varepsilon)ds_x + \int_{\partial\Omega}(x\cdot(\nabla u_\varepsilon - \nabla v_\varepsilon))\frac{\partial}{\partial\nu}\overline{u}_\varepsilon ds_x. \end{aligned} \tag{3.16}$$

Let us multiply both sides of (3.16) by $\|u_\varepsilon\|^3/\|u_\varepsilon - v_\varepsilon\|$. Noting that

$$\lim_{\varepsilon\to 0}\|u_\varepsilon\|u_\varepsilon = \lim_{\varepsilon\to 0}\|u_\varepsilon\|v_\varepsilon = 2(N-2)(N-4)\sigma_N G(\cdot,0),$$
$$\lim_{\varepsilon\to 0}\|u_\varepsilon\|\overline{u}_\varepsilon = \lim_{\varepsilon\to 0}\|u_\varepsilon\|\overline{v}_\varepsilon = 2(N-2)(N-4)\sigma_N(-\Delta G)(\cdot,0)$$

in $C^1(\omega)$ by (2.3), Lemma 10 and Lemma 3, we see that

$$\begin{aligned}
&\frac{\|u_\varepsilon\|^3}{\|u_\varepsilon - v_\varepsilon\|}\times(\text{RHS of (3.16)})\\
&= \int_{\partial\Omega}(x\cdot\nabla(\|u_\varepsilon\|v_\varepsilon))\frac{\partial}{\partial\nu}\left(\frac{\|u_\varepsilon\|^2}{\|u_\varepsilon - v_\varepsilon\|}(\overline{u}_\varepsilon - \overline{v}_\varepsilon)\right)ds_x\\
&+ \int_{\partial\Omega}\left(x\cdot\nabla\left(\frac{\|u_\varepsilon\|^2}{\|u_\varepsilon - v_\varepsilon\|}(u_\varepsilon - v_\varepsilon)\right)\right)\frac{\partial}{\partial\nu}(\|u_\varepsilon\|\overline{u}_\varepsilon)ds_x\\
&\to -8(N-4)^2(N-2)^2\sigma_N^2 b\int_{\partial\Omega}(x\cdot\nabla G)\frac{\partial(-\Delta G)}{\partial\nu}(x,0)ds_x\\
&= -8(N-4)^3(N-2)^2\sigma_N^2 bR(0).
\end{aligned}\tag{3.17}$$

On the other hand, by using (2.4), (3.8) and the dominated convergence theorem, we have

$$\begin{aligned}
&\frac{\|u_\varepsilon\|^3}{\|u_\varepsilon - v_\varepsilon\|}\times(\text{LHS of (3.16)})\\
&= A_\varepsilon\times\varepsilon\|u_\varepsilon\|^2\times\int_\Omega h_\varepsilon(x)\frac{\|u_\varepsilon\|}{\|u_\varepsilon - v_\varepsilon\|}(u_\varepsilon - v_\varepsilon)dx\\
&= A_\varepsilon\times\varepsilon\|u_\varepsilon\|^2\times(p_\varepsilon+1)\|u_\varepsilon\|^{1-(p_\varepsilon-1)N/4+p_\varepsilon}\times\\
&\times\int_0^1\int_{\Omega_\varepsilon}[t\frac{1}{\|u_\varepsilon\|}u_\varepsilon(\frac{y}{\|u_\varepsilon\|^{(p_\varepsilon-1)/4}}) + (1-t)\frac{1}{\|u_\varepsilon\|}v_\varepsilon(\frac{y}{\|u_\varepsilon\|^{(p_\varepsilon-1)/4}})]^{p_\varepsilon}w_\varepsilon(y)dydt\\
&\to \frac{c_0(N-4)^2}{4N}(\lim_{\varepsilon\to 0}\varepsilon\|u_\varepsilon\|^2)(p+1)\int_{\mathbb{R}^N}U^p w(y)dy\\
&= C(N)\int_{\mathbb{R}^N}\frac{1}{(1+|y|^2)^{(N+4)/2}}\frac{b(1-|y|^2)}{(1+|y|^2)^{(N-2)/2}}dy\\
&= C(N)b\int_{\mathbb{R}^N}\frac{1-|y|^2}{(1+|y|^2)^{N+1}}dy = 0
\end{aligned}\tag{3.18}$$

where $C(N)$ is a constant depending only on N. Here we have used (2.1) and Theorem 7 (2.5). Hence by (3.17) and (3.18), we have $b = 0$.

Step 3. $w\equiv 0$ leads to a contradiction.

By step1 and step 2, we deduce that the limit function $\lim_{\varepsilon\to 0}w_\varepsilon = w\equiv 0$. Since $\|w_\varepsilon\| = 1$, there exists $x_\varepsilon\in\Omega_\varepsilon$ such that $w_\varepsilon(x_\varepsilon) = 1$ and $|x_\varepsilon|\to\infty$

because the above convergence $w_\varepsilon \to w \equiv 0$ is uniformly on compact sets of $\mathbb{R}^N$. But this is not possible because of Lemma 9 (3.8).

Thus we have proved Theorem 1. $\square$

4 A nondegeneracy result

In this section, we will prove Theorem 2. First, we observe that the first eigenvalue $\lambda_{1,\varepsilon}$ of the linearized operator $\mathcal{L}_\varepsilon = \Delta^2 - c_0 p_\varepsilon u_\varepsilon^{p_\varepsilon - 1} Id$ is negative. Indeed, by a variational characterization of $\lambda_{1,\varepsilon}$, we have

$$\begin{aligned}
\lambda_{1,\varepsilon} &= \inf_{\phi \in H^2 \cap H_0^1(\Omega)} \frac{(\mathcal{L}_\varepsilon \phi, \phi)_{L^2(\Omega)}}{\int_\Omega \phi^2 dx} \\
&= \inf_{\phi \in H^2 \cap H_0^1(\Omega)} \frac{\int_\Omega |\Delta \phi|^2 dx - c_0 p_\varepsilon \int_\Omega u_\varepsilon^{p_\varepsilon - 1} \phi^2 dx}{\int_\Omega \phi^2 dx} \\
&\le \frac{\int_\Omega |\Delta u_\varepsilon|^2 dx - c_0 p_\varepsilon \int_\Omega u_\varepsilon^{p_\varepsilon + 1} dx}{\int_\Omega u_\varepsilon^2 dx} = \frac{c_0(1 - p_\varepsilon) \int_\Omega u_\varepsilon^{p_\varepsilon + 1} dx}{\int_\Omega u_\varepsilon^2 dx} < 0.
\end{aligned}$$

Now, the unique solution u_ε to (P_ε) is obtained by a mountain pass theorem applied to the functional

$$J_\varepsilon(u) = \frac{1}{2} \int_\Omega |\Delta u|^2 dx - \frac{c_0}{p_\varepsilon + 1} \int_\Omega |u|^{p_\varepsilon + 1} dx$$

defined on $H^2 \cap H_0^1(\Omega)$. Thus by Hofer's theorem ([13]), the Morse index of u_ε is at most 1. Since we see

$$D^2 J_\varepsilon(\phi_{1,\varepsilon}, \phi_{1,\varepsilon}) = \int_\Omega |\Delta \phi_{1,\varepsilon}|^2 dx - c_0 p_\varepsilon \int_\Omega u_\varepsilon^{p_\varepsilon - 1} \phi_{1,\varepsilon}^2 dx = \lambda_{1,\varepsilon} \int_\Omega \phi_{1,\varepsilon}^2 dx < 0$$

for the first eigenfunction $\phi_{1,\varepsilon}$, we must have that the second eigenvalue $\lambda_{2,\varepsilon}$ of $\mathcal{L}_\varepsilon$ satisfies $\lambda_{2,\varepsilon} \ge 0$. At this point, we have only to prove that

Claim: $\lambda_{2,\varepsilon} > 0$ for sufficiently small $\varepsilon > 0$.

Proof: Suppose the contrary that $\lambda_{2,\varepsilon} = 0$ and there exists a solution w_ε to

$$\begin{cases} \Delta^2 w_\varepsilon = c_0 p_\varepsilon u_\varepsilon^{p_\varepsilon - 1} w_\varepsilon & \text{in } \Omega, \\ w_\varepsilon = \Delta w_\varepsilon = 0 & \text{on } \partial\Omega \end{cases} \tag{4.1}$$

for $\varepsilon \downarrow 0$. We may assume that $\|w_\varepsilon\| = \|u_\varepsilon\|$ without losing generality. We set

$$\tilde{w}_\varepsilon(y) := \frac{1}{\|u_\varepsilon\|} w_\varepsilon \left(\frac{y}{\|u_\varepsilon\|^{(p_\varepsilon - 1)/4}} \right), \quad y \in \Omega_\varepsilon = \|u_\varepsilon\|^{(p_\varepsilon - 1)/4} \Omega.$$

We obtain

$$\begin{cases} \Delta^2 \tilde{w}_\varepsilon = c_0 p_\varepsilon \tilde{u}_\varepsilon^{p_\varepsilon - 1} \tilde{w}_\varepsilon & \text{in } \Omega_\varepsilon, \\ \tilde{w}_\varepsilon = \Delta \tilde{w}_\varepsilon = 0 & \text{on } \partial\Omega_\varepsilon, \\ \|\tilde{w}_\varepsilon\|_{L^\infty(\Omega_\varepsilon)} = 1. \end{cases} \tag{4.2}$$

By standard elliptic estimates, $\tilde{w}_\varepsilon$ converges to some function w_0 uniformly on compact sets on $\mathbb{R}^N$. As in the previous section, we also know $\int_{\Omega_\varepsilon} |\Delta \tilde{w}_\varepsilon|^2 dy \leq C$ and thus $\int_{\mathbb{R}^N} |\Delta w_0|^2 dy \leq C$ for some $C > 0$. Passing to the limit in (4.2) with noting (2.6), we obtain that w_0 solves

$$\begin{cases} \Delta^2 w_0 = c_0 p U^{p-1} w_0 & \text{in } \mathbb{R}^N, \\ \|w_0\|_{L^\infty(\mathbb{R}^N)} \leq 1. \end{cases}$$

Thus again by Lemma 8, we have

$$w_0 = \sum_{j=1}^N a_j \frac{y_j}{(1+|y|^2)^{(N-2)/2}} + b \frac{1-|y|^2}{(1+|y|^2)^{(N-2)/2}} \tag{4.3}$$

for some a_j $(j = 1, \cdots, N), b \in \mathbb{R}$.

Now, we recall the following fact, which is a special case of more general result; see Lemma 13 in Appendix.

Lemma 11 *Let Ω be a smooth bounded domain satisfying (H1),(H2)'. Then any solution w_ε to (4.1) is symmetric with respect to the hyperplane $\{x_i = 0\}, (i = 1, \cdots, N)$.*

Thanks to lemma 11, we see $a_j = 0$ for all $j = 1, \cdots, N$ in (4.3), because from the symmetry of the solution w_ε to (4.1), w_0 also has to be symmetric with respect to the hyperplane $\{y_j = 0\}$ for $j = 1, \cdots, N$.

Next we will prove that $b = 0$ in (4.3). First we show an identity, which is obtained similarly as in [11].

Lemma 12 *Let u_ε be a solution of (P_ε) and w_ε be a solution of (4.1). Then we have*

$$\int_{\partial\Omega} \left(\frac{\partial \overline{u}_\varepsilon}{\partial \nu} \frac{\partial w_\varepsilon}{\partial \nu} + \frac{\partial u_\varepsilon}{\partial \nu} \frac{\partial \overline{w}_\varepsilon}{\partial \nu} \right) (x \cdot \nu) ds_x = 0, \tag{4.4}$$

here $\overline{u}_\varepsilon = -\Delta u_\varepsilon$ and $\overline{w}_\varepsilon = -\Delta w_\varepsilon$.

Proof: Set $\eta_\varepsilon = x \cdot \nabla u_\varepsilon$. By direct computation, we have

$$\Delta \eta_\varepsilon = 2\Delta u_\varepsilon + x \cdot \nabla(\Delta u_\varepsilon),$$
$$\Delta^2 \eta_\varepsilon = 4\Delta^2 u_\varepsilon + x \cdot \nabla(\Delta^2 u_\varepsilon),$$

thus

$$\Delta^2\eta_\varepsilon = 4c_0 u_\varepsilon^{p_\varepsilon} + c_0 p_\varepsilon u_\varepsilon^{p_\varepsilon - 1}\eta_\varepsilon, \quad \text{in } \Omega.$$

Multiplying this equation by w_ε, (4.1) by η_ε, and subtracting, we have

$$\int_\Omega \left((\Delta^2\eta_\varepsilon)w_\varepsilon - (\Delta^2 w_\varepsilon)\eta_\varepsilon\right) dx = \int_\Omega 4c_0 u_\varepsilon^{p_\varepsilon} w_\varepsilon dx. \tag{4.5}$$

Green's formula implies that

$$\int_\Omega c_0 u_\varepsilon^{p_\varepsilon} w_\varepsilon dx = \int_\Omega \Delta^2 u_\varepsilon \cdot w_\varepsilon dx = \int_\Omega \Delta^2 w_\varepsilon \cdot u_\varepsilon dx = \int_\Omega c_0 p_\varepsilon u_\varepsilon^{p_\varepsilon} w_\varepsilon dx,$$

so we have

$$\int_\Omega u_\varepsilon^{p_\varepsilon} w_\varepsilon dx = 0. \tag{4.6}$$

On the other hand, Navier boundary condition implies

$$\eta_\varepsilon = (x\cdot\nu)\frac{\partial u_\varepsilon}{\partial\nu}, \quad \Delta\eta_\varepsilon = x\cdot\nabla(\Delta u_\varepsilon) = (x\cdot\nu)\frac{\partial(\Delta u_\varepsilon)}{\partial\nu}$$

on $\partial\Omega$. Thus we obtain

$$\begin{aligned}
&\int_\Omega \left((\Delta^2\eta_\varepsilon)w_\varepsilon - (\Delta^2 w_\varepsilon)\eta_\varepsilon\right) dx \\
&= \int_\Omega (\Delta(\Delta\eta_\varepsilon)\cdot w_\varepsilon - \Delta\eta_\varepsilon\cdot\Delta w_\varepsilon)dx - \int_\Omega (\Delta(\Delta w_\varepsilon)\cdot\eta_\varepsilon - \Delta w_\varepsilon\cdot\Delta\eta_\varepsilon)dx \\
&= \int_{\partial\Omega}\left(\frac{\partial}{\partial\nu}(\Delta\eta_\varepsilon)w_\varepsilon - (\Delta\eta_\varepsilon)\frac{\partial w_\varepsilon}{\partial\nu}\right) ds_x - \int_{\partial\Omega}\left(\frac{\partial}{\partial\nu}(\Delta w_\varepsilon)\eta_\varepsilon - (\Delta w_\varepsilon)\frac{\partial\eta_\varepsilon}{\partial\nu}\right) ds_x \\
&= \int_{\partial\Omega}\left((-\Delta\eta_\varepsilon)\frac{\partial w_\varepsilon}{\partial\nu} + \frac{\partial(-\Delta w_\varepsilon)}{\partial\nu}\eta_\varepsilon\right) ds_x \\
&= \int_{\partial\Omega}\left(\frac{\partial(-\Delta u_\varepsilon)}{\partial\nu}\frac{\partial w_\varepsilon}{\partial\nu} + \frac{\partial(-\Delta w_\varepsilon)}{\partial\nu}\frac{\partial u_\varepsilon}{\partial\nu}\right)(x\cdot\nu)ds_x.
\end{aligned}\tag{4.7}$$

Then by (4.5), (4.6) and (4.7), we obtain (4.4). □

Using (4.2) and arguing as in Lemma 9, we again have the estimate

$$|\tilde{w}_\varepsilon(y)| \le C\frac{1}{|y|^{N-4}} \quad \text{for } y\in\Omega_\varepsilon\cap\{|y|\ge\delta\}$$

for some $\delta > 0$. By this estimate, we obtain

$$\|u_\varepsilon\| w_\varepsilon \to -2(N-2)(N-4)\sigma_N b G(\cdot,0) \quad \text{in } C^3(\omega) \tag{4.8}$$

where $\omega \subset \Omega$ is any neighborhood of $\partial\Omega$ not containing the origin. The proof of this convergence result is very similar to that of Lemma 10, so we omit it.

Now, we multiply both sides of (4.4) by $\|u_\varepsilon\|^2$ to get

$$\int_{\partial\Omega} \left(\frac{\partial(\|u_\varepsilon\|\overline{u}_\varepsilon)}{\partial\nu} \frac{\partial(\|u_\varepsilon\|w_\varepsilon)}{\partial\nu} + \frac{\partial(\|u_\varepsilon\|u_\varepsilon)}{\partial\nu} \frac{\partial(\|u_\varepsilon\|\overline{w}_\varepsilon)}{\partial\nu} \right) (x \cdot \nu) ds_x = 0. \quad (4.9)$$

By using (2.3), (4.8) and Lemma 3, LHS of (4.9) converges to

$$LHS \to -8(N-2)^2(N-4)^3\sigma_N^2 bR(0)$$

as $\varepsilon \to 0$. Therefore we have $b = 0$.

Thus we have proved that $\tilde{w}_\varepsilon \to w_0 \equiv 0$ uniformly on compact sets of $\mathbb{R}^N$. Now, the same reason of Step 3 in the previous section is applicable since $\|\tilde{w}_\varepsilon\|_{L^\infty(\Omega_\varepsilon)} = 1$, therefore we have a desired contradiction. This ends the proof of Theorem 2. □

5 Appendix

In this appendix, we show a lemma which is an extension of Theorem 2.1 in [6] to the polyharmonic operator. For this purpose, we recall some definitions.

We say that a $K \times K$ matrix $H = (H_{ij}(x))_{1 \le i,j \le K}$ with all entries in $C(\overline{\Omega})$ is *cooperative* if $H_{ij}(x) \ge 0$ for all $i \ne j, 1 \le i,j \le K$ and $x \in \overline{\Omega}$. A matrix $H = (H_{ij}(x))$ is called *fully coupled* if for all nonempty sets $I, J \subset \{1, \cdots, K\}$ with $I \cup J = \{1, \cdots, K\}$ and $I \cap J = \phi$, there exist some $i \in I, j \in J$ and $x \in \Omega$ such that $H_{ij}(x) \ne 0$. Let L be a diagonal $K \times K$ matrix of strictly elliptic second order operators and $H = (H_{ij}(x)), H_{ij} \in C(\overline{\Omega})$. We say $\phi =^t (\phi_1, \cdots, \phi_K) \in (W^{2,N}_{loc}(\Omega) \cap C(\overline{\Omega}))^K$ is a *positive strict supersolution* to the system of the second order linear elliptic equations

$$L\psi = H\psi, \quad \psi =^t (\psi_1, \cdots, \psi_K) : \overline{\Omega} \to \mathbb{R}^K,$$

if $\phi_j(x) \ge 0$, $\left((L-H)\phi\right)_j(x) \ge 0$ for all $j = 1, \cdots, K$ and $x \in \Omega$, and either $\phi \not\equiv \mathbf{0}$ on $\partial\Omega$ or $(L-H)\phi \not\equiv \mathbf{0}$ in Ω.

In [15] Theorem 1.1, it is proved that if L is as above and H is cooperative and fully coupled, if there is a positive strict supersolution to the system $L\psi = H\psi$ in Ω, and if Ω satisfies a uniform exterior cone condition, then $\psi = \mathbf{0}$ is the unique solution to

$$L\psi = H\psi \text{ in } \Omega, \quad \psi = \mathbf{0} \text{ on } \partial\Omega.$$

Lemma 13 *Let $K \in \mathbb{N}$. Let u be a smooth solution of*

$$\begin{cases} (-\Delta)^K u = f(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = \Delta u = \cdots \Delta^{K-1} u = 0 & \text{on } \partial\Omega \end{cases} \tag{5.1}$$

where Ω is a smooth bounded domain satisfying (H1),(H2)', $f \in C^1(\mathbb{R}_+)$ and $f'(u) \geq 0$ for $u > 0$. Then any solution of the linearized problem

$$\begin{cases} (-\Delta)^K v = f'(u) v & \text{in } \Omega, \\ v = \Delta v = \cdots \Delta^{K-1} v = 0 & \text{on } \partial\Omega \end{cases} \tag{5.2}$$

is also symmetric with respect to the hyperplane $\{x_i = 0\}, (i = 1, \cdots, N)$.

Proof: We rewrite the equation (5.1) to

$$\begin{cases} -\Delta u_j = u_{j+1} =: f_j(\mathbf{u}) & \text{in } \Omega, \ (j = 1, \cdots, K-1) \\ -\Delta u_K = f(u_1) =: f_K(\mathbf{u}) & \text{in } \Omega, \\ u_j > 0 & \text{in } \Omega, \ (j = 1, \cdots, K) \\ u_j = 0 & \text{on } \partial\Omega \ (j = 1, \cdots, K) \end{cases} \tag{5.3}$$

where $u_1 = u$ and $\mathbf{u} =^t (u_1, u_2, \cdots, u_K)$. Also, setting $v = v_1$, we can rewrite the equation (5.2) to

$$\begin{cases} -\Delta v_j = v_{j+1} & \text{in } \Omega, \ (j = 1, \cdots, K-1) \\ -\Delta v_K = f'(u_1) v_1 & \text{in } \Omega, \\ v_j = 0 & \text{on } \partial\Omega \ (j = 1, \cdots, K) \end{cases} \tag{5.4}$$

which is, in matrix form,

$$L\mathbf{v} = H\mathbf{v}, \quad \mathbf{v} =^t (v_1, \cdots, v_K)$$

for

$$L = \begin{pmatrix} -\Delta & & 0 \\ & \ddots & \\ 0 & & -\Delta \end{pmatrix} \quad \text{and} \quad H = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & & & \ddots & 1 \\ f'(u_1) & 0 & \cdots & \cdots & 0 \end{pmatrix}$$

Note that H is cooperative and fully coupled.

Now, fix $1 \leq i \leq K$. By our assumption of f, we easily see

$$\frac{\partial f_j}{\partial u_i}(\mathbf{u}) \geq 0, \quad j \neq i, \ 1 \leq j \leq K.$$

Therefore we can apply Lemma 4.3 in [16] to get that any solution $\mathbf{u}$ of (5.3) is symmetric with respect to the hyperplanes $\{x_i = 0\}$ and $\frac{\partial u_j}{\partial x_i} > 0$ on $\Omega_i^- = \{x \in \Omega | x_i < 0\}$ for any $j = 1, \cdots, K$. Note also that since the domain Ω is smooth and symmetric, Ω_i^- satisfies a uniform exterior cone condition for any $i = 1, \cdots, N$.

Set $\phi_j = \frac{\partial u_j}{\partial x_i}$ in Ω_i^-. From above, we see $\phi_j > 0$ in Ω_i^-. Also by elliptic regularity, we see $\phi_j \in W^{2,N}_{loc}(\Omega_i^-) \cap C(\overline{\Omega_i^-})$, and $\phi_j \not\equiv 0$ on $\partial\Omega \cap \partial\Omega_i^-$ by the Hopf lemma. Moreover, by differentiating the equation of (5.3) with respect to x_i, we have

$$\begin{cases} -\Delta(\frac{\partial u_j}{\partial x_i}) = \frac{\partial u_{j+1}}{\partial x_i} & \text{in } \Omega_i^-, \ (j = 1, \cdots, K-1) \\ -\Delta(\frac{\partial u_K}{\partial x_i}) = f'(u_1)\frac{\partial u_1}{\partial x_i} & \text{in } \Omega_i^-, \end{cases}$$

which is in matrix form, $L\phi = H\phi$ in Ω_i^- for $\phi = {}^t(\phi_1, \cdots, \phi_K)$. Therefore, ϕ is a positive strict supersolution to the system $L\psi = H\psi$ in Ω_i^- in the sense described above.

At this point, we can apply Theorem 1.1 of Sweers [15] to get that the system

$$\begin{cases} L\psi &= H\psi \quad \text{in } \Omega_i^-, \\ \psi &= 0 \quad \text{on } \partial\Omega_i^- \end{cases} \tag{5.5}$$

has the only solution $\psi \equiv \mathbf{0}$.

Now, set

$$\hat{\psi}_j(x) = v_j(x_1, \cdots, x_i, \cdots, x_N) - v_j(x_1, \cdots, -x_i, \cdots, x_N), \quad x \in \Omega_i^-$$

for $j = 1, \cdots, K$, where $\mathbf{v} = {}^t(v_1, \cdots, v_K)$ is a solution of (5.4). By the symmetry of $\mathbf{u}$, we have

$$f'(u_1(x)) = f'(u_1(x_1, \cdots, -x_i, \cdots, x_N)),$$

so $\hat{\psi} = {}^t(\hat{\psi}_1, \cdots, \hat{\psi}_K)$ is a solution of (5.5). Thus $\hat{\psi} \equiv \mathbf{0}$ and $\mathbf{v}$ is symmetric with respect to the hyperplane $\{x_i = 0\}$. □

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