A nondegeneracy result for least energy solutions to a biharmonic problem with nearly critical exponent

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Dedicated to professor Toshitaka Nagai on the occasion of his sixties birthday

2000 Mathematics Subject Classification: 35B33, 35J35, 35J40.
Key words: nondegeneracy, critical exponent, least energy solution

Abstract

Consider the problem $\Delta^2 u = c_0 K(x) u^{p_\varepsilon},\ u > 0$ in Ω, $u = \Delta u = 0$ on $\partial\Omega$, where Ω is a smooth bounded domain in $\mathbb{R}^N (N \geq 5)$, $c_0 = (N-4)(N-2)N(N+2), p = (N+4)/(N-4), p_\varepsilon = p - \varepsilon$ and K is a smooth positive function on $\overline{\Omega}$.

Under some assumptions on the coefficient function K, which include the nondegeneracy of its unique maximum point as a critical point of Hess$K$, we prove that least energy solutions of the problem are nondegenerate for $\varepsilon > 0$ small.

1 Introduction

Consider the problem

$$\begin{cases} \Delta^2 u = c_0 K(x) u^{p_\varepsilon} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N (N \geq 5)$ is a smooth bounded domain, $c_0 = (N-4)(N-2)N(N+2)$, $p_\varepsilon = p - \varepsilon, p = (N+4)/(N-4)$ is the critical Sobolev exponent with respect to the Sobolev embedding $H^2 \cap H_0^1(\Omega) \hookrightarrow L^{p+1}(\Omega)$, and $\varepsilon > 0$ is a small parameter. Here, K is a positive function in $C^2(\overline{\Omega})$.

We put an assumption on the coefficient function K:

(K): $K \in C^2(\overline{\Omega}), 0 < K(x) \leq 1, K^{-1}(\max_{\overline{\Omega}} K) = \{x_0\} \subset \Omega$ with $K(x_0) = 1$, and x_0 is a nondegenerate critical point of K.

In the following, as solutions of (1.1) we consider only least energy solutions u_ε such that

$$\frac{\int_\Omega |\Delta u_\varepsilon|^2 dx}{\left(\int_\Omega K(x)|u_\varepsilon|^{p_\varepsilon+1} dx\right)^{\frac{2}{p_\varepsilon+1}}} = \inf_{u \in H^2 \cap H_0^1(\Omega)} \frac{\int_\Omega |\Delta u|^2 dx}{\left(\int_\Omega K(x)|u|^{p_\varepsilon+1} dx\right)^{\frac{2}{p_\varepsilon+1}}}.$$

We easily check that least energy solutions blow up in the sense that $\|u_\varepsilon\|_{L^\infty(\Omega)} = u_\varepsilon(x_\varepsilon) \to +\infty$ as $\varepsilon \to 0$, and that the maximum point x_ε of u_ε converges to a maximum point of K in $\overline{\Omega}$. Therefore we have $x_\varepsilon \to x_0$ as $\varepsilon \to 0$, here by the assumption (K), x_0 is the unique interior maximum point of K.

In this note, we prove the nondegeneracy of least energy solutions to (1.1) when $\varepsilon > 0$ is sufficiently small, under the assumption (K). Here as usual, the nondegeneracy of u_ε for small ε means that the problem

$$\begin{cases} \Delta^2 v_\varepsilon = c_0 p_\varepsilon K(x) u_\varepsilon^{p_\varepsilon - 1} v_\varepsilon \quad \text{in } \Omega, \\ v_\varepsilon = \Delta v_\varepsilon = 0 \quad \text{on } \partial\Omega \end{cases} \tag{1.2}$$

admits no solution except for the trivial one for $\varepsilon > 0$ small enough.

Theorem 1.1 *Let $\Omega \subset \mathbb{R}^N (N \geq 5)$ be a smooth bounded domain. Under the assumption (K), least energy solution u_ε to (1.1) is nondegenerate for $\varepsilon > 0$ small.*

The precise asymptotic behavior of least energy solutions as $\varepsilon \to 0$ when $K \not\equiv 1$ was obtained in [6] under the assumption (K). Using this result, we prove Theorem 1.1 along the line of [7] and [8], the original idea of which comes from [4].

2 Preliminaries

In this section, we recall some facts which are needed in the sequel. Let $G = G(x,z)$ denote the Green function of Δ^2 under the Navier boundary condition:

$$\begin{cases} \Delta^2 G(\cdot, z) &= \delta_z & \text{in } \Omega, \\ G(\cdot, z) &= \Delta G(\cdot, z) = 0 & \text{on } \partial\Omega. \end{cases}$$

We decompose G as $G(x,z) = \Gamma(x,z) - H(x,z)$, where $\Gamma(x,z)$ is the fundamental solution of Δ^2:

$$\Gamma(x,z) = \begin{cases} \frac{1}{(N-4)(N-2)\sigma_N} |x-z|^{4-N}, & N \geq 5, \\ \frac{1}{\sigma_4} \log |x-z|^{-1}, & N = 4, \end{cases}$$

where σ_N is the volume of the $(N-1)$ dimensional unit sphere in $\mathbb{R}^N$ and $H(x,z)$ is the regular part of the Green function. Finally, let $R(z) = H(z,z)$ denote the Robin function of the Green function of Δ^2 with the Navier boundary condition. By the maximum principle, we have $R > 0$ on Ω and $R(z) \to +\infty$ as z tends to the boundary of Ω. In the following, we set $\overline{G} = -\Delta G$. Then $\overline{G}$ is the Green function of $-\Delta$ under the Dirichlet boundary condition, and satisfy

$$\begin{cases} -\Delta G = \overline{G}, \ -\Delta \overline{G} = \delta_z & \text{in } \Omega, \\ G > 0, \ \overline{G} > 0 & \text{in } \Omega, \\ G = \overline{G} = 0 & \text{on } \partial\Omega. \end{cases}$$

Lemma 2.1 *For any $z \in \Omega$, there holds*

$$\int_{\partial\Omega} ((x-z)\cdot\nu_x) \left(\frac{\partial G}{\partial \nu_x}\right) \left(\frac{\partial \overline{G}}{\partial \nu_x}\right) (x,z) ds_x = (N-4)R(z), \tag{2.1}$$

$$\int_{\partial\Omega} \frac{\partial G}{\partial \nu_x}(x,z) \frac{\partial \overline{G}}{\partial \nu_x}(x,z) \nu_i(x) ds_x = \frac{\partial R}{\partial z_i}(z), \quad (i = 1, \cdots, N), \tag{2.2}$$

$$\int_{\partial\Omega} \left(\frac{\partial \overline{G}}{\partial x_i}\right) \frac{\partial}{\partial z_j} \left(\frac{\partial G}{\partial \nu_x}\right)(x,z) ds_x + \int_{\partial\Omega} \left(\frac{\partial G}{\partial x_i}\right) \frac{\partial}{\partial z_j} \left(\frac{\partial \overline{G}}{\partial \nu_x}\right)(x,z) ds_x$$
$$= \frac{\partial^2 R}{\partial z_i \partial z_j}(z), \quad (i,j = 1, \cdots, N). \tag{2.3}$$

Here ν_x is the outer unit normal at $x \in \partial\Omega$.

Proof. See [3]:Lemma 3.1 and Lemma 3.3. Note that our sign convention is different from that of [3]. By differentiating (2.2) with respect to z_j, noting that $\left(\frac{\partial G}{\partial \nu_x}(x,z)\right)\nu_i(x) = \frac{\partial G}{\partial x_i}(x,z)$, $\left(\frac{\partial \overline{G}}{\partial \nu_x}(x,z)\right)\nu_i(x) = \frac{\partial \overline{G}}{\partial x_i}(x,z)$ on $\partial\Omega$, we see that (2.3) holds. □

Lemma 2.2 *Let u_ε be a solution to (1.1) and v_ε be a solution to (1.2). Denote $\overline{u}_\varepsilon = -\Delta u_\varepsilon$ and $\overline{v}_\varepsilon = -\Delta v_\varepsilon$. Then the following identities hold true:*

$$\int_{\partial\Omega} ((x-z)\cdot\nu_x) \left\{ \left(\frac{\partial u_\varepsilon}{\partial \nu_x}\right)\left(\frac{\partial \overline{v}_\varepsilon}{\partial \nu_x}\right) + \left(\frac{\partial \overline{u}_\varepsilon}{\partial \nu_x}\right)\left(\frac{\partial v_\varepsilon}{\partial \nu_x}\right) \right\} ds_x$$
$$= c_0 \int_\Omega u_\varepsilon^{p_\varepsilon} v_\varepsilon (x-z)\cdot\nabla K(x) dx \tag{2.4}$$

for any $z \in \mathbb{R}^N$ and

$$\int_{\partial\Omega} \left\{ \left(\frac{\partial \overline{u}_\varepsilon}{\partial x_i}\right)\left(\frac{\partial v_\varepsilon}{\partial \nu_x}\right) + \left(\frac{\partial u_\varepsilon}{\partial x_i}\right)\left(\frac{\partial \overline{v}_\varepsilon}{\partial \nu_x}\right) \right\} ds_x = c_0 \int_\Omega \left(\frac{\partial K}{\partial x_i}\right) u_\varepsilon^{p_\varepsilon} v_\varepsilon dx \tag{2.5}$$

for $i = 1, 2, \cdots, N$.

Proof. For smooth f, g, we have the formula

$$\int_\Omega \left((\Delta^2 f)g - (\Delta^2 g)f\right) dx$$
$$= \int_{\partial\Omega} \left(\frac{\partial \Delta f}{\partial \nu_x}\right) g - \left(\frac{\partial \Delta g}{\partial \nu_x}\right) f ds_x + \int_{\partial\Omega} \left(\frac{\partial f}{\partial \nu_x}\right) \Delta g - \left(\frac{\partial g}{\partial \nu_x}\right) \Delta f ds_x. \quad (2.6)$$

Set $w_\varepsilon(x) = (x-z)\cdot\nabla u_\varepsilon(x) + \alpha_\varepsilon u_\varepsilon(x)$ where $\alpha_\varepsilon = \frac{4}{p_\varepsilon - 1}$. Direct computation yields that

$$\Delta^2 w_\varepsilon = (\alpha_\varepsilon + 4)c_0 K(x) u_\varepsilon^{p_\varepsilon} + c_0 p_\varepsilon K(x) u_\varepsilon^{p_\varepsilon - 1}(x-z)\cdot\nabla u_\varepsilon + c_0 u_\varepsilon^{p_\varepsilon}(x-z)\cdot\nabla K(x).$$

Since v_ε satisfies $\Delta^2 v_\varepsilon = c_0 p_\varepsilon u_\varepsilon^{p_\varepsilon - 1} v_\varepsilon$, we have

$$(\Delta^2 w_\varepsilon)v_\varepsilon - (\Delta^2 v_\varepsilon)w_\varepsilon = (\alpha_\varepsilon + 4 - p_\varepsilon\alpha_\varepsilon)c_0 u_\varepsilon^{p_\varepsilon} v_\varepsilon = 0.$$

Integrating this identity on Ω with the formula (2.6), and noting that

$$w_\varepsilon(x) = (x-z)\cdot\nu_x\left(\frac{\partial u_\varepsilon}{\partial \nu_x}\right), \quad \Delta w_\varepsilon(x) = (x-z)\cdot\nu_x\left(\frac{\partial \Delta u_\varepsilon}{\partial \nu_x}\right)$$

for $x \in \partial\Omega$, we have (2.4).

On the other hand, differentiating the equation in (1.1) with respect to x_i, we have

$$\Delta^2\left(\frac{\partial u_\varepsilon}{\partial x_i}\right) = c_0 p_\varepsilon u_\varepsilon^{p_\varepsilon - 1}\left(\frac{\partial u_\varepsilon}{\partial x_i}\right) \quad \text{in } \Omega.$$

Multiplying this by v_ε, and the equation of v_ε by $\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)$ and subtracting, we obtain

$$(\Delta^2 v_\varepsilon)\left(\frac{\partial u_\varepsilon}{\partial x_i}\right) - \left(\Delta^2\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)\right) v_\varepsilon = 0.$$

Finally, integration by parts formula (2.6) yields (2.5). $\square$

Next is the asymptotic result by [6]. In what follows, we use a symbol $\|\cdot\|$ to denote the L^∞ norm of functions.

Theorem 2.3 *Let $\Omega \subset \mathbb{R}^N, N \geq 5$ be a smooth bounded domain. Let u_ε be a least energy solution to $(P_{\varepsilon,K})$ for $\varepsilon > 0$ and let $x_\varepsilon \in \Omega$ be a point such that $u_\varepsilon(x_\varepsilon) = \|u_\varepsilon\|$. Assume (K). Then after passing to a subsequence, the following estimate holds true:*

There exists a constant $C > 0$ independent of ε such that for any $R_\varepsilon \to \infty$ with $r_\varepsilon = R_\varepsilon \|u_\varepsilon\|^{-\frac{p_\varepsilon - 1}{4}} \to 0$,

$$\begin{cases} u_\varepsilon(x) \le C \frac{\|u_\varepsilon\|}{\left(1+\|u_\varepsilon\|^{\frac{4}{N-4}}|x-x_\varepsilon|^2\right)^{\frac{N-4}{2}}}, & \text{for } |x - x_\varepsilon| \le r_\varepsilon, \\ u_\varepsilon(x) \le \frac{C}{\|u_\varepsilon\|} \frac{1}{|x-x_\varepsilon|^{N-4}}, & \text{for } \{|x - x_\varepsilon| > r_\varepsilon\} \cap \Omega. \end{cases} \tag{2.7}$$

Furthermore, as $\varepsilon \to 0$,

$$(1) \begin{cases} |x_\varepsilon - x_0| = O(\|u_\varepsilon\|^{-2}) & N = 5, \\ |x_\varepsilon - x_0| = o(\|u_\varepsilon\|^{-\frac{2}{N-4}}) & N \ge 6, \end{cases} \tag{2.8}$$

$$(2) \|u_\varepsilon\|^\varepsilon \to 1, \tag{2.9}$$

$$(3) \|u_\varepsilon\| u_\varepsilon(x) \to 2(N-4)(N-2)\sigma_N G(x, x_0) \quad \text{in } C^3_{loc}(\overline{\Omega} \setminus \{x_0\}), \tag{2.10}$$

$$(4) \begin{cases} \varepsilon \|u_\varepsilon\|^2 \to \frac{2^{15}}{21}\pi R(x_0) & N = 5, \\ \varepsilon \|u_\varepsilon\|^2 \to -\frac{1}{4}\Delta K(x_0) + 480\pi^3 R(x_0) & N = 6, \\ \varepsilon \|u_\varepsilon\|^{\frac{4}{N-4}} \to -\frac{2}{(N-2)(N-4)}\Delta K(x_0) & N \ge 7. \end{cases} \tag{2.11}$$

Now, consider the scaled function

$$\tilde{u}_\varepsilon(y) := \frac{1}{\|u_\varepsilon\|} u_\varepsilon \left(\frac{y}{\|u_\varepsilon\|^{\frac{p_\varepsilon - 1}{4}}} + x_\varepsilon \right), \quad y \in \Omega_\varepsilon := \|u_\varepsilon\|^{\frac{p_\varepsilon - 1}{4}} (\Omega - \varepsilon). \tag{2.12}$$

$\tilde{u}_\varepsilon$ satisfies $0 < \tilde{u}_\varepsilon \le 1, \tilde{u}_\varepsilon(0) = 1$, and

$$\begin{cases} \Delta^2 \tilde{u}_\varepsilon = c_0 K(\frac{y}{\|u_\varepsilon\|^{\frac{p_\varepsilon - 1}{4}}} + x_\varepsilon) \tilde{u}_\varepsilon^{p_\varepsilon} & \text{in } \Omega_\varepsilon, \\ \tilde{u}_\varepsilon = \Delta \tilde{u}_\varepsilon = 0 & \text{on } \partial\Omega_\varepsilon. \end{cases}$$

Since $\|u_\varepsilon\| \to \infty$ as $\varepsilon \to 0$ and x_ε does not approach to $\partial\Omega$, we see $\Omega_\varepsilon \to \mathbb{R}^N$. By standard elliptic estimates, we have a subsequence denoted also by $\tilde{u}_\varepsilon$ that

$$\tilde{u}_\varepsilon \to U \quad \text{compact uniformly in } \mathbb{R}^N \tag{2.13}$$

as $\varepsilon \to 0$ for some function U. Passing to the limit, we obtain that U is a solution of

$$\begin{cases} \Delta^2 U = c_0 U^p \quad \text{in } \mathbb{R}^N, \\ 0 < U \le 1, \ U(0) = 1, \\ \lim_{|y| \to \infty} U(y) = 0. \end{cases}$$

According to the uniqueness theorem by Chang Shou Lin [5], we obtain

$$U(y) = \left(\frac{1}{1+|y|^2}\right)^{\frac{N-4}{2}}. \tag{2.14}$$

In terms of $\tilde{u}_\varepsilon$ in (2.12), the estimate (2.7) reads

$$\tilde{u}_\varepsilon(y) \le \begin{cases} CU(y) & \text{for } |y| \le R_\varepsilon, \\ C\frac{1}{|y|^{N-4}} & \text{for } \{|y| > R_\varepsilon\} \cap \Omega_\varepsilon, \end{cases} \tag{2.15}$$

where $R_\varepsilon \to \infty$ is any sequence as in the above.

Here, we recall a theorem by Bartsch, Weth and Willem [1].

Lemma 2.4 *Let v_0 be a solution to*

$$\begin{cases} \Delta^2 v_0 = c_0 p U^{p-1} v_0 \quad \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |\Delta v_0|^2 dy < \infty. \end{cases}$$

Then there exist a_j $(j = 1, 2, \cdots, N), b \in \mathbb{R}$ such that v_0 can be written as

$$v_0 = \sum_{j=1}^{N} a_j \frac{y_j}{(1+|y|^2)^{(N-2)/2}} + b\frac{1-|y|^2}{(1+|y|^2)^{(N-2)/2}}.$$

3 Proof of Theorem 1.1

In this section, we will prove Theorem 1.1.

We argue by contradiction. We assume there exists a non-trivial solution v_ε to (1.2) satisfying $\|v_\varepsilon\| = \|u_\varepsilon\|$ for any $\varepsilon > 0$.

Consider the scaled function

$$\tilde{v}_\varepsilon(y) = \frac{1}{\|u_\varepsilon\|} v_\varepsilon\left(\frac{y}{\|u_\varepsilon\|^{\frac{p_\varepsilon - 1}{4}}} + x_\varepsilon\right), \quad y \in \Omega_\varepsilon = \|u_\varepsilon\|^{\frac{p_\varepsilon-1}{4}}(\Omega - x_\varepsilon). \tag{3.1}$$

We see $0 < \tilde{v}_\varepsilon \le 1$ and $\tilde{v}_\varepsilon$ satisfies

$$\begin{cases} \Delta^2 \tilde{v}_\varepsilon = c_0 p_\varepsilon K\left(\frac{y}{\|u_\varepsilon\|^{\frac{p_\varepsilon-1}{4}}} + x_\varepsilon\right) \tilde{u}_\varepsilon^{p_\varepsilon - 1} \tilde{v}_\varepsilon \quad \text{in } \Omega_\varepsilon, \\ \tilde{v}_\varepsilon = \Delta \tilde{v}_\varepsilon = 0 \quad \text{on } \partial\Omega_\varepsilon, \\ \|\tilde{v}_\varepsilon\|_{L^\infty(\Omega_\varepsilon)} = 1. \end{cases} \tag{3.2}$$

By $\|\tilde{v}_\varepsilon\|_{L^\infty(\Omega_\varepsilon)} = 1$, elliptic estimate implies that

$$\tilde{v}_\varepsilon \to v_0 \quad \text{compact uniformly in } \mathbb{R}^N \tag{3.3}$$

for some v_0 and v_0 satisfies

$$\Delta^2 v_0 = c_0 p U^{p-1} v_0 \quad \text{in } \mathbb{R}^N.$$

Also by arguing as in [7], we have

$$\int_{\Omega_\varepsilon} |\Delta \tilde{v}_\varepsilon|^2 dy \le C \tag{3.4}$$

for some $C > 0$ independent of $\varepsilon > 0$ small. By (3.4) and Fatou's lemma, we also have

$$\int_{\mathbb{R}^N} |\Delta v_0|^2 dy \le C.$$

Thus by Lemma 2.4, we have

$$v_0 = \sum_{j=1}^{N} a_j \frac{y_j}{(1+|y|^2)^{(N-2)/2}} + b \frac{1-|y|^2}{(1+|y|^2)^{(N-2)/2}}. \tag{3.5}$$

In the following, we divide the proof into three steps.

Step 1. $b = 0$.

Step 2. $a_j = 0, j = 1, \cdots, N$.

Step 3. $v_0 = 0$ leads to a contradiction.

First, by using the Kelvin transformation and a local supremum estimate for weak solutions to a linear biharmonic equation by Caristi and Mitidieri [2], we can obtain the pointwise estimate for the scaled function $\tilde{v}_\varepsilon$, just as in [7] Lemma 3.1.

Lemma 3.1 *Let $\tilde{v}_\varepsilon$ be a solution of (3.2). Then we have the estimate*

$$|\tilde{v}_\varepsilon(y)| \le CU(y), \quad \forall y \in \Omega_\varepsilon \tag{3.6}$$

for some $C > 0$.

Also by Lemma 3.1 and Theorem 2.3 (2.7), we have the following convergence result. For a proof, see Lemma 3.2 in [7].

Lemma 3.2 *Let $\omega \subset \Omega$ be any neighborhood of $\partial\Omega$ not containing x_0. Then we have*

$$\|u_\varepsilon\| v_\varepsilon \to -2(N-2)(N-4)\sigma_N b G(\cdot, x_0) \quad \text{in } C^3(\omega). \tag{3.7}$$

Proof of Step 1. Here, we prove only the case $N \geq 7$. Proof of the cases $N = 5$ and $N = 6$ will be done by a similar argument; see [8] for the second order $-\Delta$ case.

Putting $z = x_0$ in (2.4) and multiplying $\|u_\varepsilon\|^{4/(N-4)}$, we have

$$\begin{aligned}
&\|u_\varepsilon\|^{\frac{4}{N-4}-2} \int_{\partial\Omega} ((x - x_0) \cdot \nu_x) \left(\frac{\partial \|u_\varepsilon\| u_\varepsilon}{\partial \nu_x} \right) \left(\frac{\partial \|u_\varepsilon\| \overline{v}_\varepsilon}{\partial \nu_x} \right) ds_x \\
&+ \|u_\varepsilon\|^{\frac{4}{N-4}-2} \int_{\partial\Omega} ((x - x_0) \cdot \nu_x) \left(\frac{\partial \|u_\varepsilon\| \overline{u}_\varepsilon}{\partial \nu_x} \right) \left(\frac{\partial \|u_\varepsilon\| v_\varepsilon}{\partial \nu_x} \right) ds_x \\
&= \|u_\varepsilon\|^{\frac{4}{N-4}} c_0 \int_\Omega u_\varepsilon^{p_\varepsilon} v_\varepsilon (x - z) \cdot \nabla K(x) dx.
\end{aligned} \tag{3.8}$$

As $\frac{4}{N-4} < 2$ if $N \geq 7$, LHS of (3.8) converges to 0 as $\varepsilon \to 0$. On the other hand, by Taylor's formula and the change of variables, we write

$$\text{(RHS) of (3.8)} =: C_1 + C_2 + C_3 + C_4$$

where, putting $b_{ij} = \frac{\partial^2 K}{\partial x_i \partial x_j}(x_0)$,

$$C_1 = c_0 \|u_\varepsilon\|^{\frac{4}{N-4} + p_\varepsilon + 1 - (\frac{p_\varepsilon - 1}{4})N - (\frac{p_\varepsilon - 1}{2})} \int_{\Omega_\varepsilon} \tilde{u}_\varepsilon^{p_\varepsilon} \tilde{v}_\varepsilon(y) \sum_{i,j=1}^N b_{ij} y_i y_j dy,$$

$$C_2 = 2c_0 \|u_\varepsilon\|^{\frac{4}{N-4} + p_\varepsilon + 1 - (\frac{p_\varepsilon - 1}{4})N - (\frac{p_\varepsilon - 1}{4})} \int_{\Omega_\varepsilon} \tilde{u}_\varepsilon^{p_\varepsilon} \tilde{v}_\varepsilon(y) \sum_{i,j=1}^N b_{ij} y_i (x_{\varepsilon j} - x_{0j}) dy,$$

$$C_3 = c_0 \|u_\varepsilon\|^{\frac{4}{N-4} + p_\varepsilon + 1 - (\frac{p_\varepsilon - 1}{4})N} \int_{\Omega_\varepsilon} \tilde{u}_\varepsilon^{p_\varepsilon} \tilde{v}_\varepsilon(y) \sum_{i,j=1}^N b_{ij} (x_{\varepsilon i} - x_{0i})(x_{\varepsilon j} - x_{0j}) dy,$$

$$C_4 = c_0 \|u_\varepsilon\|^{\frac{4}{N-4} + p_\varepsilon + 1 - (\frac{p_\varepsilon - 1}{4})N} \int_{\Omega_\varepsilon} \tilde{u}_\varepsilon^{p_\varepsilon} \tilde{v}_\varepsilon(y) \left(O \left| \frac{y}{\|u_\varepsilon\|^{\frac{p_\varepsilon - 1}{4}}} + x_\varepsilon - x_0 \right|^3 \right) dy.$$

By (2.15), (3.6), (2.9), (2.8) and the dominated convergence theorem, we see

$$C_2 = O(\|u_\varepsilon\|^{\frac{2}{N-4}+\frac{N-3}{4}\varepsilon}) \times O\left(\int_{\mathbb{R}^N} U^p v_0(y)|y|dy + o(1)\right) \times o(\|u_\varepsilon\|^{-\frac{2}{N-4}}) = o(1),$$

$$C_3 = O(\|u_\varepsilon\|^{\frac{4}{N-4}+\frac{N-4}{4}\varepsilon}) \times O\left(\int_{\mathbb{R}^N} U^p v_0(y)dy + o(1)\right) \times o(\|u_\varepsilon\|^{-\frac{4}{N-4}}) = o(1),$$

$$\begin{aligned} C_4 &= O(\|u_\varepsilon\|^{\frac{4}{N-4}+\frac{N-4}{4}\varepsilon}) \int_{\Omega_\varepsilon} \tilde{u}_\varepsilon^{p_\varepsilon} \tilde{v}_\varepsilon(y) \left(O\left(\left| \frac{y}{\|u_\varepsilon\|^{\frac{p_\varepsilon - 1}{2}}} \right|^3 \right) + O(|x_\varepsilon - x_0|^3) \right) \\ &= O(\|u_\varepsilon\|^{\frac{4}{N-4}}) \times O(\|u_\varepsilon\|^{-\frac{6}{N-4}}) \times O\left(\int_{\mathbb{R}^N} U^p v_0(y)(|y|^3+1)dy + o(1)\right) \\ &= O(\|u_\varepsilon\|^{-\frac{2}{N-4}}) \end{aligned}$$

as $\varepsilon \to 0$. As for C_1, we see

$$C_1 = c_0 \|u_\varepsilon\|^{(\frac{N-2}{4})\varepsilon} \int_{\Omega_\varepsilon} \tilde{u}_\varepsilon^{p_\varepsilon} \tilde{v}_\varepsilon(y) \sum_{i,j=1}^N b_{ij} y_i y_j dy$$

$$\to c_0 \int_{\mathbb{R}^N} U^p(y) v_0(y) \sum_{i,j=1}^N b_{ij} y_i y_j dy = \frac{c_0}{N} b \Delta K(x_0) \int_{\mathbb{R}^N} U^p(y) \frac{1-|y|^2}{(1+|y|^2)^{N/2}} |y|^2 dy.$$

Thus letting $\varepsilon \to 0$ in (3.8), we have

$$0 = \Delta K(x_0) \times b.$$

Hence we obtain $b = 0$, because our nondegeneracy assumption of x_0 assures that $\Delta K(x_0) < 0$ strictly.

Proof of Step 2.

In this step, we prove $a_j = 0, j = 1, 2, \cdots, N$ in (3.5) by using the next lemma.

Lemma 3.3 *Assume $b = 0$ and $a = (a_1, \cdots, a_N) \neq 0$ in (3.5). Then we have*

$$\|u_\varepsilon\|^{\frac{N-2}{N-4}} v_\varepsilon \to 2(N-2)\sigma_N \sum_{j=1}^N a_j \left(\frac{\partial G}{\partial z_j}(x,z) \right) \Big|_{z=x_0}$$

in $C^3_{loc}(\overline{\Omega} \setminus \{x_0\})$.

Proof. Since $-\Delta \overline{v}_\varepsilon = c_0 p_\varepsilon K(x) u_\varepsilon^{p_\varepsilon - 1} v_\varepsilon$ in Ω, $\overline{v}_\varepsilon = 0$ on $\partial\Omega$, the Green representation formula implies that

$$\overline{v}_\varepsilon(x) = c_0 p_\varepsilon \int_\Omega \overline{G}(x,z) K(z) u_\varepsilon^{p_\varepsilon - 1}(z) v_\varepsilon(z) dz \tag{3.9}$$

for any $x \in \overline{\Omega} \setminus \{x_0\}$, here $\overline{G}(x,z) = -\Delta_x G(x,z)$ is the Green function of $-\Delta$ under the Dirichlet boundary condition. By a change of variables, we see

$$\begin{aligned} & c_0 p_\varepsilon \int_\Omega \overline{G}(x,z) K(z) u_\varepsilon^{p_\varepsilon - 1}(z) v_\varepsilon(z) dz \\ &= c_0 p_\varepsilon \|u_\varepsilon\|^{p_\varepsilon - (\frac{p_\varepsilon - 1}{4})N} \int_{\Omega_\varepsilon} \overline{G}_\varepsilon(x,y) K_\varepsilon(y) \tilde{u}_\varepsilon^{p_\varepsilon - 1} \tilde{v}_\varepsilon(y) dy \end{aligned}$$

where $\overline{G}_\varepsilon(x,y) = \overline{G}(x, \frac{y}{\|u_\varepsilon\|^{\frac{p_\varepsilon - 1}{4}}} + x_\varepsilon)$ and $K_\varepsilon(y) = K(\frac{y}{\|u_\varepsilon\|^{\frac{p_\varepsilon - 1}{4}}} + x_\varepsilon)$. By (2.13) and (3.3), we obtain

$$K_\varepsilon(y) \tilde{u}_\varepsilon^{p_\varepsilon - 1} \tilde{v}_\varepsilon(y) \to \sum_{j=1}^N a_j \left(\frac{\partial}{\partial y_j} \frac{-1}{(N+4)} U^p(y) \right)$$

uniformly on compact subsets of $\mathbb{R}^N$.

Now, let us consider the following linear first order PDE

$$\sum_{j=1}^N a_j \frac{\partial w_\varepsilon}{\partial y_j} = \tilde{u}_\varepsilon^{p_\varepsilon - 1} \tilde{v}_\varepsilon(y), \quad y \in \mathbb{R}^N$$

with the initial condition $w_\varepsilon|_{\Gamma_a} = \frac{-1}{(N+4)} U^p(y)$, where $\Gamma_a = \{x \in \mathbb{R}^N | x \cdot a = 0\}$. Here, the right hand side is assumed to be 0 outside of Ω_ε. By the unique solvability, we have the solution w_ε of this problem with the estimate $w_\varepsilon(y) = O(|y|^{-(N+3)})$ as $|y| \to \infty$, since $\tilde{u}_\varepsilon^{p_\varepsilon - 1} \tilde{v}_\varepsilon(y) = O(|y|^{-(N+4)})$ by (2.15) and (3.6). Also we have

$$w_\varepsilon \to \frac{-1}{(N+4)} U^p \quad \text{uniformly on compact subsets on } \mathbb{R}^N$$

and

$$\int_{\Omega_\varepsilon} w_\varepsilon(y) dy \to \frac{-1}{(N+4)} \int_{\mathbb{R}^N} U^p dy = \left(\frac{-1}{N+4} \right) \left(\frac{2\sigma_N}{N(N+2)} \right)$$

by the dominated convergence theorem. Using integration by parts, we have

$$\overline{v}_\varepsilon(x) = c_0 p_\varepsilon \|u_\varepsilon\|^{p_\varepsilon - (\frac{p_\varepsilon - 1}{4})N} \int_{\Omega_\varepsilon} \overline{G}_\varepsilon(x,y) K_\varepsilon(y) \sum_{j=1}^N a_j \frac{\partial w_\varepsilon}{\partial y_j} dy$$

$$= -c_0 p_\varepsilon \|u_\varepsilon\|^{p_\varepsilon - (\frac{p_\varepsilon - 1}{4})N} \sum_{j=1}^N a_j \int_{\Omega_\varepsilon} \frac{\partial}{\partial y_j} \left\{ \overline{G}_\varepsilon(x,y) K_\varepsilon(y) \right\} w_\varepsilon(y) dy$$

$$= -c_0 p_\varepsilon \|u_\varepsilon\|^{p_\varepsilon - (\frac{p_\varepsilon - 1}{4})N - (\frac{p_\varepsilon - 1}{4})} \sum_{j=1}^N a_j \int_{\Omega_\varepsilon} \frac{\partial}{\partial z_j} \left\{ \overline{G}(x,z) K(z) \right\} \Big|_{z = \frac{y}{\|u_\varepsilon\|^{\frac{p_\varepsilon - 1}{4}}} + x_\varepsilon} w_\varepsilon(y) dy.$$

Note that $p_\varepsilon - (\frac{p_\varepsilon - 1}{4})N - (\frac{p_\varepsilon - 1}{4}) = -(\frac{N-2}{N-4}) + \varepsilon(\frac{N-3}{4})$. Now, we see

$$\begin{aligned} &\frac{\partial}{\partial z_j} \left\{ \overline{G}(x,z) K(z) \right\} \Big|_{z = \frac{y}{\|u_\varepsilon\|^{\frac{p_\varepsilon - 1}{4}}} + x_\varepsilon} \\ &\to \left(\frac{\partial \overline{G}}{\partial z_j}(x, x_0) \right) K(x_0) + \overline{G}(x, x_0) \left(\frac{\partial K}{\partial z_j}(x_0) \right) \\ &= \frac{\partial \overline{G}}{\partial z_j}(x, x_0) \end{aligned}$$

uniformly on compact subsets of $\mathbb{R}^N$ as $\varepsilon \to 0$, since x_0 is a critical point of K with $K(x_0) = 1$. Also we note that $\frac{N}{N-2} + p_\varepsilon - (\frac{p_\varepsilon - 1}{2})N - (\frac{p_\varepsilon - 1}{2}) = (\frac{N-1}{2})\varepsilon$. Therefore, we have the convergence

$$\begin{aligned} \|u_\varepsilon\|^{\frac{N-2}{N-4}} \overline{v}_\varepsilon(x) \to & - c_0 p \left(\frac{-1}{N+4} \right) \left(\frac{2\sigma_N}{N(N+2)} \right) \sum_{j=1}^N a_j \left(\frac{\partial \overline{G}}{\partial z_j}(x,z) \right) \Big|_{z=x_0} \\ &= 2(N-2)\sigma_N \sum_{j=1}^N a_j \left(\frac{\partial \overline{G}}{\partial z_j}(x,z) \right) \Big|_{z=x_0} \end{aligned}$$

for any $x \in \overline{\Omega} \setminus \{x_0\}$. Elliptic estimates implies this convergence holds true in $C^1_{loc}(\overline{\Omega} \setminus \{x_0\})$. This proves Lemma. □

Now, assume the contrary that $a = (a_1, \cdots, a_N) \neq 0$. We multiply both sides of (2.5) in Lemma 2.2 by $\|u_\varepsilon\|^{(N-2)/(N-4)} \times \|u_\varepsilon\|^{-1}$ to get

$$\|u_\varepsilon\|^{-2}\left[\int_{\partial\Omega}\left(\frac{\partial\|u_\varepsilon\|\overline{u}_\varepsilon}{\partial x_i}\right)\left(\frac{\partial\|u_\varepsilon\|^{\frac{N-2}{N-4}}v_\varepsilon}{\partial\nu_x}\right)ds_x + \left(\frac{\partial\|u_\varepsilon\|u_\varepsilon}{\partial x_i}\right)\left(\frac{\partial\|u_\varepsilon\|^{\frac{N-2}{N-4}}\overline{v}_\varepsilon}{\partial\nu_x}\right)ds_x\right]$$
$$= \|u_\varepsilon\|^{-1+\frac{N-2}{N-4}}c_0\int_\Omega\left(\frac{\partial K}{\partial x_i}\right)u_\varepsilon^{p_\varepsilon}v_\varepsilon dx \tag{3.10}$$

As $\varepsilon \to 0$, we see that

$$\int_{\partial\Omega}\left(\frac{\partial\|u_\varepsilon\|\overline{u}_\varepsilon}{\partial x_i}\right)\left(\frac{\partial\|u_\varepsilon\|^{\frac{N-2}{N-4}}v_\varepsilon}{\partial\nu_x}\right)ds_x + \left(\frac{\partial\|u_\varepsilon\|u_\varepsilon}{\partial x_i}\right)\left(\frac{\partial\|u_\varepsilon\|^{\frac{N-2}{N-4}}\overline{v}_\varepsilon}{\partial\nu_x}\right)ds_x$$

tends to

$$\begin{aligned}
&4(N-4)(N-2)^2\sigma_N^2\sum_{j=1}^N a_j\times\\
&\int_{\partial\Omega}\left\{\left(\frac{\partial\overline{G}}{\partial x_i}\right)\frac{\partial}{\partial\nu_x}\left(\frac{\partial G}{\partial z_j}\right)(x,x_0) + \left(\frac{\partial G}{\partial x_i}\right)\frac{\partial}{\partial\nu_x}\left(\frac{\partial\overline{G}}{\partial z_j}\right)(x,x_0)\right\}ds_x\\
&= 4(N-4)(N-2)^2\sigma_N^2\sum_{j=1}^N a_j\frac{\partial^2 R}{\partial z_i\partial z_j}(z)\Big|_{z=x_0},
\end{aligned}$$

here we have used Theorem 2.3 (2.10), Lemma 3.3 and Lemma 2.1 (2.3). Thus we have (LHS) of (3.10) tends to 0 as $\varepsilon \to 0$.

On the other hand, again we solve the linear PDE

$$\sum_{j=1}^N a_j\frac{\partial w_\varepsilon}{\partial y_j} = \tilde{u}_\varepsilon^{p_\varepsilon}\tilde{v}_\varepsilon(y), \quad y \in \mathbb{R}^N \tag{3.11}$$

with the initial condition $w_\varepsilon|_{\Gamma_a} = \frac{-1}{2N}U^{p+1}(y)$, where $\Gamma_a = \{x \in \mathbb{R}^N | x\cdot a = 0\}$. Here as before, the RHS of (3.11) is understood as 0 outside of Ω_ε. The solution w_ε satisfies the estimate $w_\varepsilon(y) = O(|y|^{-2N+1})$ as $|y| \to \infty$, since $\tilde{u}_\varepsilon^{p_\varepsilon}\tilde{v}_\varepsilon(y) = O(U^{p_\varepsilon+1}(y)) = O(|y|^{-2N})$ by (2.15) and (3.6). As before, we have

$$w_\varepsilon \to \frac{-1}{2N}U^{p+1} \quad \text{uniformly on compact subsets on } \mathbb{R}^N$$

and

$$\int_{\Omega_\varepsilon} w_\varepsilon(y)dy \to \frac{-1}{2N}\int_{\mathbb{R}^N}U^{p+1}dy = \frac{-1}{2N}\sigma_N C_N$$

by the dominated convergence theorem, where $C_N = \int_0^\infty \frac{r^{N-1}}{(1+r^2)^N} dr = \frac{\Gamma(N/2)^2}{2\Gamma(N)}$. Thus, (RHS of (2.5)) $\times \|u_\varepsilon\|^{\frac{N}{N-2}-1}$ is

$$
\begin{aligned}
& c_0 \|u_\varepsilon\|^{-1+\frac{N-2}{N-4}} \int_\Omega \left(\frac{\partial K}{\partial x_i}\right) u_\varepsilon^{p_\varepsilon} v_\varepsilon dx \\
&= c_0 \|u_\varepsilon\|^{\frac{N-2}{N-4}+p_\varepsilon-(\frac{p_\varepsilon-1}{4})N} \int_{\Omega_\varepsilon} \left(\frac{\partial K}{\partial x_i}\right) \Big(\frac{y}{\|u_\varepsilon\|^{\frac{p_\varepsilon-1}{4}}} + x_\varepsilon\Big) \tilde{u}_\varepsilon^{p_\varepsilon} \tilde{v}_\varepsilon dy \\
&= c_0 \|u_\varepsilon\|^{\frac{N-2}{N-4}+p_\varepsilon-(\frac{p_\varepsilon-1}{4})N} \int_{\Omega_\varepsilon} \left(\frac{\partial K}{\partial x_i}\right) \Big(\frac{y}{\|u_\varepsilon\|^{\frac{p_\varepsilon-1}{4}}} + x_\varepsilon\Big) \sum_{j=1}^N a_j \frac{\partial w_\varepsilon}{\partial y_j} dy \\
&= -c_0 \|u_\varepsilon\|^{\frac{N-2}{N-4}+p_\varepsilon-(\frac{p_\varepsilon-1}{4})N} \sum_{j=1}^N a_j \int_{\Omega_\varepsilon} \frac{\partial}{\partial y_j} \left\{ \left(\frac{\partial K}{\partial x_i}\right) \Big(\frac{y}{\|u_\varepsilon\|^{\frac{p_\varepsilon-1}{4}}} + x_\varepsilon\Big) \right\} w_\varepsilon(y) dy \\
&= -c_0 \|u_\varepsilon\|^{\frac{N-2}{N-4}+p_\varepsilon-(\frac{p_\varepsilon-1}{4})N-(\frac{p_\varepsilon-1}{4})} \sum_{j=1}^N a_j \int_{\Omega_\varepsilon} \left(\frac{\partial^2 K}{\partial x_i \partial x_j}\right)(x)\Big|_{x=\frac{y}{\|u_\varepsilon\|^{\frac{p_\varepsilon-1}{4}}}+x_\varepsilon} w_\varepsilon(y) dy \\
&\to -c_0 \left(\lim_{\varepsilon\to 0} \|u_\varepsilon\|^{(\frac{N-3}{4})\varepsilon}\right) \sum_{j=1}^N a_j \frac{\partial^2 K}{\partial x_i \partial x_j}(x_0) \left(\lim_{\varepsilon\to 0} \int_{\Omega_\varepsilon} w_\varepsilon(y) dy\right) \\
&= \frac{N}{2} c_0 \sigma_N C_N \sum_{j=1}^N a_j \frac{\partial^2 K}{\partial x_i \partial x_j}(x_0).
\end{aligned}
$$

Thus we have

$$
\sum_{j=1}^N a_j \frac{\partial^2 K}{\partial x_i \partial x_j}(x_0) = 0.
$$

By our assumption of the nondegeneracy of x_0, the matrix $\left(\frac{\partial^2 K}{\partial x_i \partial x_j}\right)(x_0)$ is invertible. Therefore we obtain that $a_j = 0$ for all $j = 1, \cdots, N$. Thus we have proved Step 2. □

Proof of Step 3.

By Step 1 and Step 2, we have obtained that the limit function $\lim_{\varepsilon\to 0} \tilde{v}_\varepsilon = v_0 \equiv 0$. Since $\|\tilde{v}_\varepsilon\|_{L^\infty(\Omega_\varepsilon)} = 1$, there exists $y_\varepsilon \in \Omega_\varepsilon$ such that $\tilde{v}_\varepsilon(y_\varepsilon) = 1$ and $|y_\varepsilon| \to \infty$, because the above convergence $\tilde{v}_\varepsilon \to v_0 \equiv 0$ is uniform on compact sets of $\mathbb{R}^N$. But this is not possible because of Lemma 3.1. This proves Theorem 1.1. □

Acknowledgements. Part of this work was supported by JSPS Grant-in-Aid for Scientific Research (C), No. 20540216.

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