Classification of torus manifolds with codimension one extended actions

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Abstract. The goal of this paper is to classify torus manifolds $(M^{2n},\ T^n)$ with codimension one extended G-actions $(M^{2n},\ G)$ up to essential isomorphism, where G is a compact, connected Lie group whose maximal torus is T^n. For technical reasons, in this paper, we do not assume that torus manifolds are omnioriented. As a result, we have the following two results: (1) if there is no exceptional orbit in extended G-actions, there are five kinds of $(M^{2n},\ G)$'s (Theorem 1.1, 1.2); (2) otherwise, there are two kinds of $(M^{2n},\ G)$'s (Theorem 1.3). As a corollary of these results, we also have that if M^{2n} is a non-singular toric variety or a quasitoric manifold with codimension one extended G-actions, then M^{2n} is a complex projective bundle over a product of complex projective spaces.

1. Introduction

This paper is a continuation of [**12**] and [**13**] devoted to the study of the extended G-actions on torus manifolds $(M^{2n},\ T^n)$, where a *torus manifold* is an even dimensional oriented manifold M^{2n} acted on by a half-dimensional torus T^n with non-empty fixed point set, and G is a compact, connected Lie group whose maximal torus is T^n. In the first paper [**12**], we classified the homogeneous (*unoriented*) torus manifolds and their transformation groups up to essential isomorphism, where here an unoriented torus manifold means a torus manifold which is not assumed omniorientations. By using classical Lie theory, we proved such torus manifolds are only products of even dimensional spheres and complex projective spaces divided by finite groups. In the second paper [**13**], we classified quasitoric manifolds with codimension 1 extended G-actions up to essential isomorphism and studied relations with moment-angle manifolds. In order to classify such quasitoric manifolds, we classified more general class which involve them, i.e., simply connected torus manifolds with codimension 1 extended G-actions whose two singular orbits are also simply connected torus manifolds. To classify such torus manifolds, we used the part of the Uchida's method in [**17**]. The Uchida's method is the strong method to classify codimension 1 compact Lie group actions up to essential isomorphism. In the case that we apply the Uchida's method to classify codimension 1 actions,

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we usually need to divide our proof into two cases (compare the method of [**11**, Section 7, 8] and that of [**11**, Section 10, 11]). In particular, for torus manifolds with codimension 1 extended G-actions, (as we mentioned in [**13**]) we divide our proof into the following two cases: the case that two singular orbits of G-actions are torus manifolds; and the case that one of two singular orbits of G-actions is not a torus manifold. Remark that if a singular orbit is a torus manifold then this is a homogeneous torus manifold (see Lemma 2.1); therefore, we know such singular orbit type by using [**12**] (see Lemma 2.2). In the previous paper [**13**], we only studied the former case because of its purpose. However, in general, the latter case also occurs (see [**13**, Example 3.5]). The goal of this paper is to classify all unoriented torus manifolds with codimension one extended G-actions up to essential isomorphism. In particular, in this paper, we put a special emphasis on the proof of the latter case, because the proof of the former case is almost similar to that of [**13**] even if in the case of the classification of such unoriented torus manifolds.

Now we state the main result of this paper. Put $m_j,\ l_i \in \mathbb{N}\cup\{0\}$ for $j = 1,\ \cdots,\ b$ and $i = 1,\ \cdots,\ a$, and $\mathcal{A} \subset \prod_{j=1}^{b}\mathbb{Z}_2$. We get the following three theorems (see Section 3.1, 5.1 and 8.1 for detail).

Theorem 1.1 (Theorem 3.1). *Suppose a torus manifold M has a codimension one extended G-action. If there are two singular orbits and both of them are torus manifolds, then $(M,\ G)$ is essentially isomorphic to*

$$\left(\prod_{j=1}^{b} S^{2m_j} \times_{\mathcal{A}} N,\quad \prod_{j=1}^{b} SO(2m_j+1)\times H\right),$$

such that $(N,\ H)$ is one of the following three types:

N	H
$\left(\prod_{i=1}^{a} S^{2l_i+1}\right)\times_{T^a} S(\mathbb{C}^k_{\mathfrak{a}}\oplus\mathbb{R})$	$\prod_{i=1}^{a} SU(l_i+1)\times U(k)$
$\left(\prod_{i=1}^{a-1} S^{2l_i+1}\right)\times_{T^{a-1}} P(\mathbb{C}^{k_1}_{\mathfrak{b}}\oplus\mathbb{C}^{k_2})$	$\prod_{i=1}^{a-1} SU(l_i+1)\times S(U(k_1)\times U(k_2))$
$\prod_{i=1}^{a} \mathbb{C}P(l_i)\times S(\mathbb{R}^{2k}\oplus\mathbb{R})$	$\prod_{i=1}^{a} SU(l_i+1)\times SO(2k)$

where $k,\ k_1,\ k_2 \in \mathbb{N}$.

Theorem 1.2 (Theorem 8.1). *Suppose a torus manifold M has a codimension one extended G-action. If there are two singular orbits and one of them is not a torus manifold, then $(M,\ G)$ is essentially isomorphic to*

$$\left(\prod_{j=1}^{b-1} S^{2m_j} \times_{\mathcal{A}} N,\quad \prod_{j=1}^{b-1} SO(2m_j+1)\times H\right),$$

such that $(N,\ H)$ is one of the following two types:

N	H
$\prod_{i=1}^{a} S^{2l_i+1}\times_{T^a} S(\mathbb{C}^{k_1}_{\mathfrak{c}}\oplus\mathbb{R}^{2k_2-1})$	$\prod_{i=1}^{a} SU(l_i+1)\times U(k_1)\times SO(2k_2-1)$
$\prod_{i=1}^{a} \mathbb{C}P(l_i)\times S(\mathbb{R}^{2k_1}\oplus\mathbb{R}^{2k_2-1})$	$\prod_{i=1}^{a} SU(l_i+1)\times SO(2k_1)\times SO(2k_2-1)$

where $k_1 \in \mathbb{N}$, $k_2 \geq 2$.

Theorem 1.3 (Theorem 5.1). *Suppose a torus manifold M has a codimension one extended G-action. If there is an exceptional orbit, then $(M,\ G)$ is essentially* *isomorphic to*

$$\left(\prod_{j=1}^{b} S^{2m_j} \times_{\mathcal{A}\times\mathbb{Z}_2} N,\quad \prod_{j=1}^{b} SO(2m_j+1)\times H\right),$$

such that $(N,\ H)$ *is one of the following two types:*

N	H
$\left(\prod_{i=1}^{a} S^{2l_i+1}\right) \times_{T^a} S(\mathbb{C}_{\mathfrak{c}}^{k_1}\oplus\mathbb{R})$	$\prod_{i=1}^{a} SU(l_i+1)\times U(k_1)$
$\prod_{i=1}^{a} \mathbb{C}P(l_i)\times S(\mathbb{R}^{2k_1}\oplus\mathbb{R})$	$\prod_{i=1}^{a} SU(l_i+1)\times SO(2k_1)$

where $k_1\in\mathbb{N}$.

From the above theorems (see Section 3.1, 5.1 and 8.1 for detail), the finite group $\mathcal{A}$ or $\mathcal{A}\times\mathbb{Z}_2$ acts only on $\prod_{j=1}^{b} S^{2m_j}$ and on the fibre of N (*not* on $\prod_{i=1}^{a} S^{2l_i+1}$ and $\prod_{i=1}^{a}\mathbb{C}P(l_i)$). Therefore, we also have that if an unoriented torus manifold has a codimension one extended G-action, then this manifold is a fibre bundle over the homogeneous torus manifold $\prod_{j=1}^{b} S^{2m_j}/\mathcal{A}'\times\prod_{i=1}^{a}\mathbb{C}P(l_i)$ (see [**12**]) whose fibre is a complex projective space $\mathbb{C}P(l)$, an even dimensional sphere S^{2m}, or an even dimensional real projective space $\mathbb{R}P(2m)$. Therefore, we can easily show the following corollary:

Corollary 1.4. If a non-singular toric variety or a quasitoric manifold M has a codimension one extended G-action, then $(M,\ G)$ is essentially isomorphic to

$$\left(\prod_{i=1}^{a-1} S^{2l_i+1}\times_{T^{a-1}} P(\mathbb{C}_{\mathfrak{b}}^{k_1}\oplus\mathbb{C}^{k_2}),\quad \prod_{i=1}^{a-1} SU(l_i+1)\times S(U(k_1)\times U(k_2))\right).$$

Remark that the manifold in Corollary 1.4 is equivariantly diffeomorphic to

$$\prod_{i=1}^{a-1}\mathbb{C}_o^{l_i+1}\times_{(\mathbb{C}^*)^{a-1}} P(\mathbb{C}_{\mathfrak{b}}^{k_1}\oplus\mathbb{C}^{k_2}),$$

where $\mathbb{C}_o^{l_i+1}=\mathbb{C}^{l_i+1}-\{o\}$ and $\mathbb{C}^*=\mathbb{C}-\{0\}$ (removed the origin).

The organization of this paper is as follows. In Section 2, we first set up some notation and basic facts from [**12, 13**]. Then we know that, in order to classify codimension one extended actions, we need to consider the three cases: the cases (1), (2) and (3). In Section 3, we classify the case (1), i.e., two singular orbits are torus manifolds. The proof of this case is similar to the previous classification in [**13**]. Hence, we can apply the arguments of the proofs in [**13**], and show Theorem 1.1. In Section 4, we give a preparation for the cases (2) and (3). In particular, we know the following key fact in this section: in order to classify the cases (2) and (3), we may only study the isotropy subgroup $K_2\subset G$ and its slice representation σ_2 by Remark 4.2, where G/K_1 is the singular orbit of the codimension 1 extended action which is a torus manifold and G/K_2 is the other non-principal orbit. In Section 5, we state the main theorem and give the remark for the case (2), i.e., one of non-principal orbits is an exceptional orbit. In particular, we divide this case into (2)-(a) and (2)-(b). In Section 6 and 7, we study the cases (2)-(a) and (2)-(b) respectively, and prove Theorem 1.3. In Section 8, we state the main theorem of the case (3), i.e., one of the non-principal orbits is not a torus manifold but a singular orbit, and we divide this case into the cases (3)-(a) and (3)-(b). In Section 9 and 10, we classify the case (3)-(a) and the case (3)-(b) respectively, and prove Theorem 1.2.

2. Review of the previous papers

In order to classify codimension 1 extended actions, in this section, we recall the basic facts (see [**1, 8, 12, 13, 17**] for detail).

2.1. Definition of torus manifold. We start with recalling the definition of the torus manifold. Let $(M^{2n},\ T^n)$ be a pair $2n$-dimensional, compact, connected manifold M^{2n} and a half dimensional torus T^n. We call $(M^{2n},\ T^n)$ a *torus manifold* if it satisfies that

(1) T^n-action on M^{2n} is amost effective, i.e., the intersection of all isotropy subgroups is a finite subgroup in T^n;
(2) its fixed point set is non-empty, i.e., $M^T \neq \emptyset$.

A torus manifold $(M^{2n},\ T^n)$ is often denoted by $(M,\ T)$ or M simply.

In the paper [**8**], the definition of torus manifolds involves the choice of orientations of manifold M and its *characteristic submanifolds* called *omniorientation* on M. Because we will classify extended actions up to essential isomorphism in this paper, we do not need to choice an omniorientation on M. Moreover, the T-action on M does not need to be effective (also see [**12**, Remark 2.1]). We also call such torus manifold an *unoriented torus manifold* in this paper.

Note that M^T is finite for unoriented torus manifolds $(M,\ T)$ as well as the torus manifolds in the sense of [**8**].

2.2. Codimension 1 extended G-actions and their singular orbits. Let $(M,\ T)$ be a torus manifold, and let G be a compact, connected Lie group whose maximal torus is T. We next recall the basic facts for extended G-actions of $(M,\ T)$.

Suppose a T^n-action on M^{2n} extends to a G-action on M with codimension 1 principal orbits, i.e., $(2n-1)$-dimensional orbit. Then we call $(M,\ T)$ has a *codimension* 1 *extended G-action*, and such extended G-action on M is denoted by $(M,\ G)$. We will classify such $(M,\ G)$ up to *essential isomorphism*, where here we say that $(M,\ G)$ is *essentially isomorphic* to $(M',\ G')$ if these induced effective actions are weak equivariantly diffeomorphic (see [**12, 13**] for detail)

For codimension 1 extended actions of $(M,\ T)$, the following lemma holds (see [**13**, Lemma 3.1, 3.2]).

LEMMA 2.1. *Suppose that an (unoriented) torus manifold $(M^{2n},\ T^n)$ has a codimension 1 extended G-action. Then a G-orbit G/K_1 of T-fixed points is a singular orbit in $(M,\ G)$, i.e., $\dim G/K_1 < 2n-1$.*

Furthermore, there is some subtorus $T' \subset T$ such that $(G/K_1,\ T')$ is an (unoriented) torus manifold.

PROOF. In the previous paper [**13**], we assume the orientation of the torus manifold. However, we can also apply the proofs of [**13**, Lemma 3.1, 3.2] to the case of unoriented torus manifolds. Hence, we can prove this lemma with a method similar to the proofs of [**13**, Lemma 3.1, 3.2]. □

In this paper, we do not need to consider the orientation on the singular orbit G/K_1. Hence, we can directly apply the main result in [**12**] to a homogeneous torus manifold G/K_1. Moreover, we get the following lemma using the argument in [**13**, Section 3.3].

Lemma 2.2. *Suppose that a torus manifold* $(M^{2n},\ T^n)$ *extends to a codimension 1 extended action. Then this codimension 1 extended action is essentially isomorphic to* $(M,\ G)$ *which satisfies that, for its singular orbit* G/K_1 *with* $M^T \cap G/K_1 \neq \emptyset$, *the pair* $(G,\ K_1)$ *is as follows:*

$$\left(\prod_{i=1}^{a} SU(l_i+1) \times \prod_{j=1}^{b} SO(2m_j+1) \times G_1'',\ \prod_{i=1}^{a} S(U(l_i)\times U(1)) \times \mathcal{S}_1 \times G_1''\right),$$

where $\prod_{j=1}^{b} SO(2m_j) \subset \mathcal{S}_1 \subset \prod_{j=1}^{b} S(O(2m_j)\times O(1))$ *and* G_1'' *is a product of connected, simple Lie groups and tori.*

2.3. Slice representations of G/K_1. In this subsection, we study the slice representation of G/K_1 for the case of unoriented torus manifolds.

Due to Lemma 2.1, G/K_1 is a torus manifold. Therefore, we can put $\dim G/K_1 = 2n-2k_1$ for $k_1 \geq 1$. Let $K_1' = \prod_{i=1}^{a} S(U(l_i)\times U(1)) \times \mathcal{S}_1$ (see Lemma 2.2), and let $Z(X)$ be the centralizer of a subgroup X in $O(2k_1)$, i.e., $Z(X) = \{g \in O(2k_1) \mid gx = xg \text{ for all } x \in X\}$.

Using a method similar to the proof of [**13**, Lemma 5.6], we have the following lemma for $K_1 = K_1' \times G_1''$ and the slice representation $\sigma_1 : K_1 \to O(2k_1)$.

Lemma 2.3. *For* G_1'' *in* $K_1 = K_1' \times G_1''$, *there are the following two cases:*

$$G_1'' = SU(k_1)\times T^1, \quad \textit{and} \quad G_1'' = SO(2k_1).$$

For these two cases, the slice representation $\sigma_1 : K_1 = K_1' \times G_1'' \to O(2k_1)$ *satisfies the following list:*

	G_1''	$\sigma_1(G_1'')$	$\sigma_1(K_1')$
(1)	$SU(k_1)\times T^1$	$U(k_1) \subset O(2k_1)$	$Z(U(k_1)) \simeq T^1 \subset U(k_1) \subset O(2k_1)$
(2)	$SO(2k_1)$	$SO(2k_1)$	$Z(SO(2k_1)) = \{\pm I_{2k_1}\} \subset O(2k_1)$

where the right list means $\sigma_1(K_1') \subset Z(U(k_1))$ *for (1) and* $\sigma_1(K_1') \subset Z(SO(2k_1))$ *for (2).*

Furthermore, the image of σ_1 *is in* $SO(2k_1)$, *i.e.,* $\sigma_1 : K_1 \to SO(2k_1)$.

Remark 2.4. We give the following four remarks.

1. If $G_1'' = SO(2k_1)$ then we can regard $k_1 \geq 2$, because $SO(2) \simeq T^1 = SU(1)\times T^1$ for $k_1 = 1$.
2. In the previous paper [**13**], $\sigma_1(K_1')$ is always connected, because K_1 is connected. However, in this paper, K_1 is not always connected (because G_1'' is connected but K_1' might not be connected for $K_1 = K_1' \times G_1''$).
3. Moreover, in [**13**], we can assume $m_j \geq 2$ (where m_j is defined in Lemma 2.2). However, in this paper, we assume $m_j \geq 1$.
4. $K_1/K \cong K_1^o/K^o \cong S^{2k_1-1}$ because $\dim G/K_1 = 2n-2k_1$ and the tubular neighborhood of G/K_1 is $G\times_{K_1} D^{2k_1}$ such that K_1 acts transitively on $\partial D^{2k_1} (\cong K_1/K)$ through σ_1.

In order to apply the same arguments in [**13**] to the case $m_j \geq 1$, we need to study the case $m_j = 1$. Let $r_j : \prod_{j=1}^{b} SO(2m_j+1) \to SO(2m_j+1)$ be the natural projection to the j-th factor. Note that we have for $\mathcal{S}_1$ in Lemma 2.2

$$r_j(\mathcal{S}_1) = SO(2m_j) \quad \text{or} \quad S(O(2m_j)\times O(1)) \subset SO(2m_j+1).$$

The following lemma holds.

Lemma 2.5. *Assume* $m_j = 1$. *Then there are the following two cases:*

(1) *if* $r_j(\mathcal{S}_1) = SO(2m_j)(= SO(2))$, *then we can regard* $SO(2m_j+1) = SO(3)$ *as* $SU(2) = SU(l_{a+1}+1)$ *up to essential isomorphism;*
(2) *if* $r_j(\mathcal{S}_1) = S(O(2m_j) \times O(1))(= S(O(2) \times O(1)))$, *then there is some inclusion* $\iota : S(O(2) \times O(1)) \to \mathcal{S}_1$ *such that* $\mathrm{Im}\ \iota \cap SO(2m_j) = SO(2m_j)$, *and for the slice representation* σ_1 *we have*

$$\sigma_1 \circ \iota(S(O(2) \times O(1))) \subset \{\pm I_{2k_1}\} \subset SO(2k_1).$$

Proof. The first statement can be easily proved using the fact that $SO(3) \approx SU(2)$, i.e., these Lie algebras are same (also see [**13**, Section 2.2]). We may only prove the second statement.

Suppose $r_j(\mathcal{S}_1) = S(O(2m_j) \times O(1))(= S(O(2) \times O(1)))$. Because $\prod_{j=1}^{b} SO(2m_j) \subset \mathcal{S}_1 \subset \prod_{j=1}^{b} S(O(2m_j) \times O(1))$, we can easily see that there is some inclusion $\iota : S(O(2) \times O(1)) \to \mathcal{S}_1$ such that $\mathrm{Im}\ \iota \cap SO(2m_j) = SO(2m_j)$. We may only prove that this inclusion ι satisfies $\sigma_1 \circ \iota(S(O(2) \times O(1))) \subset \{\pm I_{2k_1}\} \subset SO(2k_1)$. Let

$$J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in O(2).$$

Then $J^2 = I_2$. Therefore, by the right list in Lemma 2.3, we have that

$$\sigma_1 \circ \iota\left(\begin{pmatrix} J & 0 \\ 0 & -1 \end{pmatrix}\right) \in \{\pm I_{2k_1}\} \subset U(2k_1) \subset SO(2k_1). \tag{2.1}$$

If $G_1'' = SO(2k_1)$, then the statement $\sigma_1 \circ \iota(S(O(2) \times O(1))) \subset \{\pm I_{2k_1}\}$ is straightforward because of Lemma 2.3. Assume $G_1'' = SU(k_1) \times T^1$. Now we can put

$$\sigma_1 \circ \iota\left(\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}\right) = A^{\gamma} \in SO(2) \subset SO(2k_1)$$

for $A \in SO(2)$, where $\gamma \in \mathbb{Z}$ and $A^{\gamma} \in SO(2) \simeq T^1 \subset U(k_1) \subset SO(2k_1)$ for the diagonal subgroup $T^1 \subset U(k_1)$, because $SO(2)$ is the abelian group. Hence, by Eq. (2.1), we have that

$$\begin{aligned} A^{\gamma} &= \sigma_1 \circ \iota\left(\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}\right) \\ &= \sigma_1 \circ \iota\left(\begin{pmatrix} J & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} J & 0 \\ 0 & -1 \end{pmatrix}\right) \\ &= \sigma_1 \circ \iota\left(\begin{pmatrix} A^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right) \\ &= A^{-\gamma}. \end{aligned}$$

It follows that $\gamma = 0$; hence, we have the second statement. □

If $m_j \geq 2$, then we can easily show that $\sigma_1 \circ \iota \circ r_j(\mathcal{S}_1) \subset \{\pm I_{2k_1}\}$. Hence, by the above Lemma 2.5, we can regard $\sigma_1(\mathcal{S}_1) \subset \{\pm I_{2k_1}\}$. This corresponds with the previous fact that $\sigma_s(\prod_{j=1}^{b} SO(2m_j)) = \{e\}$ in [**13**, Lemma 7.1, 7.2, 7.3]. Because $\{\pm I_{2k_1}\}$ is the center of $SO(2k_1)$, we can use the same argument in the previous paper [**13**, Section 7] for analyzing the slice representation σ_1.

2.4. Global structures of codimension 1 extended actions. We next study the global structure of $(M,\ G)$. Even if M is non-oriented, the following structure theorem holds (see [**1**, 8.2 Theorem in Chapter IV]):

THEOREM 2.6. *Let $(M,\ G)$ be a G-manifold M with codimension 1 orbits. If every orbit is principal, then M is a G/K-bundle over S^1. Otherwise, there are two non-principal orbits G/K_1 and G/K_2 such that $K_1 \cap K_2 \supset K$ (K is the principal isotropy subgroup). Furthermore, there exists a closed, invariant tubular neighborhood X_s of G/K_s for $s = 1,\ 2$ such that*

$$M = X_1 \cup X_2 \quad \textit{and} \quad X_1 \cap X_2 = \partial X_1 = \partial X_2 = G/K,$$

where ∂X_s means the boundary of X_s.

Because of Lemma 2.1, we can assume G/K_1 is a singular orbit and a torus manifold throughout this paper; moreover, because of Theorem 2.6, there are the following three cases:

(1) G/K_2 is a torus manifold (automatically G/K_2 is a singular orbit);
(2) G/K_2 is an exceptional orbit, namely, $\dim G/K_2 = \dim G/K = 2n-1$;
(3) G/K_2 is not a torus manifold but a singular orbit.

We call the above cases *the case* (1), (2) and (3), respectively. Remark that if M is simply connected, then we do not need to consider the case (2) (see [**13**, Theorem 2.6] or [**17**, Lemma 1.2.1]). From the next section, we start to classify for each above case.

Before we go to the next section, we introduce the following Lemma 2.7 for the attaching map of $f : \partial X_1 \to \partial X_2$. In the final part of the classification, we compute the attaching maps f from ∂X_1 to ∂X_2, and construct the manifold $M(f) = X_1 \cup_f X_2$ by using f. For two attaching maps f and f', we know whether $M(f)$ and $M(f')$ are diffeomorphic or not, by making use of the following Uchida's criterion (see [**17**, Lemma 5.3.1]).

LEMMA 2.7 (Uchida's criterion). *Let $f, f' : \partial X_1 \to \partial X_2$ be G-equivariant diffeomorphisms. Then $M(f)$ is equivariantly diffeomorphic to $M(f')$ as G-manifolds, if one of the following conditions are satisfied:*

(1) *f is G-diffeotopic to f';*
(2) *$f^{-1}f'$ is extendable to a G-equivariant diffeomorphism on X_1;*
(3) *$f'f^{-1}$ is extendable to a G-equivariant diffeomorphism on X_2.*

We remark that this criterion also holds for non-orientable cases.

Using the above criterion (1), we can take the attaching map f from the group $N(K;G)/N(K;G)^o$, where $N(K;G)$ is the normalizer of K in G and $N(K;G)^o$ is its identity component (also see [**13**, Section 8.1])).

3. The case (1): two singular orbits are torus manifolds

The goal of this section is to classify the case (1). So, in this section, we assume that the other singular orbit G/K_2 is also a torus manifold. Then we can put $\dim G/K_s = 2n - 2k_s$ for $k_s \geq 1$ and $s = 1,\ 2$. Note that the proof of this case is similarly to that of the previous classification [**13**].

3.1. Notations and main theorem. First, we state the main theorem of this section. In order to state it, we prepare some notations. A manifold $X \times_H Y$ denotes a quotient manifold of $X \times Y$ divided by a free H-action. The manifold $\left(\prod_{i=1}^{a} S^{2l_i+1}\right) \times_{T^a} S(\mathbb{C}_{\mathfrak{a}}^k \oplus \mathbb{R})$ is the quotient manifold of $\left(\prod_{i=1}^{a} S^{2l_i+1}\right) \times S(\mathbb{C}_{\mathfrak{a}}^k \oplus \mathbb{R})$ divided by the following T^a-action: T^a acts on $\prod_{i=1}^{a} S^{2l_i+1}$ by the a-times product of the scaler S^1-action on $S^{2l_i+1} \subset \mathbb{C}^{l_i+1}$ for $i = 1, \cdots, a$ (in other words, T^a acts on $\prod_{i=1}^{a} S^{2l_i+1}$ naturally); and T^a acts on the $2k$-dimensional sphere $S(\mathbb{C}_{\mathfrak{a}}^k \oplus \mathbb{R}) \subset \mathbb{C}_{\mathfrak{a}}^k \oplus \mathbb{R}$ through the representation $\mathfrak{a} : T^a \to S^1$ such that $\mathfrak{a}(t_1, \cdots, t_a) \mapsto t_1^{\alpha_1} \cdots t_a^{\alpha_a}$ for some $\alpha_1, \cdots, \alpha_a \in \mathbb{Z}$, that is, $\mathbb{C}_{\mathfrak{a}}^k \simeq \mathbb{C}^k$ (as a vector space) is the representation space of the representation $\mathfrak{a}$ (S^1 acts on this space by the scaler multiplication). The manifold $\left(\prod_{i=1}^{a-1} S^{2l_i+1}\right) \times_{T^{a-1}} P(\mathbb{C}_{\mathfrak{b}}^{k_1} \oplus \mathbb{C}^{k_2})$ is the projectify of the complex vector bundle $\left(\prod_{i=1}^{a-1} S^{2l_i+1}\right) \times_{T^{a-1}} (\mathbb{C}_{\mathfrak{b}}^{k_1} \oplus \mathbb{C}^{k_2})$, where T^{a-1} acts on $\prod_{i=1}^{a-1} S^{2l_i+1}$ naturally; and T^{a-1} acts on the representation space $\mathbb{C}_{\mathfrak{b}}^{k_1} \simeq \mathbb{C}^{k_1}$ through the representation $\mathfrak{b} : T^{a-1} \to S^1$, and on $\mathbb{C}^{k_2}$ trivially. A group $\mathcal{A}$ is a subgroup of $\prod_{j=1}^{b} \mathbb{Z}_2$, where $\mathbb{Z}_2$ is generated by the antipodal involution on S^{2m_j} for $j = 1, \cdots, b$.

Now we may state the main theorem in this section.

THEOREM 3.1. *Suppose a torus manifold M has a codimension one extended G-action. If there are two singular orbits and both of them are torus manifolds, then (M, G) is essentially isomorphic to*

$$\left(\prod_{j=1}^{b} S^{2m_j} \times_{\mathcal{A}} N, \quad \prod_{j=1}^{b} SO(2m_j+1) \times H\right),$$

such that (N, H) is one of the followings:

	N	H
(a)	$\left(\prod_{i=1}^{a} S^{2l_i+1}\right) \times_{T^a} S(\mathbb{C}_{\mathfrak{a}}^k \oplus \mathbb{R})$	$\prod_{i=1}^{a} SU(l_i+1) \times U(k)$
(b)	$\left(\prod_{i=1}^{a-1} S^{2l_i+1}\right) \times_{T^{a-1}} P(\mathbb{C}_{\mathfrak{b}}^{k_1} \oplus \mathbb{C}^{k_2})$	$\prod_{i=1}^{a-1} SU(l_i+1) \times S(U(k_1) \times U(k_2))$
(c)	$\prod_{i=1}^{a} \mathbb{C}P(l_i) \times S(\mathbb{R}^{2k} \oplus \mathbb{R})$	$\prod_{i=1}^{a} SU(l_i+1) \times SO(2k)$

where $\mathcal{A}$ acts on $\prod_{j=1}^{b} S^{2m_j}$ as the subgroup $\prod_{j=1}^{b} \mathbb{Z}_2$ and on the fibre of N through the following representations:

- **(a):** $\sigma_{\mathbb{C}} : \mathcal{A} \to \{\pm 1\} \subset S^1$ *on* $S(\mathbb{C}_{\mathfrak{a}}^k \oplus \mathbb{R}) \cap \mathbb{C}_{\mathfrak{a}}^k$;
- **(b):** $\sigma_{\mathbb{C}} : \mathcal{A} \to \{\pm 1\} \subset S^1$ *on* $\mathbb{C}_{\mathfrak{b}}^{k_1}$*-factor in* $P(\mathbb{C}_{\mathfrak{b}}^{k_1} \oplus \mathbb{C}^{k_2})$;
- **(c):** $\sigma_{\mathbb{R}} : \mathcal{A} \to \{\pm I_{2k}\} \subset SO(2k)$ *on* $S(\mathbb{R}^{2k} \oplus \mathbb{R}) \cap \mathbb{R}^{2k}$;

respectively.

Here, G-actions on M are as follows: $\prod SO(2m_j+1)$ and $\prod SU(l_i+1)$ act naturally on $\prod S^{2m_j}$ and $\prod S^{2l_i+1}$, respectively; and $U(k)$, $S(U(k_1) \times U(k_2))$ and $SO(2k)$ act naturally on $\mathbb{C}_{\mathfrak{a}}^k$, $\mathbb{C}_{\mathfrak{b}}^{k_1} \oplus \mathbb{C}^{k_2}$ and $\mathbb{R}^{2k}$, respectively.

From the next subsection, we start to prove the above theorem.

3.2. Singular isotropy subgroups and images of their slice representations. Because G/K_2 is a torus manifold, the pair (G, K_2) as well as (G, K_1)

which is described in Lemma 2.2 satisfies the following property:

$$(G,\ K_2) = \left(\prod_{i=1}^{c} SU(l'_i+1) \times \prod_{j=1}^{d} SO(2m'_j+1) \times G''_2,\ \prod_{i=1}^{c} S(U(l'_i)\times U(1)) \times \mathcal{S}_2 \times G''_2 \right),$$

where $\prod_{j=1}^{d} SO(2m'_j) \subset \mathcal{S}_2 \subset \prod_{j=1}^{d} S(O(1)\times O(2m'_j))$ and G''_2 is a product of connected, simple Lie groups and tori.

By the same argument of [**13**, Section 6, 7] for G/K_1^o and G/K_2^o (where K_s^o is the identity component of K_s), we have

$$(G,\ K_1,\ K_2) = (\prod_{j=1}^{b} SO(2m_j+1) \times \widehat{G},\ \mathcal{S}_1 \times \widehat{K_1},\ \mathcal{S}_2 \times \widehat{K_2})$$

such that $(\widehat{G},\ \widehat{K_1},\ \widehat{K_2})$ is as follows:

(a)**:** $\widehat{G} = \prod_{i=1}^{a} SU(l_i+1) \times SU(k) \times T^1$,
$\widehat{K_1} = \widehat{K_2} = \prod_{i=1}^{a} S(U(l_i)\times U(1)) \times SU(k) \times T^1$,
where $\dim G/K_s = 2n-2k$ and $k \geq 1$ $(k_1 = k_2 = k)$;

(b)**:** $\widehat{G} = \prod_{i=1}^{a-1} SU(l_i+1) \times SU(k_1) \times SU(k_2) \times T^1$,
$\widehat{K_1} = \prod_{i=1}^{a-1} S(U(l_i)\times U(1)) \times SU(k_1) \times S(U(k_2-1)\times U(1)) \times T^1$,
$\widehat{K_2} = \prod_{i=1}^{a-1} S(U(l_i)\times U(1)) \times S(U(k_1-1)\times U(1)) \times SU(k_2) \times T^1$,
where $\dim G/K_s = 2n-2k_s$ and $k_s \geq 1$;

(c)**:** $\widehat{G} = \prod_{i=1}^{a} SU(l_i+1) \times SO(2k)$,
$\widehat{K_1} = \widehat{K_2} = \prod_{i=1}^{a} S(U(l_i)\times U(1)) \times SO(2k)$,
where $\dim G/K_s = 2n-2k$ and $k \geq 2$ $(k_1 = k_2 = k)$.

We call the above cases *the case* (1)-(a), (1)-(b) and (1)-(c), respectively.

Next, we consider the slice representation σ_s for $s = 1,\ 2$. Because of Lemma 2.3 and 2.5, we have $\sigma_s : K_s = K'_s \times G''_s \to SO(2k_s)$. Moreover, there are the following two cases:

G''_s	$\sigma_s(G''_s)$	$\sigma_s(K'_s)$
$SU(k_s)\times T^1$	$U(k_s) \subset O(2k_s)$	$Z(U(k_s)) \simeq T^1 \subset U(k_s) \subset SO(2k_s)$
$SO(2k_s)$	$SO(2k_s)$	$Z(SO(2k_s)) = \{\pm I_{2k_s}\} \subset SO(2k_s)$

where the right list means $\sigma_s(K'_s) \subset Z(U(k_s))$ or $Z(SO(2k_s))$.

In order to classify the above each case, we analyze the followings: principal isotropy subgroups $K = \sigma_1^{-1}(SO(2k_1-1)) = \sigma_2^{-1}(SO(2k_2-1))$; attaching maps $f : G/K = \partial X_1 \to \partial X_2 = G/K \in N(K;G)/N(K;G)^o$ (see Section 2.4); and constructions of G-manifolds as $M(f) = X_1 \cup_f X_2$, where X_s denotes a G-invariant tubular neighborhood of G/K_s for $s = 1,\ 2$.

3.3. The case (1)-(a). In this subsection, we study the case (1)-(a), that is,

- $G = \prod_{j=1}^{b} SO(2m_j+1) \times \prod_{i=1}^{a} SU(l_i+1) \times SU(k) \times T^1$,
- $K_1 = \mathcal{S}_1 \times \prod_{i=1}^{a} S(U(l_i)\times U(1)) \times SU(k) \times T^1$,
- $K_2 = \mathcal{S}_2 \times \prod_{i=1}^{a} S(U(l_i)\times U(1)) \times SU(k) \times T^1$

where $\prod_{j=1}^{b} SO(2m_j) \subset \mathcal{S}_s \subset \prod_{j=1}^{b} S(O(2m_j)\times O(1))$, $\dim G/K_s = 2n-2k$, and $k \geq 1$ for $s = 1,\ 2$.

In order to know the precise structure of K, we analyze slice representations $\sigma_s : K_s \to SO(2k)$. By Section 3.2, the slice representation $\sigma_s : K_s \to U(k) \subset$

$SO(2k)$ satisfies the followings:

$$\sigma_s(S(U(l_i)\times U(1))) = \{I_{2k}\} \text{ or } Z(U(k)) = T^1 \subset U(k) \text{ for all } i = 1,\ \cdots,\ a;$$
$$\sigma_s(SU(k)\times T^1) = U(k),$$

where $Z(U(k)) = T^1 \subset U(k)$ is the center of $U(k)$, i.e., the diagonal subgroup. Because $\sigma_s(\mathcal{S}_s^o) = \sigma_s(\prod_{j=1}^b SO(2m_j)) = \{1\} \subset T^1 \subset U(k)$ (by making use of Lemma 2.5) and $\mathcal{S}_s/\prod_{j=1}^b SO(2m_j) \simeq \mathcal{A}_s \subset \prod_{j=1}^b \mathbb{Z}_2$ (i.e., $\mathcal{A}_s$ is a subgroup of $\prod_{j=1}^b \mathbb{Z}_2$ generated by antipodal involutions), we have that

$$\sigma_s(\mathcal{S}_s) = \sigma_{\mathbb{C}}(\mathcal{A}_s) \subset \{\pm 1\} \subset T^1 \subset U(k),$$

where $\sigma_{\mathbb{C}} : \mathcal{A}_s \to \{\pm 1\}$. If $\sigma_s(\mathcal{S}_s) = \{1\}$ then we can apply the same argument in [**13**]. So we assume $\sigma_s(\mathcal{S}_s) = \{\pm 1\}$.

Now, the principal isotropy subgroup K is as follows ($X \equiv Y$ means that two groups X and Y are conjugate in G):

$$\begin{aligned} K \ &\equiv\ \sigma_1^{-1}(U(k-1)) = \left\{ \left(A,\ (t_1,\ \cdots,\ t_a),\ \begin{pmatrix} X & 0 \\ 0 & x \end{pmatrix},\ t \right) \ \middle|\ \sigma_1(A) x t_1^{\alpha_1}\cdots t_a^{\alpha_a} t^{\alpha} = 1 \right\} \\ &\equiv\ \sigma_2^{-1}(U(k-1)) = \left\{ \left(B,\ (t_1,\ \cdots,\ t_a),\ \begin{pmatrix} Y & 0 \\ 0 & y \end{pmatrix},\ t \right) \ \middle|\ \sigma_2(B) y t_1^{\beta_1}\cdots t_a^{\beta_a} t^{\beta} = 1 \right\}, \end{aligned}$$

where $A \in \mathcal{S}_1$, $B \in \mathcal{S}_2$; $X,\ Y \in U(k-1)$, $x,\ y \in T^1$ such that $x \det X = y \det Y = 1$; and

$$(t_1,\ \cdots,\ t_a) = \left(\begin{pmatrix} U_1 & 0 \\ 0 & t_1 \end{pmatrix},\ \cdots,\ \begin{pmatrix} U_a & 0 \\ 0 & t_a \end{pmatrix} \right) \in \prod_{i=1}^a S(U(l_i)\times U(1)).$$

First we take $A = B = e \in \mathcal{S}_1 \cap \mathcal{S}_2$ (the identity elements in $\prod_{j=1}^b SO(2m_j+1)$). Then $\sigma_1(A) = 1 = \sigma_2(B)$. Hence, by using the the same argument of [**13**, Section 7.1] for $A = B = e$, we have that

$$\alpha_i = \beta_i \in \mathbb{Z} \text{ for } i = 1,\ \cdots,\ a, \quad \text{and} \quad \alpha = \beta \in \mathbb{N}.$$

Moreover, if $k = 1$ then we can take $\alpha = \beta = 1$ (up to essential isomorphism). Next, we consider the following part in K (if $k \geq 2$):

$$\begin{aligned} \mathcal{R}_1 \ &=\ \left\{ \left(A,\ e,\ \begin{pmatrix} J & 0 \\ 0 & \sigma_1(A) \end{pmatrix},\ 1 \right) \ \middle|\ A \in \mathcal{S}_1 \right\} \\ &\equiv\ \left\{ \left(B,\ e,\ \begin{pmatrix} J & 0 \\ 0 & \sigma_2(B) \end{pmatrix},\ 1 \right) \ \middle|\ B \in \mathcal{S}_2 \right\} = \mathcal{R}_2, \end{aligned}$$

where $e \in \prod_{i=1}^a S(U(l_i)\times U(1))$ is the identity element and $J \in U(k-1)$ such that $J = I_{k-1}$ (if $\sigma_s(C_s) = 1$) or $\det J = -1$ with $J^2 = I_{k-1}$ (if $\sigma_s(C_s) = -1$) for $s = 1,\ 2$ and $C_1 = A,\ C_2 = B$. For $\mathcal{R}_1$ and $\mathcal{R}_2$, we can easily show the following isomorphisms (not identity in G):

$$\mathcal{S}_1 \simeq \mathcal{R}_1, \quad \mathcal{S}_2 \simeq \mathcal{R}_2.$$

Because $\mathcal{R}_1 \equiv \mathcal{R}_2$, we see that

$$\mathcal{S}_1 = \mathcal{S}_2 \subset K_1 \cap K_2$$

by the definition of $\mathcal{S}_s$, i.e., we can regard $K_1 = K_2$ in G. If $k = 1$, then we can apply the same above argument by taking the set $\{(A,\ e,\ \sigma_1(A))\} \equiv \{(B,\ e,\ \sigma_2(B))\}$, and we can regard $K_1 = K_2$. Therefore, by using the above argument, we have $\sigma_1 = \sigma_2$. Moreover, K is as follows:

$$K = \left\{ \left(A,\ (t_1,\ \cdots,\ t_a),\ \begin{pmatrix} X & 0 \\ 0 & x \end{pmatrix},\ t \right) \,\middle|\, \sigma_1(A) x t_1^{\alpha_1} \cdots t_a^{\alpha_a} t^{\alpha} = 1 \right\}$$

As the result of the above argument, the tubular neighborhood $X_s = G \times_{K_s} D^{2k}$ $(s = 1,\ 2)$ is equivariantly diffeomorphic to the following manifold:

$$\prod_{j=1}^{b} S^{2m_j} \times_{\mathcal{A}} \left(\prod_{i=1}^{a} S^{2l_i+1} \times_{T^a} D(\mathbb{C}^k_{\mathfrak{a}}) \right), \tag{3.1}$$

where T^a acts on $\prod_{i=1}^{a} S^{2l_i+1}$ naturally, and on $D(\mathbb{C}^k_{\mathfrak{a}})$ by $\mathfrak{a} : (t_1,\ \cdots,\ t_a) \mapsto t_1^{\alpha_1} \cdots t_a^{\alpha_a}$; and $\mathcal{A} = \mathcal{A}_1 = \mathcal{A}_2 (\simeq \mathcal{S}_1/\mathcal{S}_1^o = \mathcal{S}_2/\mathcal{S}_2^o)$ acts on $\prod_{j=1}^{b} S^{2m_j}$ as the subgroup of $\prod_{j=1}^{b} \mathbb{Z}_2$, and on $D(\mathbb{C}^k_{\mathfrak{a}})$ by $\sigma_{\mathbb{C}} : \mathcal{A} \to \{\pm 1\}$ (induced by σ_1). Here, $G = \prod_{j=1}^{b} SO(2m_j + 1) \times \prod_{i=1}^{a} SU(l_i + 1) \times SU(k) \times T^1$ acts on this manifold as follows: $\prod_{j=1}^{b} SO(2m_j + 1)$ acts on $\prod_{j=1}^{b} S^{2m_j}$ naturally; $\prod_{i=1}^{a} SU(l_i + 1)$ acts on $\prod_{i=1}^{a} S^{2l_i+1}$ naturally; and $SU(k) \times T^1$ acts on $D(\mathbb{C}^k_{\mathfrak{a}})$ by $(A,\ t) \mapsto At^{\alpha} \in U(k)$.

Next, we analyze attaching maps $f : \partial X_1 \to \partial X_2$. Because of $\alpha \in \mathbb{N}$, we have

$$N(K; G)/N(K; G)^o \simeq \prod_{j=1}^{b} \mathbb{Z}_2 \times \prod_{i=1}^{a} W_{l_i+1},$$

where $\prod_{j=1}^{b} S(O(2m_j) \times O(1))/SO(2m_j) \simeq \prod_{j=1}^{b} \mathbb{Z}_2$ and

$$W_{l_i+1} = \begin{cases} \{I_{l_i+1}\} & \text{if } l_i \geq 2 \text{ or } \alpha_i \neq 0 \\ S_2 & \text{if } l_i = 1 \text{ and } \alpha_i = 0. \end{cases}$$

Here, the above $S_2 (\simeq \mathbb{Z}_2)$ is the Weyl group of $SU(2)$. Therefore, by the same argument as [**13**, Section 8.2], we can show that $I \circ f : \partial X_1 \to \partial X_1$ is extendable to the equivariant map $X_1 \to X_1$, where $I : G/K = \partial X_2 \to \partial X_1 = G/K$ is the identity attaching map. Hence, $M(f) \cong M(I)$ for all attaching maps f by the Uchida's criterion (2). As the result, we have that if the case (1) holds for $(M,\ G)$, then such $(M,\ G)$ is only determined by the representations $\mathfrak{a} : T^a \to T^1$ and $\sigma_{\mathbb{C}} : \mathcal{A} \to \{\pm 1\}$ (up to essential isomorphism), i.e., if we fix the representations $\mathfrak{a}$ and $\sigma_{\mathbb{C}}$, then $(M,\ G)$ is unique up to essential isomorphism. Thus, we have that such manifold M and G are as follows (up to essential isomorphism):

$$M = \prod_{j=1}^{b} S^{2m_j} \times_{\mathcal{A}} \left(\prod_{i=1}^{a} S^{2l_i+1} \times_{T^a} S(\mathbb{C}^k_{\mathfrak{a}} \oplus \mathbb{R}) \right),$$

$$G = \prod_{j=1}^{b} SO(2m_j + 1) \times \prod_{i=1}^{a} SU(l_i + 1) \times U(k),$$

by computing the orbits G/K, G/K_1 and G/K_2. This corresponds with the first case of Theorem 3.1.

3.4. The case (1)-(b). In this subsection, we study the case (1)-(b), that is,

- $G = \prod_{j=1}^{b} SO(2m_j+1) \times \prod_{i=1}^{a-1} SU(l_i+1) \times SU(k_1) \times SU(k_2) \times T^1$,
- $K_1 = \mathcal{S}_1 \times \prod_{i=1}^{a-1} S(U(l_i)\times U(1)) \times SU(k_1) \times S(U(k_2-1)\times U(1)) \times T^1$,
- $K_2 = \mathcal{S}_2 \times \prod_{i=1}^{a-1} S(U(l_i)\times U(1)) \times S(U(k_1-1)\times U(1)) \times SU(k_2) \times T^1$,

where $\prod_{j=1}^{b} SO(2m_j) \subset \mathcal{S}_s \subset \prod_{j=1}^{b} S(O(2m_j)\times O(1))$, $\dim G/K_s = 2n-2k_s$ and $k_s \geq 1$ for $s = 1,\ 2$.

In order to know the precise structure of K, we analyze the slice representation $\sigma_s : K_s \to SO(2k_s)$. The following property can be proved similarly to that of the case (1)-(a) (see Section 3.2, 3.3):

$$\sigma_s(S(U(l_i)\times U(1))) = \{I_{2k_s}\} \text{ or } Z(U(k_s)) \subset U(k_s) \text{ for all } i = 1,\ \cdots,\ a-1;$$
$$\sigma_s(S(U(k_r-1)\times U(1))) = \{I_{2k_s}\} \text{ or } Z(U(k_s)) \subset U(k_s);$$
$$\sigma_s(SU(k_s)\times T^1) = U(k_s);$$
$$\sigma_s(\mathcal{S}_s) = \sigma_{\mathbb{C}}(\mathcal{A}_s) \subset \{\pm 1\},$$

where $s + r = 3$ and $s,\ r \geq 1$. Therefore, K satisfies that

$$\begin{aligned} K &\equiv \sigma_1^{-1}(U(k_1-1)) \\ &= \left\{ \left(A,\ (t_1,\ \cdots,\ t_{a-1}),\ \begin{pmatrix} C & 0 \\ 0 & c \end{pmatrix},\ \begin{pmatrix} X & 0 \\ 0 & x \end{pmatrix},\ t \right) \;\middle|\; \sigma_1(A) = t_1^{\alpha_1}\cdots t_{a-1}^{\alpha_{a-1}} x^{\alpha_a} t^{\alpha} c \right\} \\ &\equiv \sigma_2^{-1}(U(k_2-1)) \\ &= \left\{ \left(B,\ (t_1,\ \cdots,\ t_{a-1}),\ \begin{pmatrix} Y & 0 \\ 0 & y \end{pmatrix},\ \begin{pmatrix} D & 0 \\ 0 & d \end{pmatrix},\ t \right) \;\middle|\; \sigma_2(B) = t_1^{\beta_1}\cdots t_{a-1}^{\beta_{a-1}} y^{\beta_a} t^{\beta} d \right\}, \end{aligned}$$

where $A \in \mathcal{S}_1$, $B \in \mathcal{S}_2$, $(t_1,\ \cdots,\ t_{a-1}) \in \prod_{i=1}^{a-1} S(U(l_i)\times U(1))$, and $C,\ Y \in U(k_1-1)$, $D,\ X \in U(k_2-1)$, $x,\ y,\ c,\ d \in T^1$ such that $x\det X = y \det Y = c\det C = d\det D = 1$.

By the same argument as [**13**, Section 7.2] for $\sigma_1(A) = 1 = \sigma_2(B)$, we see that

$$\alpha_i = \beta_i \in \mathbb{Z} \text{ for } i = 1,\ \cdots,\ a-1, \quad \text{and} \quad \alpha_a = \beta_a = 1 \quad \text{and} \quad \alpha = \beta \in \mathbb{N}.$$

Moreover, if $k_1 = 1$ or $k_2 = 1$, then we have that $\alpha = 1$ or $\beta = 1$, respectively. Using the method similar to that demonstrated in Section 3.3, we can also prove

$$\mathcal{S}_1 = \mathcal{S}_2 \subset K_1 \cap K_2,$$

and $\sigma_1|_{\mathcal{S}_1} = \sigma_2|_{\mathcal{S}_2}$ (i.e., the restricted representations are the same representations). Hence, K is as follows:

$$K = \left\{ \left(A,\ (t_1,\ \cdots,\ t_{a-1}),\ \begin{pmatrix} Y & 0 \\ 0 & y \end{pmatrix},\ \begin{pmatrix} X & 0 \\ 0 & x \end{pmatrix},\ t \right) \;\middle|\; \sigma_1(A) t_1^{\alpha_1}\cdots t_{a-1}^{\alpha_{a-1}} xy t^{\alpha} = 1 \right\}.$$

As the result of the above argument, the tubular neighborhood $X_s = G \times_{K_s} D^{2k_s}$ $(s = 1,\ 2)$ is equivariantly diffeomorphic to the following manifold:

$$\prod_{j=1}^{b} S^{2m_j} \times_{\mathcal{A}} \left(\prod_{i=1}^{a-1} S^{2l_i+1} \times_{T^{a-1}} D(\mathbb{C}_{\mathfrak{b}}^{k_s}) \right) \times \mathbb{C}P(k_r - 1),$$

where T^{a-1} acts on $\prod_{i=1}^{a-1} S^{2l_i+1}$ naturally and on $D(\mathbb{C}_{\mathfrak{b}}^{k_s})$ by $\mathfrak{b} : (t_1,\ \cdots,\ t_{a-1}) \mapsto t_1^{\alpha_1}\cdots t_{a-1}^{\alpha_{a-1}}$, $\mathcal{A}(= \mathcal{A}_s \simeq \mathcal{S}_s/\mathcal{S}_s^o)$ acts on $\prod_{j=1}^{b} S^{2m_j}$ as the subgroup of $\prod_{j=1}^{b} \mathbb{Z}_2$ and on $D(\mathbb{C}_{\mathfrak{b}}^{k_s})$ by $\sigma_{\mathbb{C}} : \mathcal{A} \to \{\pm 1\}$, and $s + r = 3$ for $s,\ r \geq 1$. Here, $G =$

$\prod_{j=1}^{b} SO(2m_j+1) \times \prod_{i=1}^{a-1} SU(l_i+1) \times SU(k_s) \times T^1 \times SU(k_r)$ acts on this manifold as follows: $\prod_{j=1}^{b} SO(2m_j+1)$ acts on $\prod_{j=1}^{b} S^{2m_j}$ naturally; $\prod_{i=1}^{a-1} SU(l_i+1)$ acts on $\prod_{i=1}^{a-1} S^{2l_i+1}$ naturally; $SU(k_s) \times T^1$ acts on $D(\mathbb{C}_{\mathfrak{b}}^{k_s})$ by $(A,\ t) \mapsto At^{\alpha}$; and $SU(k_r)$ acts on $\mathbb{C}P(k_r-1)$ naturally.

Next, we analyze attaching maps $f : \partial X_1 \to \partial X_2$. In this case, we have

$$N(K;G)/N(K;G)^o \simeq \prod_{j=1}^{b} \mathbb{Z}_2 \times \prod_{i=1}^{a-1} W_{l_i+1}$$

where $\prod_{j=1}^{b} S(O(2m_j) \times O(1))/SO(2m_j) \simeq \prod_{j=1}^{b} \mathbb{Z}_2$ and

$$W_{l_i+1} = \begin{cases} \{I_{l_i+1}\} & \text{if } l_i \geq 2 \text{ or } \alpha_i \neq 0 \\ S_2 & \text{if } l_i = 1 \text{ and } \alpha_i = 0. \end{cases}$$

This is the same as the case (1)-(a). Therefore, we can show the Uchida's criterion (2) for $I \circ f$. Hence, by using the similar argument to that demonstrated in Section 3.3, we have that $(M,\ G)$ which satisfies the case (1)-(b) is unique up to essential isomorphism if we fix the representations $\mathfrak{b}$ and $\sigma_{\mathbb{C}}$. Thus, we have that M and G in the case (1)-(b) are as follows (up to essential isomorphism):

$$M = \prod_{j=1}^{b} S^{2m_j} \times_{\mathcal{A}} \left(\prod_{i=1}^{a-1} S^{2l_i+1} \times_{T^{a-1}} P(\mathbb{C}_{\mathfrak{b}}^{k_1} \oplus \mathbb{C}^{k_2}) \right),$$

$$G = \prod_{j=1}^{b} SO(2m_j+1) \times \prod_{i=1}^{a-1} SU(l_i+1) \times S(U(k_1) \times U(k_2)).$$

This corresponds with the second case of Theorem 3.1.

3.5. The case (1)-(c). In this subsection, we study the case (1)-(c), that is,

- $G = \prod_{j=1}^{b} SO(2m_j+1) \times \prod_{i=1}^{a} SU(l_i+1) \times SO(2k)$,
- $K_1 = \mathcal{S}_1 \times \prod_{i=1}^{a} S(U(l_i) \times U(1)) \times SO(2k)$,
- $K_2 = \mathcal{S}_2 \times \prod_{i=1}^{a} S(U(l_i) \times U(1)) \times SO(2k)$,

where $\prod_{j=1}^{b} SO(2m_j) \subset \mathcal{S}_s \subset \prod_{j=1}^{b} S(O(2m_j) \times O(1))$, $\dim G/K_s = 2n-2k$ and $k \geq 2$ for $s = 1,\ 2$.

In order to know the precise structure of K, we analyze the slice representation $\sigma_s : K_s \to SO(2k)$ as well as the cases (1) and (2). By Section 3.2, we have the followings for the slice representation $\sigma_s : K_s \to SO(2k)$:

$$\sigma_s(S(U(l_i) \times U(1))) = \{I_{2k}\} \subset SO(2k) \text{ for all } i = 1,\ \cdots,\ a;$$
$$\sigma_s(SO(2k)) = SO(2k);$$
$$\sigma_s(\mathcal{S}_s) = \sigma_{\mathbb{R}}(\mathcal{A}_s) \subset \{\pm I_{2k}\}.$$

Therefore, we have that K is the following subgroup:

$$\begin{aligned} K &\equiv \sigma_1^{-1}(SO(2k-1)) = \left\{ \left(A,\ (t_1,\ \cdots,\ t_a),\ \begin{pmatrix} E & 0 \\ 0 & e \end{pmatrix} \right) \,\middle|\, \sigma_1(A)e = 1 \right\} \\ &\equiv \sigma_2^{-1}(SO(2k-1)) = \left\{ \left(B,\ (t_1,\ \cdots,\ t_a),\ \begin{pmatrix} F & 0 \\ 0 & f \end{pmatrix} \right) \,\middle|\, \sigma_2(B)f = 1 \right\}, \end{aligned}$$

where $A \in \mathcal{S}_1$, $B \in \mathcal{S}_2$, $(t_1, \cdots, t_a) \in \prod_{i=1}^a S(U(l_i)\times U(1))$, and $E,\ F \in O(2k-1)$, $e,\ f \in O(1)$ such that $e \det E = f \det F = 1$.

Next, we consider the following part in K:

$$\begin{aligned} \mathcal{R}_1 &= \left\{ \left(A,\ (1, \cdots, 1),\ \begin{pmatrix} \sigma_1(A) I_{2k-1} & 0 \\ 0 & \sigma_1(A) \end{pmatrix} \right) \middle| \ \sigma_1(A) = \pm 1 \right\} \\ &\equiv \left\{ \left(B,\ (1, \cdots, 1),\ \begin{pmatrix} \sigma_2(B) I_{2k-1} & 0 \\ 0 & \sigma_2(B) \end{pmatrix} \right) \middle| \ \sigma_2(B) = \pm 1 \right\} = \mathcal{R}_2. \end{aligned}$$

By using $\mathcal{R}_1$, $\mathcal{R}_2$ and the method similar to that demonstrated in Section 3.3 (the case (1)-(a)), we can regard $K_1 = K_2$ in G and $\sigma_1 = \sigma_2$. Hence, K is as follows:

$$K = \prod_{i=1}^a S(U(l_i) \times U(1)) \times \left\{ \left(A,\ \begin{pmatrix} X & 0 \\ 0 & x \end{pmatrix} \right) \in \mathcal{S}_1 \times S(O(2k-1) \times O(1)) \ \middle| \ \sigma_1(A) x = 1 \right\}$$

As the result of the above argument, the tubular neighborhood $X_s = G \times_{K_s} D^{2k}$ $(s = 1,\ 2)$ is equivariantly diffeomorphic to the following manifold:

$$\prod_{i=1}^a \mathbb{C}P(l_i) \times \left(\prod_{j=1}^b S^{2m_j} \times_{\mathcal{A}} D(\mathbb{R}^{2k}) \right), \tag{3.2}$$

where $\mathcal{A}(= \mathcal{A}_s \simeq \mathcal{S}_s/\mathcal{S}_s^o)$ acts on $\prod_{j=1}^b S^{2m_j}$ as the subgroup of $\prod_{j=1}^b \mathbb{Z}_2$ and on $D(\mathbb{R}^{2k})$ by $\sigma_{\mathbb{R}} : \mathcal{A} \to \{\pm I_{2k}\}$. Here, $G = \prod_{j=1}^b SO(2m_j+1) \times \prod_{i=1}^a SU(l_i + 1) \times SO(2k)$ acts on this manifold as follows: $\prod_{i=1}^a SU(l_i+1)$ acts on $\prod_{i=1}^a \mathbb{C}P(l_i)$ naturally; $\prod_{j=1}^b SO(2m_j+1)$ acts on $\prod_{j=1}^b S^{2m_j}$ naturally; and $SO(2k)$ acts on $D(\mathbb{R}^{2k})$ naturally.

Next, we analyze attaching maps $f : \partial X_1 \to \partial X_2$. In this case, we have

$$N(K;G)/N(K;G)^o \simeq \prod_{j=1}^b \mathbb{Z}_2 \times \prod_{i=1}^a W_{l_i} \times W,$$

where $W = S(O(2k-1) \times O(1))/SO(2k-1) = \{I_{2k},\ -I_{2k}\} \simeq \mathbb{Z}_2$, and

$$W_{l_i+1} = \begin{cases} \{I_{l_i+1}\} & \text{if } l_i \geq 2 \text{ or } \alpha_i \neq 0 \\ S_2 & \text{if } l_i = 1 \text{ and } \alpha_i = 0. \end{cases}$$

Therefore, by the same argument of [**13**, Section 8.4], we can show the Uchida's criterion (2) for $I \circ f$. Hence, by using the similar argument to that demonstrated in Section 3.3, we have that $(M,\ G)$ which satisfies the case (1)-(c) is unique up to essential isomorphism if we fix the representation $\sigma_{\mathbb{R}}$. Thus, we have that M and G in the case (1)-(c) are as follows (up to essential isomorphism):

$$M = \prod_{j=1}^b S^{2m_j} \times_{\mathcal{A}} \left(\prod_{i=1}^a \mathbb{C}P(l_i) \times S(\mathbb{R}^{2k} \oplus \mathbb{R}) \right),$$

$$G = \prod_{j=1}^b SO(2m_j+1) \times \prod_{i=1}^a SU(l_i+1) \times SO(2k).$$

This corresponds with the third case of Theorem 3.1.

4. Preliminary for the cases (2) and (3)

In this section, we prepare to classify the other cases, i.e., G/K_1 is a torus manifold but G/K_2 is not a torus manifold (the cases (2) and (3)). Henceforth, we assume that G/K_2 is not a torus manifold. In this case, $G/K_2 \cap M^T = \emptyset$, by using Lemma 2.1. It follows that $T \subset G$ (a maximal torus) but $T \not\subset K_2^o$, i.e., rank $G >$ rank K_2^o.

We first prove that $\dim G/K_2^o = \dim G/K_2$ is odd. Because G/K_1 is a torus manifold, we can apply the same argument which is demonstrated in Section 3 to get K (also see Section 4.1). By using K's in Section 3, we can easily show the following formula:

$$\text{rank } G = \text{rank } K_1^o = \text{rank } K^o + 1.$$

Let $T' \subset K^o$ be a maximal torus such that $T' \subset T$. If T' is not a maximal torus in K_2^o, then we have rank $K_2^o =$ rank $K^o + 1$ by using $T' \subset K^o \subset K_2^o \subset G$ and the above formula. However, this gives a contradiction to rank $G >$ rank K_2^o. Therefore, we have rank $K_2^o =$ rank K^o. Due to [**7**, Theorem 1.1], we also have $\dim K_2^o/K^o$ is even. Thus, by considering the fibration: $K_2^o/K^o \to G/K^o \to G/K_2^o$ and $\dim G/K^o = 2n-1$, we have that $\dim G/K_2^o = \dim G/K_2$ is odd. Hence, we can put

$$\dim G/K_2 = 2n - 2k_2 + 1$$

for $k_2 \geq 1$. Remark that if $k_2 = 1$ then this is in the case (2): G/K_2 is an exceptional orbit, otherwise this is in the case (3): G/K_2 is a singular orbit.

Since our G-action on M has codimension 1 orbits, we have that

$$K_2/K \cong S^{2k_2-2}$$

in each case.

In summary, we have the following lemma.

Lemma 4.1. *Suppose a torus manifold* $(M,\ T)$ *has codimension* 1 *extended* G*-actions. If* G/K_2 *is not a torus manifold, then*

$$\dim G/K_2 = 2n - 2k_2 + 1, \quad K_2/K \cong S^{2k_2-2},$$

for $k_2 \geq 1$.

4.1. Structures of G/K_1 and their tubular neighborhoods of the cases (2) and (3).

The main part to classify the cases (2) and (3) is to determine the group K_2 and its inclusion $K_2 \subset G$. From the next section, We will determine $K_2 \subset G$ by making use of the relation $K \subset K_2 \subset G$ and the classification result of the transitive action on S^{2k_2-2}. In this subsection, we recall G, K_1 and K; and in the next subsection, we give an important remark for the attaching map.

Let $\dim G/K_1 = 2n - 2k_1$ for $k_1 \geq 1$. By Lemma 2.2, 2.3 and Remark 2.4, we have $(G,\ K_1)$ as the following two cases:

- **(a):** $G = \prod_{j=1}^{b} SO(2m_j+1) \times \prod_{i=1}^{a} SU(l_i+1) \times SU(k_1) \times T^1$, and $K_1 = \mathcal{S}_1 \times \prod_{i=1}^{a} S(U(l_i) \times U(1)) \times SU(k_1) \times T^1$ for $k_1 \geq 1$;
- **(b):** $G = \prod_{j=1}^{b} SO(2m_j+1) \times \prod_{i=1}^{a} SU(l_i+1) \times SO(2k_1)$, and $K_1 = \mathcal{S}_1 \times \prod_{i=1}^{a} S(U(l_i) \times U(1)) \times SO(2k_1)$ for $k_1 \geq 2$.

We call them *the case* (a) and (b) respectively.

4.1.1. *The case (a).* For the principal isotropy subgroup K of the case (a), we can easily show the following by using the same argument in Section 3.3:

$$K=\left\{\left(A,\ (t_1,\ \cdots,\ t_a),\ \begin{pmatrix} X & 0 \\ 0 & x \end{pmatrix},\ t\right)\ \middle|\ \sigma_1(A)xt_1^{\alpha_1}\cdots t_a^{\alpha_a}t^{\alpha}=1\right\},$$

where $A\in\mathcal{S}_1$, $(t_1,\ \cdots,\ t_a)\in\prod_{i=1}^{a}S(U(l_i)\times U(1))$, $t\in T^1$ and $X\in U(k_1-1)$ such that $x\det X=1$. Here, $\sigma_1(A)\subset\{\pm1\}\subset S^1$ and $(\alpha_1,\ \cdots,\ \alpha_a,\ \alpha)\in\mathbb{Z}^a\times\mathbb{N}$. If $k_1=1$, then we can take $\alpha=1$. Therefore, a tubular neighborhood $X_1=G\times_{K_1}D^{2k_1}$ of the case (a) is equivariantly diffeomorphic to the following manifold defined in (3.1):

$$\prod_{j=1}^{b}S^{2m_j}\times_{\mathcal{A}_1}\left(\prod_{i=1}^{a}S^{2l_i+1}\times_{T^a}D(\mathbb{C}^{k_1}_{\mathfrak{c}})\right). \tag{4.1}$$

Remark that from this section we write the representation $\mathfrak{a}:T^a\to S^1$ as $\mathfrak{c}$.

4.1.2. *The case (b).* For the principal isotropy subgroup K of the case (b), we can easily show the following by using the same argument in Section 3.5:

$$\begin{aligned} K &= \prod_{i=1}^{a}S(U(l_i)\times U(1)) \\ &\times \left\{\left(A,\ \begin{pmatrix} X & 0 \\ 0 & x \end{pmatrix}\right)\in\mathcal{S}_1\times S(O(2k_1-1)\times O(1))\ \middle|\ \sigma_1(A)x=1\right\}. \end{aligned}$$

Hence, a tubular neighborhood $X_1=G\times_{K_1}D^{2k_1}$ of the case (b) is equivariantly diffeomorphic to the following manifold defined in (3.2):

$$\prod_{i=1}^{a}\mathbb{C}P(l_i)\times\left(\prod_{j=1}^{b}S^{2m_j}\times_{\mathcal{A}_1}D(\mathbb{R}^{2k_1})\right). \tag{4.2}$$

4.2. Important remark for the attaching maps of the cases (2) and (3). Before we go to the next section, we give the following important remark for the attaching maps $f:G/K=\partial X_1\to\partial X_2=G/K$.

Remark 4.2. In each above case (a) and (b), we see that $N(K;G)/N(K;G)^o$ is the same as one of the cases (1)-(a), (1)-(b) and (1)-(c). Therefore, we have already proved the Uchida's criterion (2) for $I\circ f$ in Section 3, that is, $I\circ f:\partial X_1\to\partial X_1$ is extendable to the equivariant map $X_1\to X_1$. Hence, once we know X_1 and X_2 for the cases (2) and (3), then a G-diffeomorphism $M(f)\cong M(I)$ always holds, i.e., the constructing manifold $X_1\cup X_2$ is unique. It follows that, for the cases (2) and (3), we may only analyse the structure of $X_2=G\times_{K_2}D^{2k_2-1}$.

In the next six sections (the case (2) in Section 5, 6, 7; the case (3) in Section 8, 9, 10), we will analyze K_2 and its slice representation $\sigma_2:K_2\to O(2k_2-1)$ of the cases (2) and (3).

5. Main theorem and remarks of the case (2)

The goal of this section is to state the main theorem and give some remarks of the case (2). In this section and the next two sections, we assume that the other singular orbit G/K_2 is an exceptional orbit, i.e., $\dim G/K_2=\dim G/K=2n-1$.

Then we have $k_2 = 1$ and $K_2/K \cong S^0$ by Lemma 4.1. Due to [**1**, 3.2 Theorem (ii) in Chapter IV], we have that K is a normal subgroup of K_2, i.e.,

$$K \subset K_2 \subset N(K; G), \tag{5.1}$$

where $N(K; G)$ is the normalizer of K in G.

5.1. Main theorem of the case (2)**.** Before we state the main theorem, we prepare some notations (also see Section 3.1). Let $\mathcal{A}$ be a subgroup of $\prod_{j=1}^{b} \mathbb{Z}_2$, where $\prod_{j=1}^{b} \mathbb{Z}_2$ is generated by the antipodal involutions on S^{2m_j} for $j = 1, \cdots, b$. The subgroup Δ in $S^1 \times O(1)$ (resp. in $SO(2k_1) \times O(1)$) denotes the diagonal subgroup $\{(1,\ 1),\ (-1,\ -1)\}$ (resp. $\{(I_{2k_1},\ 1),\ (-I_{2k_1},\ -1)\}$), and the subgroup $O(1)$ in $S^1 \times O(1)$ (resp. in $SO(2k_1) \times O(1)$) denotes the subgroup $\{(1,\ 1),\ (1,\ -1)\}$ (resp. $\{(I_{2k_1},\ 1),\ (I_{2k_1},\ -1)\}$). Now we may state the main theorem of this case.

Theorem 5.1. *Suppose a torus manifold M has a codimension one extended G-action. If there is an exceptional orbit, then $(M,\ G)$ is essentially isomorphic to*

$$\left(\prod_{j=1}^{b} S^{2m_j} \times_{\mathcal{A} \times \mathbb{Z}_2} N,\quad \prod_{j=1}^{b} SO(2m_j + 1) \times H \right),$$

such that $(N,\ H)$ is one of the followings:

	N	H
(a)	$(\prod_{i=1}^{a} S^{2l_i+1}) \times_{T^a} S(\mathbb{C}_{\mathfrak{c}}^{k_1} \oplus \mathbb{R})$	$\prod_{i=1}^{a} SU(l_i + 1) \times U(k_1)$
(b)	$\prod_{i=1}^{a} \mathbb{C}P(l_i) \times S(\mathbb{R}^{2k_1} \oplus \mathbb{R})$	$\prod_{i=1}^{a} SU(l_i + 1) \times SO(2k_1)$

where $\mathcal{A}$ acts on $\prod_{j=1}^{b} S^{2m_j}$ as the subgroup $\prod_{j=1}^{b} \mathbb{Z}_2$ and on the fibre of N through the following representations:

(a): $\sigma_{\mathbb{C}} : \mathcal{A} \to \{\pm 1\} \subset S^1$ *on* $S(\mathbb{C}_{\mathfrak{c}}^{k_1} \oplus \mathbb{R}) \cap \mathbb{C}_{\mathfrak{c}}^{k_1}$*;*
(b): $\sigma_{\mathbb{R}} : \mathcal{A} \to \{\pm I_{2k_1}\} \subset SO(2k_1)$ *on* $S(\mathbb{R}^{2k_1} \oplus \mathbb{R}) \cap \mathbb{R}^{2k_1}$*;*

respectively, and $\mathbb{Z}_2$ (i.e., the second factor of $\mathcal{A} \times \mathbb{Z}_2$) acts on $\prod_{j=1}^{b} S^{2m_j}$ through the representation $\rho : \mathbb{Z}_2 \to \prod_{j=1}^{b} \mathbb{Z}_2$ which satisfies $\rho(\mathbb{Z}_2) \cap \mathcal{A} = \{1\}$ and on the fibre of N through the following representations:

(a): $\sigma_{\mathbb{C}\oplus\mathbb{R}} : \mathbb{Z}_2 \to \{\pm 1\} \times O(1) \subset S^1 \times O(1)$ *on* $\mathbb{C}_{\mathfrak{c}}^{k_1} \oplus \mathbb{R}$*;*
(b): $\sigma_{\mathbb{R}\oplus\mathbb{R}} : \mathbb{Z}_2 \to \{\pm I_{2k_1}\} \times O(1) \subset SO(2k_1) \times O(1)$ *on* $\mathbb{R}^{2k_1} \oplus \mathbb{R}$*;*

respectively, such that $\sigma (= \sigma_{\mathbb{C}\oplus\mathbb{R}},\ \sigma_{\mathbb{R}\oplus\mathbb{R}})$ and ρ satisfy one of the followings:

(i): *ρ is non-trivial, and $\sigma(\mathbb{Z}_2) = O(1)$ or $\sigma(\mathbb{Z}_2) = \Delta$;*
(ii): *ρ is trivial, and $\sigma(\mathbb{Z}_2) = \Delta$.*

Here, G-actions of (a), (b) are the same as Theorem 3.1 (a), (c), respectively.

Remark 5.2. All manifolds appeared in Theorem 5.1 (a), (b) are $\mathbb{Z}_2$-quotient of manifolds appeared in Theorem 3.1 (a), (c), respectively. These $\mathbb{Z}_2$-actions are defined by $\rho \times \sigma$ in Theorem 5.1. Moreover, we remark that torus actions of manifolds in Theorem 3.1 (a), (c) have the next property: two singular orbits are same (diffeomorphic to G/H). By the $\mathbb{Z}_2$-quotient defined in Theorem 5.1, these two same singular orbits in Theorem 3.1 (a), (c) go to just one singular orbit G/K_1. Then G/K_1 can be regarded as $\mathbb{Z}_2 \backslash G/H$ (if ρ is non-trivial) or G/H (if ρ is trivial).

In order to prove Theorem 5.1, we divide this case (2) into two cases which correspond with the case (a) and (b) for the type of K_1 (see Section 4). We call them *the case* (2)-(a) and (2)-(b), respectively, i.e., the case (2)-(a) is that G/K_2 is an exceptional orbit and G/K_1 satisfies the case (a), and the case (2)-(b) is that G/K_2 is an exceptional orbit and G/K_1 satisfies the case (b).

In Section 6 and 7, we study the case (2)-(a) and (2)-(b), respectively. Before we go to Section 6, we give some technical remarks to study the case (2).

5.2. Remarks of the case (2)**.** We have already analysed X_1 and attaching map $f : \partial X_1 \to \partial X_2$ in Section 4.1.1, 4.1.2 and 4.2; therefore, in order to construct a G-manifold $M = X_1 \cup_f X_2$, we may only analyse $X_2 = G \times_{K_2} D(\mathbb{R})$, where $D(\mathbb{R}) = D^1 \subset \mathbb{R}$ (1-dimensional disk).

Because $K_2/K \simeq S^0$ and G/K_2 is an exceptional orbit, we can easily show that the slice representation $\sigma_2 : K_2 \longrightarrow O(1)$ always satisfies the following properties:

- σ_2 is surjective;
- $\ker \sigma_2 = K$.

Hence, a tubular neighborhood $X_2 = G \times_{K_2} D(\mathbb{R})$ is only determined by the inclusion $K_2 \subset G$. Therefore, we may only analyse the inclusion $K \subset K_2 \subset N(K;G)$, (see (5.1)) in the case (2). So we first need to compute $N(K;G)$. In the remainder of this section, we compute $N(K;G)$ in the cases (2)-(a) and (2)-(b).

First we assume that K is in the case (2)-(a) (see Section 4.1.1). Because of the definition of K in the case (2)-(a), we have that $N(K;G)$ is as follows:

$$N(K;G) = \prod_{j=1}^{b} S(O(2m_j) \times O(1)) \times \prod_{i\in I} W_i \times \prod_{i'\in I'} S(U(l_{i'}) \times U(1)) \times S(U(k_1-1) \times U(1)) \times T^1,$$

where if $l_i = 1$ and $\alpha_i = 0$ then $i \in I$, otherwise $i' \in I'$ ($I \cup I' = \{1, \cdots, a\}$), and $W_i = N(S(U(1) \times U(1)); SU(2))$ for $i \in I$. However, if $l_i = 1$ and $\alpha_i = 0$ then we can regard $SU(2)$ as $SO(3)$ up to essential isomorphism ($SU(2) \approx SO(3)$). Regarding $\{1, \cdots, b\} \cup I$ as $\{1, \cdots, b\}$ and I' as $\{1, \cdots, a\}$ again, we can write $N(K;G)$ as follows:

$$(5.2)\qquad N(K;G) = \prod_{j=1}^{b} S(O(2m_j) \times O(1)) \times \prod_{i=1}^{a} S(U(l_i) \times U(1)) \times S(U(k_1-1) \times U(1)) \times T^1.$$

Next we assume that K is in the case (2)-(b) (see Section 4.1.2). By the similar argument of the case (2)-(a), we can regard $N(K;G)$ in the case (2)-(b) as follows:

$$(5.3)\qquad N(K;G) = \prod_{j=1}^{b} S(O(2m_j) \times O(1)) \times \prod_{i=1}^{a} S(U(l_i) \times U(1)) \times S(O(2k_1-1) \times O(1)).$$

In the next two sections, we will analyse $K \subset K_2 \subset N(K;G)$.

6. The case (2)-(a)

In this section, we study the case (2)-(a). From Section 4.1.1, we have that

$$G = \prod_{j=1}^{b} SO(2m_j+1) \times \prod_{i=1}^{a} SU(l_i+1) \times SU(k_1) \times T^1,$$

$$K_1 = \mathcal{S}_1 \times \prod_{i=1}^{a} S(U(l_i)\times U(1)) \times SU(k_1) \times T^1,$$

$$K = \left\{ \left(A,\ (t_1,\ \cdots,\ t_a),\ \begin{pmatrix} X & 0 \\ 0 & x \end{pmatrix},\ t \right) \ \middle|\ \sigma_1(A) t_1^{\alpha_1} \cdots t_a^{\alpha_a} x t^{\alpha} = 1 \right\}.$$

The elements of K are often denoted by $(A,\ (t_1,\ \cdots,\ t_a),\ x,\ t)$ simply.

In order to analyse the inclusion $K \subset K_2 \subset N(K;G)$ (see (5.2) for $N(K;G)$), we first define p_1 and p_2 as the following two natural projections:

$$p_1 : N(K;G) \longrightarrow \prod_{j=1}^{b} S(O(2m_j)\times O(1));$$

$$p_2 : N(K;G) \longrightarrow \prod_{i=1}^{a} S(U(l_i)\times U(1)) \times S(U(k_1-1)\times U(1)) \times T^1.$$

Then we can easily prove the following lemma.

LEMMA 6.1. *For p_1, p_2 and K, the following properties hold:*

(1) *the image of K by p_1 satisfies $p_1(K) = \mathcal{S}_1$;*
(2) *if $\sigma_1(\mathcal{S}_1) = \{1\}$, then $K = \mathcal{S}_1 \times p_2(K)$ and* $p_2(K) = \{((t_1,\cdots,t_a),x,t) \mid t_1^{\alpha_1}\cdots t_a^{\alpha_a} x t^{\alpha} = 1\}$;
(3) *if $\sigma_1(\mathcal{S}_1) = \{\pm 1\}$, then $(\mathcal{S}_1 \times p_2(K))/K \simeq \mathbb{Z}_2$ and* $p_2(K) = \{((t_1,\cdots,t_a),x,t) \mid t_1^{\alpha_1}\cdots t_a^{\alpha_a} x t^{\alpha} = \pm 1\}$.

PROOF. We define $\mathcal{R}_1 \subset K$ as follows (also see Section 3.3):

$$\mathcal{R}_1 \quad = \quad \left\{ \left(A,\ e,\ \begin{pmatrix} J & 0 \\ 0 & \sigma_1(A) \end{pmatrix},\ 1 \right) \ \middle|\ A \in \mathcal{S}_1 \right\}$$

where $e \in \prod_{i=1}^{a} S(U(l_i)\times U(1))$ is the identity element and $J \in U(k_1-1)$ such that $J = I_{k_1-1}$ (if $\sigma_1(A) = 1$) or $\det J = -1$ with $J^2 = I_{k_1-1}$ (if $\sigma_1(A) = -1$). Then $\mathcal{R}_1 \simeq \mathcal{S}_1$. It follows that the first statement holds. The second and third statements are proved by the definition of K and the first property $p_1(K) = \mathcal{S}_1$. □

Let $\mathcal{S}_2$ be a subgroup of $\prod_{j=1}^{b} S(O(2m_j)\times O(1))$ such that $\mathcal{S}_2/\mathcal{S}_1 \simeq \mathbb{Z}_2$. Because $K_2/K \simeq S^0$, for $p_s(K_2)$ and $p_s(K)$ $(s = 1,\ 2)$, one of the following four cases occurs:

(i): $p_1(K_2) = \mathcal{S}_2$, and $p_2(K_2) = p_2(K)$;
(ii): $p_1(K_2) = p_1(K) = \mathcal{S}_1$, and $p_2(K_2)/p_2(K) \simeq \mathbb{Z}_2$;
(iii): $p_1(K_2) = p_1(K) = \mathcal{S}_1$, and $p_2(K_2) = p_2(K)$;
(iv): otherwise, i.e., $p_1(K_2) = \mathcal{S}_2$, and $p_2(K_2)/p_2(K) \simeq \mathbb{Z}_2$.

We call the above cases *the case* (2)-(a)-(i), (2)-(a)-(ii), (2)-(a)-(iii) and (2)-(a)-(iv), respectively.

6.1. The case (2)-(a)-(i). Suppose the case (2)-(a)-(i) occurs, that is,

$p_1(K_2)/p_1(K) = \mathcal{S}_2/\mathcal{S}_1 \simeq \mathbb{Z}_2,$ and

$p_2(K_2) = p_2(K) = \{((t_1,\ \cdots,\ t_a),\ x,\ t) \mid t_1^{\alpha_1}\cdots t_a^{\alpha_a} x t^{\alpha} \in \sigma_1(\mathcal{S}_1) \subset \{\pm 1\}\}.$

Because $K \subset p_1(K) \times p_2(K)$ and $K_2 \subset p_1(K_2) \times p_2(K_2)$, in this case we have that

$$K \subset \mathcal{S}_1 \times p_2(K), \quad K \subset K_2 \subset \mathcal{S}_2 \times p_2(K). \tag{6.1}$$

First we assume $\sigma_1(\mathcal{S}_1) = \{\pm 1\}$, then $(\mathcal{S}_1 \times p_2(K))/K \simeq \mathbb{Z}_2$ by Lemma 6.1 (3). Therefore, $\#(\mathcal{S}_2 \times p_2(K))/K = 4$ by $\mathcal{S}_2/\mathcal{S}_1 \simeq \mathbb{Z}_2$ and $K \subset \mathcal{S}_1 \times p_2(K) \subset \mathcal{S}_2 \times p_2(K)$ (where $\#F$ is the number of the finite group F). Hence, by making use of $K_2/K \simeq S^0$ and (6.1), we also have

$$(\mathcal{S}_2 \times p_2(K))/K_2 \simeq \mathbb{Z}_2. \tag{6.2}$$

Because $p_1(K_2) = \mathcal{S}_2$, $K \subset K_2 \cap (\mathcal{S}_1 \times p_2(K)) \subset K_2$ and $K_2/K \simeq S^0$, we also have

$$K_2 \cap (\mathcal{S}_1 \times p_2(K)) = K. \tag{6.3}$$

Now we define the representation $\rho : \mathcal{S}_1 \times p_2(K) \to \mathbb{Z}_2$ as follows:

$$\rho(A,\ (t_1,\ \cdots,\ t_a),\ x,\ t) = \sigma_1(A) t_1^{\alpha_1} \cdots t_a^{\alpha_a} x t^{\alpha}.$$

By the definition of K, we have $\ker \rho = K$. Let $\widetilde{\rho} : \mathcal{S}_2 \times p_2(K) \to \mathbb{Z}_2$ be a lift of this representation, i.e., the restricted representation $\widetilde{\rho}|_{\mathcal{S}_1 \times p_2(K)}$ coincides with ρ. Because $\mathcal{S}_2/\mathcal{S}_1 \simeq \mathbb{Z}_2$ and $\rho|_{p_2(K)} = \widetilde{\rho}|_{p_2(K)}$, this lift $\widetilde{\rho}$ is only determined by a representation $\sigma_2 : \mathcal{S}_2 \to \mathbb{Z}_2$ such that $\sigma_2|_{\mathcal{S}_1} = \sigma_1$. Hence, we have the following lemma.

Lemma 6.2. *Let $\widetilde{\rho} : \mathcal{S}_2 \times p_2(K) \to \mathbb{Z}_2$ be a lift of ρ. Then there is a representation $\sigma_2 : \mathcal{S}_2 \to \mathbb{Z}_2$ such that $\sigma_2|_{\mathcal{S}_1} = \sigma_1$ and $\widetilde{\rho}$ is denoted as follows:*

$$\widetilde{\rho}(B,\ (t_1,\ \cdots,\ t_a),\ x,\ t) = \sigma_2(B) t_1^{\alpha_1} \cdots t_a^{\alpha_a} x t^{\alpha}.$$

On the other hand, by (6.2), there is the following sequence:

$$K_2 \xrightarrow{\widetilde{i}} \mathcal{S}_2 \times p_2(K) \xrightarrow{\widetilde{r}} \mathbb{Z}_2,$$

where $\widetilde{i}$ is an inclusion and $\widetilde{r}$ is the surjective representation induced by the projective representation $\mathcal{S}_2 \times p_2(K) \to (\mathcal{S}_2 \times p_2(K))/K_2$. Let r be the restricted representation $\widetilde{r}|_{\mathcal{S}_1 \times p_2(K)}$. Using (6.3), we see that the representation r is induced by the natural surjection $\mathcal{S}_1 \times p_2(K) \to (\mathcal{S}_1 \times p_2(K))/K$, i.e., there is the following restricted sequence:

$$K \xrightarrow{i} \mathcal{S}_1 \times p_2(K) \xrightarrow{r} \mathbb{Z}_2,$$

where i is the natural inclusion. By the definition of K, we have that r can be identified with ρ. Therefore, by Lemma 6.2, we have that $\widetilde{r} = \widetilde{\rho}$ for some $\sigma_2 : \mathcal{S}_2 \to \mathbb{Z}_2$. In other wards, there is the representation $\sigma_2 : \mathcal{S}_2 \to \mathbb{Z}_2$ such that

$$\begin{aligned} K_2 &= \ker \widetilde{\rho} (= \ker \widetilde{r}) \\ &= \left\{ \left(B,\ (t_1,\ \cdots,\ t_a),\ \begin{pmatrix} X & 0 \\ 0 & x \end{pmatrix},\ t \right) \ \middle|\ \sigma_2(B) t_1^{\alpha_1} \cdots t_a^{\alpha_a} x t^{\alpha} = 1 \right\}. \end{aligned} \tag{6.4}$$

Next we assume $\sigma_1(\mathcal{S}_1) = \{1\}$, then $K = \mathcal{S}_1 \times p_2(K)$ by Lemma 6.1 (2). Hence, by using (6.1) and the assumption of the case (2)-(a)-(i), we have

$$K_2 = \mathcal{S}_2 \times p_2(K).$$

We can regard this case as the case that $\sigma_2(\mathcal{S}_2) = \{1\}$ in Eq.(6.4).

Therefore, in the case (2)-(a)-(i), the inclusion $K_2 \subset N(K;G)$ is completely determined by the subgroup $\mathcal{S}_2 \subset \prod_{j=1}^{b} S(O(2m_j) \times O(1))$ and the representation $\sigma_2 : \mathcal{S}_2 \to \mathbb{Z}_2$ such that $\sigma_2|_{\mathcal{S}_1} = \sigma_1$. Hence, using (4.1) and Remark 4.2, we can easily check the following manifold corresponds with $M = X_1 \cup X_2$ up to essential isomorphism (by computing orbit types of the natural $G = \prod_{j=1}^{b} SO(2m_j+1) \times \prod_{i=1}^{a} SU(l_i+1) \times U(k_1)$ action):

$$M = \prod_{j=1}^{b} S^{2m_j} \times_{\mathcal{A}\times\mathbb{Z}_2} \left(\prod_{i=1}^{a} S^{2l_i+1} \times_{T^a} S(\mathbb{C}_{\mathfrak{c}}^{k_1} \oplus \mathbb{R}) \right),$$

where $\mathcal{A} = \mathcal{A}_1 \simeq \mathcal{S}_1 / \prod_{j=1}^{b} SO(2m_j)$ and the $\mathcal{A}$-quotient is defined by the following actions:

- on $\prod_{j=1}^{b} S^{2m_j}$ as the subgroup of $\prod_{j=1}^{b} \mathbb{Z}_2$;
- on $\mathbb{C}_{\mathfrak{c}}^{k_1}$ through the representation $\sigma_1 : \mathcal{A} \to \mathbb{Z}_2$ (this representation is induced by $\sigma_1 : \mathcal{S}_1 \to \mathbb{Z}_2$);
- on $\mathbb{R}$ trivially,

and the $\mathbb{Z}_2$-quotient is defined by the following actions:

- on $\prod_{j=1}^{b} S^{2m_j}$ by a non-trivial representation $\rho : \mathbb{Z}_2 \to \prod_{j=1}^{b} \mathbb{Z}_2$ which satisfies $\rho(\mathbb{Z}_2) \cap \mathcal{A} = \{1\}$ (this corresponds with $p_1(K_2) = \mathcal{S}_2$);
- on $\mathbb{C}_{\mathfrak{c}}^{k_1}$ by a representation $\sigma_2 : \mathbb{Z}_2 \to \{\pm 1\}$, where if σ_1 is trivial then σ_2 is also trivial (this corresponds with that if $\sigma_1(\mathcal{S}_1) = \{1\}$ then $\sigma_2(\mathcal{S}_2) = \{1\}$);
- on $\mathbb{R}$ by the natural representation (this corresponds with the existence of an exceptional orbit).

Remark that $\mathcal{A} \times \mathbb{Z}_2$ acts on $\prod_{j=1}^{b} S^{2m_j}$ freely because ρ is non-trivial and satisfies $\rho(\mathbb{Z}_2) \cap \mathcal{A} = \{1\}$; therefore, M is a manifold, more precisely a fibre bundle over $\prod_{j=1}^{b} S^{2m_j} / (\mathcal{A} \times \mathbb{Z}_2)$ with the fibre $\prod_{i=1}^{a} S^{2l_i+1} \times_{T^a} S(\mathbb{C}_{\mathfrak{c}}^{k_1} \oplus \mathbb{R})$.

6.2. The case (2)-(a)-(ii). Suppose the case (2)-(a)-(ii) occurs, that is,

$$p_1(K_2) = p_1(K) = \mathcal{S}_1 \quad \text{and}$$
$$p_2(K_2)/p_2(K) \simeq \mathbb{Z}_2.$$

First we prove the following lemma.

Lemma 6.3. *Suppose that $p_2(K_2)/p_2(K) \simeq \mathbb{Z}_2$. Then the inclusion $p_2(K_2) \subset p_2(N(K;G))$ is unique. Furthermore, we have*

$$p_2(K_2) = \left\{ \left((t_1, \cdots, t_a), \begin{pmatrix} X & 0 \\ 0 & x \end{pmatrix}, t \right) \;\middle|\; t_1^{\alpha_1} \cdots t_a^{\alpha_a} x t^{\alpha} \in \mathbb{Z}_m \right\},$$

where $m = 2$ if $\sigma_1(\mathcal{S}_1) = \{1\}$ or $m = 4$ if $\sigma_1(\mathcal{S}_1) = \{\pm 1\}$.

Proof. We first remark that $\sigma_1(\mathcal{S}_1) \subset \{\pm 1\}$, by Section 4.1.1.

Consider the following surjective representation $\sigma : p_2(N(K;G)) = \prod_{i=1}^{a} S(U(l_i) \times U(1)) \times S(U(k_1-1) \times U(1)) \times T^1 \to S^1$ (induced by the slice representation σ_1):

$$\sigma\left((t_1, \cdots, t_a), \begin{pmatrix} X & 0 \\ 0 & x \end{pmatrix}, t \right) = t_1^{\alpha_1} \cdots t_a^{\alpha_a} x t^{\alpha} \quad (\text{or} \quad t_1^{\alpha_1} \cdots t_a^{\alpha_a} t \quad \text{for} \quad k_1 = 1).$$

Then we see that $p_2(K) = \sigma^{-1}(\sigma_1(\mathcal{S}_1)) \subset p_2(K_2)$ by the definition of K. Therefore, we have that $\sigma(p_2(K)) = \sigma_1(\mathcal{S}_1) \subset \sigma(p_2(K_2))$.

Because $p_2(K_2)/p_2(K) \simeq \mathbb{Z}_2$, we also have that

$$\sigma(p_2(K_2))/\sigma(p_2(K)) \subset \mathbb{Z}_2(\subset \sigma(p_2(N(K;G)))/\sigma(p_2(K)) \simeq S^1).$$

If $\sigma(p_2(K_2))/\sigma(p_2(K)) = \{1\} \subset \mathbb{Z}_2$, then $\sigma(p_2(K_2)) = \sigma(p_2(K)) = \sigma_1(\mathcal{S}_1)$. Hence, we have $p_2(K_2) \subset \sigma^{-1}(\sigma_1(\mathcal{S}_1)) = p_2(K)$. However, this gives a contradiction, because $p_2(K_2)/p_2(K) \simeq \mathbb{Z}_2$. Therefore, we have

$$\sigma(p_2(K_2))/\sigma(p_2(K)) = \mathbb{Z}_2 \subset S^1.$$

It follows that there are the following two cases:

- if $\sigma_1(\mathcal{S}_1) = \sigma(p_2(K)) = \{1\}$, then $\sigma(p_2(K_2)) = \mathbb{Z}_2 = \{\pm 1\} \subset S^1$; hence, $p_2(K_2) \subset \sigma^{-1}(\mathbb{Z}_2)$;
- if $\sigma_1(\mathcal{S}_1) = \sigma(p_2(K)) = \{\pm 1\}$, then $\sigma(p_2(K_2)) = \mathbb{Z}_4 = \{\pm 1,\ \pm i\} \subset S^1$; hence, $p_2(K_2) \subset \sigma^{-1}(\mathbb{Z}_4)$.

Because $p_2(K_2)/p_2(K) \simeq \mathbb{Z}_2$ and $p_2(K) = \sigma^{-1}(\sigma_1(\mathcal{S}_1))$, we can easily show that $p_2(K_2) = \sigma^{-1}(\mathbb{Z}_2)$ for the case $\sigma_1(\mathcal{S}_1) = \{1\}$ and $p_2(K_2) = \sigma^{-1}(\mathbb{Z}_4)$ for the case $\sigma_1(\mathcal{S}_1) = \{\pm 1\}$. Hence, we have that the inclusion of $p_2(K_2) \subset p_2(N(K;G))$ is uniquely determined by $\sigma^{-1}(\mathbb{Z}_2)$ or $\sigma^{-1}(\mathbb{Z}_4)$. Thus, we have the statement of this lemma. □

Assume $\sigma_1(\mathcal{S}_1) = \{\pm 1\}$. Then $m = 4$ by Lemma 6.3. Let $\widetilde{\rho} : \mathcal{S}_1 \times p_2(K_2) \to \mathbb{Z}_4$ be the following representation:

$$\widetilde{\rho}(A,\ (t_1,\ \cdots,\ t_a),\ x,\ t) = \sigma_1(A) t_1^{\alpha_1} \cdots t_a^{\alpha_a} x t^{\alpha} \in \mathbb{Z}_4.$$

We can easily show that $\widetilde{\rho}^{-1}(\mathbb{Z}_2) = \mathcal{S}_1 \times p_2(K)$ and $\ker \widetilde{\rho} = K$. Because $K_2/K \simeq S^0$, we also have $\widetilde{\rho}(K_2) = \mathbb{Z}_2 \subset \mathbb{Z}_4$. Hence, we have $K_2 \subset \widetilde{\rho}^{-1}(\mathbb{Z}_2) = \mathcal{S}_1 \times p_2(K)$. However, this gives a contradiction, because $p_2(K_2)/p_2(K) \simeq \mathbb{Z}_2$. Therefore, we have $\sigma_1(\mathcal{S}_1) = \{1\}$.

Thus, we have $K = \mathcal{S}_1 \times p_2(K)$ by Lemma 6.1 (2). Because $K_2/K \simeq S^0 \simeq \mathbb{Z}_2 \simeq p_2(K_2)/p_2(K)$ and $K \subset K_2 \subset \mathcal{S}_1 \times p_2(K_2)$, the subgroup $K_2 \subset N(K;G)$ in the case (2)-(a)-(ii) is as follows:

$$K_2 = \mathcal{S}_1 \times p_2(K_2),$$

where $p_2(K_2)$ is the group in Lemma 6.3 with $m = 2$. Hence, with a method similar to that demonstrated in the case (2)-(a)-(i) (Section 6.1), we can easily check the following manifold corresponds with $M = X_1 \cup X_2$ up to essential isomorphism:

$$M = \prod_{j=1}^{b} S^{2m_j} \times_{\mathcal{A}\times\mathbb{Z}_2} \left(\prod_{i=1}^{a} S^{2l_i+1} \times_{T^a} S(\mathbb{C}_{\mathfrak{c}}^{k_1} \oplus \mathbb{R}) \right),$$

where the $\mathcal{A}(=\mathcal{A}_1)$-quotient is defined by the following actions:

- on $\prod_{j=1}^{b} S^{2m_j}$ as the subgroup of $\prod_{j=1}^{b} \mathbb{Z}_2$;
- on $\mathbb{C}_{\mathfrak{c}}^{k_1}$ trivially (this corresponds with that $\sigma_1(\mathcal{S}_1) = \{1\}$);
- on $\mathbb{R}$ trivially

and the $\mathbb{Z}_2$-quotient is defined by the following actions:

- on $\prod_{j=1}^{b} S^{2m_j}$ trivially (this corresponds with $p_1(K_2) = \mathcal{S}_1$);
- on $\mathbb{C}_{\mathfrak{c}}^{k_1}$ by a non-trivial representation $\sigma_2 : \mathbb{Z}_2 \to \{\pm 1\}$ (this corresponds with $p_2(K_2)/p_2(K) \simeq \mathbb{Z}_2$);
- on $\mathbb{R}$ by the natural representation.

Remark that M is also G-equivariantly diffeomorphic to the following manifold (where $G = \prod_{j=1}^{b} SO(2m_j+1) \times \prod_{i=1}^{a} SU(l_i+1) \times U(k_1)$):

$$\prod_{j=1}^{b} S^{2m_j}/\mathcal{A} \times \left(\prod_{i=1}^{a} S^{2l_i+1} \times_{T^a} P(\mathbb{C}_{\mathfrak{c}}^{k_1} \oplus \mathbb{R}) \right),$$

where $P(\mathbb{C}_{\mathfrak{c}}^{k_1} \oplus \mathbb{R})$ is the real projective space.

6.3. The case (2)-(a)-(iii). Suppose the case (2)-(a)-(iii) occurs, that is,

$$p_1(K_2) = p_1(K) = \mathcal{S}_1 \quad \text{and}$$
$$p_2(K_2) = p_2(K).$$

For this case (iii), the following lemma holds.

LEMMA 6.4. *Suppose that the case (iii) occurs. Then* $\sigma_1(\mathcal{S}_1) = \{\pm 1\}$, *and* $p_2(K_2) \subset p_2(N(K;G))$ *is*

$$p_2(K_2) = p_2(K) = \left\{ \left((t_1, \cdots, t_a), \begin{pmatrix} X & 0 \\ 0 & x \end{pmatrix}, t \right) \;\middle|\; t_1^{\alpha_1} \cdots t_a^{\alpha_a} x t^{\alpha} = \pm 1 \right\}.$$

Furthermore, we have $K_2 = \mathcal{S}_1 \times p_2(K)$.

PROOF. If $\sigma_1(\mathcal{S}_1) = \{1\}$, then we have $K = \mathcal{S}_1 \times p_2(K)$ by Lemma 6.1 (2). It follows that $K = p_1(K_2) \times p_2(K_2) \supset K_2$ by the assumption of the case (iii). This gives a contradiction to $K_2/K \simeq S^0$. Hence, we have $\sigma_1(\mathcal{S}_1) = \{\pm 1\}$. Moreover, we have that

$$p_2(K_2) = p_2(K) = \left\{ \left((t_1, \cdots, t_a), \begin{pmatrix} X & 0 \\ 0 & x \end{pmatrix}, t \right) \;\middle|\; t_1^{\alpha_1} \cdots t_a^{\alpha_a} x t^{\alpha} = \pm 1 \right\}$$

by Lemma 6.1 (3).

By the definition of p_1, p_2 and the assumptions of the case (iii), we have the following relation:

$$K \subset K_2 \subset \mathcal{S}_1 \times p_2(K).$$

By the definition of K, we can easily check that $(\mathcal{S}_1 \times p_2(K))/K \simeq \mathbb{Z}_2$. Therefore, we have $K_2 = \mathcal{S}_1 \times p_2(K)$. □

By using Lemma 6.4 and a method similar to that demonstrated in the previous cases, we can easily check that the following manifold corresponds with $M = X_1 \cup X_2$ up to essential isomorphism:

$$M = \prod_{j=1}^{b} S^{2m_j} \times_{\mathcal{A} \times \mathbb{Z}_2} \left(\prod_{i=1}^{a} S^{2l_i+1} \times_{T^a} S(\mathbb{C}_{\mathfrak{c}}^{k_1} \oplus \mathbb{R}) \right),$$

where the $\mathcal{A}(= \mathcal{A}_1)$-quotient is defined by the following actions:

- on $\prod_{j=1}^{b} S^{2m_j}$ as the subgroup of $\prod_{j=1}^{b} \mathbb{Z}_2$;
- on $\mathbb{C}_{\mathfrak{c}}^{k_1}$ through the non-trivial representation $\sigma_1 : \mathcal{A} \to \mathbb{Z}_2$ (this corresponds with $\sigma_1(\mathcal{S}_1) = \{\pm 1\}$);
- on $\mathbb{R}$ trivially,

and the $\mathbb{Z}_2$-quotient is defined by the following actions:

- on $\prod_{j=1}^{b} S^{2m_j}$ trivially (this corresponds with $p_1(K_2) = \mathcal{S}_1$);

- on $\mathbb{C}_{\mathfrak{c}}^{k_1}$ by a non-trivial representation $\sigma_2 : \mathbb{Z}_2 \to \{\pm 1\}$ (this is known by using $\sigma_1(\mathcal{S}_1) = \{\pm 1\}$ and $p_2(K_2) = p_2(K)$);
- on $\mathbb{R}$ by the natural representation.

Remark that M is also G-equivariantly diffeomorphic to the following manifold as well as the case (2)-(a)-(ii):

$$\prod_{j=1}^{b} S^{2m_j} \times_{\mathcal{A}} \left(\prod_{i=1}^{a} S^{2l_i+1} \times_{T^a} P(\mathbb{C}_{\mathfrak{c}}^{k_1} \oplus \mathbb{R}) \right).$$

6.4. The case (2)-(a)-(iv). Suppose the case (2)-(a)-(iv) occurs, that is,

$$p_1(K_2)/p_1(K) = \mathcal{S}_2/\mathcal{S}_1 \simeq \mathbb{Z}_2 \quad \text{and}$$
$$p_2(K_2)/p_2(K) \simeq \mathbb{Z}_2.$$

First, we remark that Lemma 6.3 can be used in this case because of $p_2(K_2)/p_2(K) \simeq \mathbb{Z}_2$.

Assume $\sigma_1(\mathcal{S}_1) = \{\pm 1\}$. Then we have $m = 4$ for the group $p_2(K_2)$ in Lemma 6.3. Let $\widetilde{\rho} : \mathcal{S}_2 \times p_2(K_2) \to \mathbb{Z}_4$ be the following representation:

$$\widetilde{\rho}(B,\ (t_1,\ \cdots,\ t_a),\ x,\ t) = \sigma_2(B) t_1^{\alpha_1} \cdots t_a^{\alpha_a} x t^{\alpha} \in \mathbb{Z}_4,$$

where $\sigma_2 : \mathcal{S}_2 \to \mathbb{Z}_4$ is some representation such that $\sigma_2|_{\mathcal{S}_1} = \sigma_1$. Then we can easily show that $\sigma_2 : \mathcal{S}_2 \to \mathbb{Z}_2$, i.e., the image of $\mathcal{S}_2$ is in $\mathbb{Z}_2 \subset \mathbb{Z}_4$, because $\prod_{j=1}^{b} SO(2m_j) \subset \ker \sigma_1 \subset \ker \sigma_2$ and $\mathcal{S}_2 / \prod_{j=1}^{b} SO(2m_j) \subset \prod_{j=1}^{b} \mathbb{Z}_2$. Hence, we have that

$$\ker \widetilde{\rho} \subset \mathcal{S}_2 \times p_2(K). \tag{6.5}$$

Consider the restricted representation $\widetilde{\rho}|_{K_2} : K_2 \to \mathbb{Z}_4$. Then we have the following sequence:

$$K \subset \ker \widetilde{\rho}|_{K_2} = \ker \widetilde{\rho} \cap K_2 \subset K_2$$

because of the definitions of K and σ_2. Therefore, by $K_2/K \simeq S^0$, we have that $\ker \widetilde{\rho}|_{K_2} = K_2$ or K. If $\ker \widetilde{\rho}|_{K_2} = K_2$, then we have $p_2(K_2) = p_2(K)$ by (6.5). This gives a contradiction to $p_2(K_2)/p_2(K) = \mathbb{Z}_2$. Hence, we have $\ker \widetilde{\rho}|_{K_2} = K$. Then we have

$$K_2/K = K_2/\ker \widetilde{\rho}|_{K_2} \simeq \widetilde{\rho}(K_2) = \mathbb{Z}_2 = \{\pm 1\} \subset \mathbb{Z}_4.$$

It also follows that $K_2 \subset \widetilde{\rho}^{-1}(\mathbb{Z}_2) = \mathcal{S}_2 \times p_2(K)$; therefore, $p_2(K_2) = p_2(K)$. This also gives a contradiction to $p_2(K_2)/p_2(K) = \mathbb{Z}_2$. Thus, we have $\sigma_1(\mathcal{S}_1) = \{1\}$. Hence, by Lemma 6.1 (2), we have $K = \mathcal{S}_1 \times p_2(K)$.

Because $K_2/K = K_2/(\mathcal{S}_1 \times p_2(K)) \simeq \mathbb{Z}_2$ and $(\mathcal{S}_2 \times p_2(K_2))/K \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ (the assumption of (2)-(a)-(iv)), we have that

$$(\mathcal{S}_2 \times p_2(K_2))/K_2 \simeq \mathbb{Z}_2. \tag{6.6}$$

Again, we define $\widetilde{\rho} : \mathcal{S}_2 \times p_2(K_2) \to \mathbb{Z}_2 \times \mathbb{Z}_2$ as follows:

$$\widetilde{\rho}(B,\ (t_1,\ \cdots,\ t_a),\ x,\ t) = (\sigma_2(B),\ t_1^{\alpha_1} \cdots t_a^{\alpha_a} x t^{\alpha}) \in \mathbb{Z}_2 \times \mathbb{Z}_2,$$

where $\sigma_2 : \mathcal{S}_2 \to \mathbb{Z}_2$ such that $\sigma_2|_{\mathcal{S}_1} = \sigma_1$. Then we have $\ker \widetilde{\rho} = K$ by $K = \mathcal{S}_1 \times p_2(K)$ and $\sigma_1(\mathcal{S}_1) = \{1\}$. Because $K_2/K \simeq \mathbb{Z}_2$, we have that $\widetilde{\rho}(K_2) \simeq \mathbb{Z}_2$, i.e., $\widetilde{\rho}(K_2) = \mathbb{Z}_2 \times \{1\}$, $\{1\} \times \mathbb{Z}_2$ or $\Delta = \{(1,\ 1),\ (-1,\ -1)\}$. If $\widetilde{\rho}(K_2) = \mathbb{Z}_2 \times \{1\}$ or $\{1\} \times \mathbb{Z}_2$, then this gives a contradiction to that $p_2(K_2)/p_2(K) = \mathbb{Z}_2$ or $p_1(K_2) = \mathcal{S}_1$, respectively. Therefore, we have that $\widetilde{\rho}(K_2) = \Delta$. Moreover, by using (6.6), the subgroup $K_2 \subset \mathcal{S}_2 \times p_2(K_2)$ can be denoted as follows:

$$\widetilde{\rho}^{-1}(\Delta) = K_2 = \{(B,\ t_1,\ \cdots,\ t_a,\ x,\ t) \mid \sigma_2(B)t_1^{\alpha_a}\cdots t_a^{\alpha_a}xt^{\alpha} = 1\},$$

where $\sigma_2 : \mathcal{S}_2 \to \mathbb{Z}_2$ such that $\ker\sigma_2 = \mathcal{S}_1$. Hence $K_2 \subset N(K;G)$ is completely determined by $\mathcal{S}_2$ and σ_2 in the case (2)-(a)-(iv).

With a method similar to that demonstrated in the previous cases, we can easily check the following manifold corresponds with $M = X_1 \cup X_2$ up to essential isomorphism:

$$M = \prod_{j=1}^{b} S^{2m_j} \times_{\mathcal{A}\times\mathbb{Z}_2} \left(\prod_{i=1}^{a} S^{2l_i+1} \times_{T^a} S(\mathbb{C}_{\mathfrak{c}}^{k_1} \oplus \mathbb{R})\right),$$

where the $\mathcal{A}(=\mathcal{A}_1)$-quotient is defined by the following actions:

- on $\prod_{j=1}^{b} S^{2m_j}$ as the subgroup of $\prod_{j=1}^{b}\mathbb{Z}_2$;
- on $\mathbb{C}_{\mathfrak{c}}^{k_1}$ trivially (this corresponds with $\sigma_1(\mathcal{S}_1) = \{1\}$);
- on $\mathbb{R}$ trivially,

and the $\mathbb{Z}_2$-quotient is defined by the following actions:

- on $\prod_{j=1}^{b} S^{2m_j}$ by a non-trivial representation $\rho : \mathbb{Z}_2 \to \prod_{j=1}^{b}\mathbb{Z}_2$ which satisfies $\rho(\mathbb{Z}_2) \cap \mathcal{A} = \{1\}$ (this corresponds with $p_1(K_2) = \mathcal{S}_2$);
- on $\mathbb{C}_{\mathfrak{c}}^{k_1}$ by a non-trivial representation $\sigma_2 : \mathbb{Z}_2 \to \{\pm 1\}$ (this corresponds with $p_2(K_2)/p_2(K) \simeq \mathbb{Z}_2$);
- on $\mathbb{R}$ by the natural representation.

Remark that M is also G-equivariantly diffeomorphic to the following manifold as well as the previous cases:

$$\prod_{j=1}^{b} S^{2m_j}/\mathcal{A} \times_{\mathbb{Z}_2} \left(\prod_{i=1}^{a} S^{2l_i+1} \times_{T^a} S(\mathbb{C}_{\mathfrak{c}}^{k_1} \oplus \mathbb{R})\right).$$

6.5. Summary of the results from (2)-(a)-(i) to (iv). In summary, we can state the result of the case (2)-(a) as follows. Let $(M,\ G)$ be the pair in the case (2)-(a). Then $(M,\ G)$ is essentially isomorphism to the followings:

$$M = \prod_{j=1}^{b} S^{2m_j} \times_{\mathcal{A}\times\mathbb{Z}_2} \left(\prod_{i=1}^{a} S^{2l_i+1} \times_{T^a} S(\mathbb{C}_{\mathfrak{a}}^{k_1} \oplus \mathbb{R})\right),$$

$$G = \prod_{j=1}^{b} SO(2m_j+1) \times \prod_{i=1}^{a} SU(l_i+1) \times U(k_1),$$

where G acts on M naturally, and the $\mathcal{A}(=\mathcal{A}_1)$-quotient is defined by the following actions:

- on $\prod_{j=1}^{b} S^{2m_j}$ as the subgroup of $\prod_{j=1}^{b}\mathbb{Z}_2$;
- on $\mathbb{C}_{\mathfrak{c}}^{k_1}$ through the representation $\sigma_1(=\sigma_{\mathbb{C}}) : \mathcal{A} \to \mathbb{Z}_2$;
- on $\mathbb{R}$ trivially,

and the $\mathbb{Z}_2$-quotient is defined by the following actions:

- on $\prod_{j=1}^{b} S^{2m_j}$ by a representation $\rho : \mathbb{Z}_2 \to \prod_{j=1}^{b}\mathbb{Z}_2$ which satisfies $\rho(\mathbb{Z}_2) \cap \mathcal{A} = \{1\}$;
- on $\mathbb{C}_{\mathfrak{c}}^{k_1}$ by a representation $\sigma_2 : \mathbb{Z}_2 \to \{\pm 1\}$;

- on $\mathbb{R}$ by the natural representation $\kappa : \mathbb{Z}_2 \to O(1)$,

where ρ or σ_2 is always non-trivial, i.e., the case that both of two representations ρ and σ_2 are trivial does not occur. This corresponds with the first manifold in Theorem 5.1, where $\sigma_2 \oplus \kappa = \sigma_{\mathbb{C}\oplus\mathbb{R}}$.

7. The case (2)-(b)

In this section, we study the case (2)-(b). From Section 4.1.2, we have that

$$
\begin{aligned}
G &= \prod_{j=1}^{b} SO(2m_j+1) \times \prod_{i=1}^{a} SU(l_i+1) \times SO(2k_1),\\
K_1 &= \mathcal{S}_1 \times \prod_{i=1}^{a} S(U(l_i)\times U(1)) \times SO(2k_1),\\
K &= \prod_{i=1}^{a} S(U(l_i)\times U(1))\\
&\quad \times \left\{ \left(A,\ \begin{pmatrix} X & 0 \\ 0 & x \end{pmatrix} \right) \in \mathcal{S}_1 \times S(O(2k_1-1)\times O(1)) \;\middle|\; \sigma_1(A)x = 1 \right\}\\
&= \prod_{i=1}^{a} S(U(l_i)\times U(1)) \times K'.
\end{aligned}
$$

An element in $\mathcal{S}_1 \times S(O(2k_1-1)\times O(1))$ is often denoted by $(A,\ x)$. In this case, by Section 5.2, we may only analyze the inclusion of K_2 such that $K \subset K_2 \subset N(K;G)$ and $N(K;G)$ is known as (5.3).

Because $K \subset K_2 \subset N(K;G)$ and $K_2/K \simeq S^0$, we have that

$$K_2 = \prod_{i=1}^{a} S(U(l_i)\times U(1)) \times K_2',$$

where $K_2' \subset \prod_{j=1}^{b} S(O(2m_j)\times O(1)) \times S(O(2k_1-1)\times O(1))$ such that $K_2'/K' \simeq \mathbb{Z}_2$.

Let p_1 and p_2 be the following two natural projections:

$$p_1 : \prod_{j=1}^{b} S(O(2m_j)\times O(1)) \times S(O(2k_1-1)\times O(1)) \longrightarrow \prod_{j=1}^{b} S(O(2m_j)\times O(1));$$

$$p_2 : \prod_{j=1}^{b} S(O(2m_j)\times O(1)) \times S(O(2k_1-1)\times O(1)) \longrightarrow S(O(2k_1-1)\times O(1)).$$

Similarly to Lemma 6.1, we can easily prove the following lemma.

Lemma 7.1. *For p_1, p_2 and K', the following properties hold:*

(1) *the image of K' by p_1 satisfies $p_1(K') = \mathcal{S}_1$;*
(2) *if $\sigma_1(\mathcal{S}_1) = \{1\}$, then $p_2(K') = SO(2k_1-1)$ and $K' = \mathcal{S}_1 \times SO(2k_1-1)$;*
(3) *if $\sigma_1(\mathcal{S}_1) = \{\pm 1\}$, then $p_2(K') = S(O(2k_1-1)\times O(1))$ and $(\mathcal{S}_1 \times S(O(2k_1-1)\times O(1)))/K' \simeq \mathbb{Z}_2$.*

Proof. The first statement is proved by making use of the subgroup $\mathcal{R}_1$ in Section 3.5. The second and third statements are proved by the definition of K' and the first property $p_1(K') = \mathcal{S}_1$. □

Because $K_2'/K' \simeq \mathbb{Z}_2$, similarly to the case (2)-(a), one of the following four cases occurs:

(i): $p_1(K_2') = \mathcal{S}_2$, and $p_2(K_2') = p_2(K')$;
(ii): $p_1(K_2') = p_1(K') = \mathcal{S}_1$, and $p_2(K_2')/p_2(K') \simeq \mathbb{Z}_2$;
(iii): $p_1(K_2') = p_1(K') = \mathcal{S}_1$, and $p_2(K_2') = p_2(K')$;
(iv): otherwise, i.e., $p_1(K_2') = \mathcal{S}_2$, and $p_2(K_2')/p_2(K') \simeq \mathbb{Z}_2$,

where $\mathcal{S}_2$ is a subgroup of $\prod_{j=1}^{b} S(O(2m_j) \times O(1))$ such that $\mathcal{S}_2/\mathcal{S}_1 \simeq \mathbb{Z}_2$. We call the above cases *the case* (2)-(b)-(i), (2)-(b)-(ii), (2)-(b)-(iii) and (2)-(b)-(iv), respectively.

7.1. The case (2)-(b)-(i). Suppose the case (2)-(b)-(i) occurs, that is,

$$p_1(K_2') = \mathcal{S}_2 \quad \text{and} \quad p_2(K_2') = p_2(K').$$

If $\sigma_1(\mathcal{S}_1) = \{\pm 1\}$, then K' is defined as $\ker \rho$, where $\rho : \mathcal{S}_1 \times p_2(K') \to \mathbb{Z}_2$ is the following representation:

$$\rho(A, x) = \sigma_1(A)x \in \mathbb{Z}_2.$$

Consider the lift of this representation $\widetilde{\rho} : \mathcal{S}_2 \times p_2(K') \to \mathbb{Z}_2$. Similarly to the proof of Lemma 6.2, we can easily show that this lift is only determined by $\sigma_2 : \mathcal{S}_2 \to \mathbb{Z}_2$ such that $\sigma_2|_{\mathcal{S}_1} = \sigma_1$, i.e., a representation $\widetilde{\rho} : \mathcal{S}_2 \times p_2(K') \to \mathbb{Z}_2$ is denoted by

$$\widetilde{\rho}(B,\ x) = \sigma_2(B)x.$$

On the other hand, there is the following induced representation form $K_2' \subset \mathcal{S}_2 \times p_2(K')$:

$$\mathcal{S}_2 \times p_2(K') \xrightarrow{\widetilde{r}} (\mathcal{S}_2 \times p_2(K'))/K_2' \simeq \mathbb{Z}_2.$$

With a method similar to that demonstrated in Section 6.1, $\widetilde{r}$ can be identified with $\widetilde{\rho}$, i.e., the lift of ρ. Hence, there is a representation $\sigma_2 : \mathcal{S}_2 \to \mathbb{Z}_2$ such that $\widetilde{r}(B,\ x) = \sigma_2(B)x$. Because $\ker \widetilde{r} = K_2'$, we have that

$$(7.1)\qquad K_2' = \left\{ \left(B,\ \begin{pmatrix} X & 0 \\ 0 & x \end{pmatrix} \right) \in \mathcal{S}_2 \times S(O(2k_1 - 1) \times O(1)) \ \middle|\ \sigma_2(B)x = 1 \right\},$$

where $\sigma_2 : \mathcal{S}_2 \to \mathbb{Z}_2$ is a representation such that $\sigma_2|_{\mathcal{S}_1} = \sigma_1$.

If $\sigma_1(\mathcal{S}_1) = \{1\}$, then we have

$$K = \prod_{i=1}^{a} S(U(l_i) \times U(1)) \times \mathcal{S}_1 \times SO(2k_1 - 1),$$

by Lemma 7.1 (2). Because of the assumptions of the case (2)-(b)-(i), we have that

$$K_2 = \prod_{i=1}^{a} S(U(l_i) \times U(1)) \times \mathcal{S}_2 \times SO(2k_1 - 1).$$

Therefore, we can regard K_2' of this case as that with $\sigma_2(\mathcal{S}_2) = 1$ in Eq.(7.1).

Hence, we have that $M = X_1 \cup X_2$ is as follows, with the method similar to that demonstrated in Section 6.1:

$$M = \prod_{i=1}^{a} \mathbb{C}P(l_i) \times \left(\prod_{j=1}^{b} S^{2m_j} \times_{\mathcal{A} \times \mathbb{Z}_2} S(\mathbb{R}^{2k_1} \oplus \mathbb{R}) \right),$$

where the $\mathcal{A} \times \mathbb{Z}_2$-quotient is defined by the same rule as that in the case (2)-(a)-(i) (see Section 6.1 by replacing $\mathbb{C}_{\mathfrak{c}}^{k_1}$ as $\mathbb{R}^{2k_1}$), and $G = \prod_{i=1}^{a} SU(l_i+1) \times \prod_{j=1}^{b} SO(2m_j+1) \times SO(2k_1)$ acts naturally on this manifold.

7.2. The case (2)-(b)-(ii). Suppose the case (2)-(b)-(ii) occurs, that is,

$$p_1(K_2') = p_1(K') = \mathcal{S}_1 \quad \text{and} \quad p_2(K_2')/p_2(K') \simeq \mathbb{Z}_2.$$

If $\sigma_1(\mathcal{S}_1) = \{\pm 1\}$, then we have $p_2(K') = S(O(2k_1-1) \times O(1)) \supset p_2(K_2')$ by Lemma 7.1 (3). This gives a contradiction to $p_2(K_2')/p_2(K') \simeq \mathbb{Z}_2$. Therefore, we have $\sigma_1(\mathcal{S}_1) = \{1\}$. Hence, we have that $K' = \mathcal{S}_1 \times SO(2k_1-1)$ by Lemma 7.1 (2).

Because $p_2(K_2')/p_2(K') \simeq \mathbb{Z}_2$ and $p_2(K') = SO(2k_1-1)$, we have the following sequence:

$$K' = \mathcal{S}_1 \times SO(2k_1-1) \subset K_2' \subset \mathcal{S}_1 \times p_2(K_2') = \mathcal{S}_1 \times S(O(2k_1-1) \times O(1)).$$

Because $K_2'/K' \simeq \mathbb{Z}_2$, we have that $K_2' = \mathcal{S}_1 \times S(O(2k_1-1) \times O(1))$ and

$$K_2 = \prod_{i=1}^{a} S(U(l_i) \times U(1)) \times \mathcal{S}_1 \times S(O(2k_1-1) \times O(1)),$$

$$K = \prod_{i=1}^{a} S(U(l_i) \times U(1)) \times \mathcal{S}_1 \times SO(2k_1-1).$$

Hence, similarly to Section 7.1, we have that $M = X_1 \cup X_2$ is as follows:

$$M = \prod_{i=1}^{a} \mathbb{C}P(l_i) \times \left(\prod_{j=1}^{b} S^{2m_j} \times_{\mathcal{A} \times \mathbb{Z}_2} S(\mathbb{R}^{2k_1} \oplus \mathbb{R}) \right),$$

where the $\mathcal{A} \times \mathbb{Z}_2$-quotient is defined by the same rule as that in the case (2)-(a)-(ii) (see Section 6.2 by replacing $\mathbb{C}_{\mathfrak{c}}^{k_1}$ as $\mathbb{R}^{2k_1}$). This manifold is also G-equivariantly diffeomorphic to the following manifold:

$$\prod_{i=1}^{a} \mathbb{C}P(l_i) \times \prod_{j=1}^{b} S^{2m_j}/\mathcal{A} \times P(\mathbb{R}^{2k_1} \oplus \mathbb{R}),$$

where $G = \prod_{i=1}^{a} SU(l_i+1) \times \prod_{j=1}^{b} SO(2m_j+1) \times SO(2k_1)$.

7.3. The case (2)-(b)-(iii). Suppose the case (2)-(b)-(iii) occurs, that is,

$$p_1(K_2') = p_1(K') = \mathcal{S}_1 \quad \text{and} \quad p_2(K_2') = p_2(K').$$

If $\sigma_1(\mathcal{S}_1) = \{1\}$, then we have $K' = \mathcal{S}_1 \times SO(2k_1-1) = p_1(K') \times p_2(K') = p_1(K_2') \times p_2(K_2') \supset K_2'$. This gives a contradiction to $K_2'/K' \simeq \mathbb{Z}_2$. Therefore, we have $\sigma_1(\mathcal{S}_1) = \{\pm 1\}$.

Hence, we have the following sequence:

$$K' \subset K_2' \subset p_1(K_2') \times p_2(K_2') = \mathcal{S}_1 \times S(O(2k_1-1) \times O(1)).$$

By Lemma 7.1 (3) and $K_2'/K' \simeq \mathbb{Z}_2$, we also have that $K_2' = \mathcal{S}_1 \times S(O(2k_1-1) \times O(1))$ and

$$K_2 = \prod_{i=1}^{a} S(U(l_i) \times U(1)) \times \mathcal{S}_1 \times S(O(2k_1-1) \times O(1)),$$

$$K = \prod_{i=1}^{a} S(U(l_i) \times U(1)) \times \{(A,\ x) \mid \sigma_1(A)x = 1\}.$$

Hence, similarly to Section 7.1, we have that $M = X_1 \cup X_2$ is as follows:

$$M = \prod_{i=1}^{a} \mathbb{C}P(l_i) \times \left(\prod_{j=1}^{b} S^{2m_j} \times_{\mathcal{A}\times\mathbb{Z}_2} S(\mathbb{R}^{2k_1} \oplus \mathbb{R}) \right),$$

where the $\mathcal{A}\times\mathbb{Z}_2$-quotient is defined by the same rule as that in the case (2)-(a)-(iii) (see Section 6.3 by replacing $\mathbb{C}_{\mathfrak{c}}^{k_1}$ as $\mathbb{R}^{2k_1}$). This manifold is also G-equivariantly diffeomorphic to the following manifold as well as the case (2)-(b)-(ii):

$$\prod_{i=1}^{a} \mathbb{C}P(l_i) \times \prod_{j=1}^{b} S^{2m_j} \times_{\mathcal{A}} P(\mathbb{R}^{2k_1} \oplus \mathbb{R}).$$

7.4. The case (2)-(b)-(iv). Suppose the case (2)-(b)-(iv) occurs, that is,

$$p_1(K_2') = \mathcal{S}_2 \quad \text{and} \quad p_2(K_2')/p_2(K') \simeq \mathbb{Z}_2.$$

By the same reason in the case (2)-(b)-(ii) (see Section 7.2), we have $\sigma_1(\mathcal{S}_1) = \{1\}$. Hence, we have that $K' = \mathcal{S}_1 \times SO(2k_1 - 1)$ by Lemma 7.1 (2).

Because $p_2(K_2')/p_2(K') \simeq \mathbb{Z}_2$ and $p_2(K') = SO(2k_1 - 1)$, we have the following sequence:

$$K' = \mathcal{S}_1 \times SO(2k_1 - 1) \subset K_2' \subset \mathcal{S}_2 \times p_2(K_2') = \mathcal{S}_2 \times S(O(2k_1 - 1) \times O(1)).$$

Because $K_2'/K' \simeq \mathbb{Z}_2$, we also have the following inclusion map:

$$i : K_2'/K' \simeq \mathbb{Z}_2 \longrightarrow (\mathcal{S}_2 \times S(O(2k_1 - 1) \times O(1)))/K' \simeq \mathbb{Z}_2 \times \mathbb{Z}_2.$$

Therefore, there are three types of the inclusion i, i.e., $i(K_2'/K') = \mathbb{Z}_2 \times \{1\}$, $\{1\} \times \mathbb{Z}_2$ or Δ, where Δ is the diagonal subgroup in $\mathbb{Z}_2 \times \mathbb{Z}_2$. Assume $i(K_2'/K') = \mathbb{Z}_2 \times \{1\}$ or $\{1\} \times \mathbb{Z}_2$. This gives a contradiction to $p_2(K_2')/p_2(K') \simeq \mathbb{Z}_2$ or $p_1(K_2')/p_1(K') \simeq \mathbb{Z}_2$, respectively. Therefore, we have that

$$i(K_2'/K') = \Delta. \tag{7.2}$$

Let $\widetilde{\rho} : \mathcal{S}_2 \times p_2(K_2') \to \mathbb{Z}_2 \times \mathbb{Z}_2$ be a representation such that $\widetilde{\rho}(B,\ x) = (\sigma_2(B),\ x)$ for some $\sigma_2 : \mathcal{S}_2 \to \mathbb{Z}_2$ with $\sigma_2|_{\mathcal{S}_1} = \sigma_1$. By (7.2), we have that $K_2' = \widetilde{\rho}^{-1}(\Delta)$. It follows that

$$K_2' = \left\{ \left(B,\ \begin{pmatrix} X & 0 \\ 0 & x \end{pmatrix} \right) \in \mathcal{S}_2 \times S(O(2k_1 - 1) \times O(1)) \ \middle|\ \sigma_2(B)x = 1 \right\}.$$

Because $K_2 = \prod_{i=1}^{a} S(U(l_i) \times U(1)) \times K_2'$ and $K = \prod_{i=1}^{a} S(U(l_i) \times U(1)) \times \mathcal{S}_1 \times SO(2k_1 - 1)$, (similarly to Section 7.1) we have that $M = X_1 \cup X_2$ is as follows:

$$M = \prod_{i=1}^{a} \mathbb{C}P(l_i) \times \left(\prod_{j=1}^{b} S^{2m_j} \times_{\mathcal{A}\times\mathbb{Z}_2} S(\mathbb{R}^{2k_1} \oplus \mathbb{R}) \right),$$

where the $\mathcal{A}\times\mathbb{Z}_2$-quotient is defined by the same rule as that in the case (2)-(a)-(iv) (see Section 6.4 by replacing $\mathbb{C}_{\mathfrak{c}}^{k_1}$ as $\mathbb{R}^{2k_1}$).

7.5. Summary of the results from (2)-(b)-(i) to (iv). In summary, we can state the result of the case (2)-(b) as follows. Let $(M,\ G)$ be the pair in the case (2)-(b). Then $(M,\ G)$ is essentially isomorphism to the followings:

$$M = \prod_{j=1}^{b} S^{2m_j} \times_{\mathcal{A}\times\mathbb{Z}_2} \left(\prod_{i=1}^{a} \mathbb{C}P(l_i) \times S(\mathbb{R}^{2k_1} \oplus \mathbb{R}) \right),$$
$$G = \prod_{j=1}^{b} SO(2m_j+1) \times \prod_{i=1}^{a} SU(l_i+1) \times SO(2k_1),$$

where G acts on M naturally, and the $\mathcal{A}(=\mathcal{A}_1)$-quotient is defined by the following actions:

- on $\prod_{j=1}^{b} S^{2m_j}$ as the subgroup of $\prod_{j=1}^{b} \mathbb{Z}_2$;
- on $\mathbb{R}^{2k_1}$ through the representation $\sigma_1(=\sigma_{\mathbb{R}}) : \mathcal{A} \to \mathbb{Z}_2 = \{\pm I_{2k_1}\}$;
- on $\mathbb{R}$ trivially,

and the $\mathbb{Z}_2$-quotient is defined by the following actions:

- on $\prod_{j=1}^{b} S^{2m_j}$ by a representation $\rho : \mathbb{Z}_2 \to \prod_{j=1}^{b} \mathbb{Z}_2$ which satisfies $\rho(\mathbb{Z}_2) \cap \mathcal{A} = \{1\}$;
- on $\mathbb{R}^{2k_1}$ by a representation $\sigma_2 : \mathbb{Z}_2 \to \{\pm I_{2k_1}\}$;
- on $\mathbb{R}$ by the natural representation $\kappa : \mathbb{Z}_2 \to O(1)$,

where ρ or σ_2 is always non-trivial as well as the case (2)-(a). This corresponds with the second manifold in Theorem 5.1, where $\sigma_2 \oplus \kappa = \sigma_{\mathbb{R}\oplus\mathbb{R}}$.

8. Main theorem of the case (3) and preparations

From this section, we start to classify the final case, i.e., the case (3): G/K_2 is not a torus manifold but a singular orbit (see Section 2.4). The goal of this section is to state the main theorem and a preparation to classify the case (3). From this section, we assume that the orbit G/K_2 is not a torus manifold but a singular orbit. By Lemma 4.1 and the assumption of this case, we have for $k_2 \geq 2$

$$\dim G/K_2 = 2n - 2k_2 + 1, \tag{8.1}$$

and

$$K_2/K \cong S^{2k_2-2}. \tag{8.2}$$

By using (8.1) and (8.2), the slice representation of K_2 in the case (3) is

$$\sigma_2 : K_2 \longrightarrow O(2k_2 - 1). \tag{8.3}$$

In the case (3) as well as the case (2), there are the following two cases:

(3)-(a): *the case* (3)-(a), i.e., G/K_1 satisfies the case (a) (see Section 4.1.1);
(3)-(b): *the case* (3)-(b), i.e., G/K_1 satisfies the case (b) (see Section 4.1.2).

8.1. Main theorem and notations. First we state the main theorem of the case (3). Before we state it, we prepare some notations (also see Section 3.1 and 5.1). Let $\mathcal{A}$ be a subgroup of $\prod_{j=1}^{b} \mathbb{Z}_2$, where $\prod_{j=1}^{b} \mathbb{Z}_2$ is the following group: the first $(b-1)$ factors $\prod_{j=1}^{b-1} \mathbb{Z}_2$ are generated by the antipodal involutions on S^{2m_j} for $j = 1, \cdots, b-1$ and the b-th factor $\mathbb{Z}_2$ is $\{\pm I_{2k_2-1}\}$. The quotient manifold $\left(\prod_{i=1}^{a} S^{2l_i+1}\right) \times_{T^a} S(\mathbb{C}^{k_1}_{\mathfrak{c}} \oplus \mathbb{R}^{2k_2-1})$ is defined similarly as $\left(\prod_{i=1}^{a} S^{2l_i+1}\right) \times_{T^a} S(\mathbb{C}^{k}_{\mathfrak{a}} \oplus$

$\mathbb{R}$) (see Section 3.1), where $\mathbb{C}^{k_1}_{\mathfrak{c}}(\simeq \mathbb{C}^{k_1})$ is a representation space of a representation $\mathfrak{c} : T^a \to S^1$.

Now we may state the main theorem of this section.

THEOREM 8.1. *Suppose a torus manifold M has a codimension one extended G-action. If there are two singular orbits and one of them is not a torus manifold, then $(M,\ G)$ is essentially isomorphic to*

$$\left(\prod_{j=1}^{b-1} S^{2m_j} \times_{\mathcal{A}} N, \quad \prod_{j=1}^{b-1} SO(2m_j+1) \times H\right),$$

such that $(N,\ H)$ is one of the followings:

	N	H
(a)	$(\prod_{i=1}^{a} S^{2l_i+1}) \times_{T^a} S(\mathbb{C}^{k_1}_{\mathfrak{c}} \oplus \mathbb{R}^{2k_2-1})$	$\prod_{i=1}^{a} SU(l_i+1) \times U(k_1) \times SO(2k_2-1)$
(b)	$\prod_{i=1}^{a} \mathbb{C}P(l_i) \times S(\mathbb{R}^{2k_1} \oplus \mathbb{R}^{2k_2-1})$	$\prod_{i=1}^{a} SU(l_i+1) \times SO(2k_1) \times SO(2k_2-1)$

where $\mathcal{A}$ acts on $\prod_{j=1}^{b-1} S^{2m_j} \times \mathbb{R}^{2k_2-1}$ as a subgroup of $\prod_{j=1}^{b} \mathbb{Z}_2$ and on the fibre of N through the following representations:

(a): $\sigma_{\mathbb{C}} : \mathcal{A} \to \{\pm 1\} \subset S^1$ *on* $S(\mathbb{C}^{k_1}_{\mathfrak{c}} \oplus \mathbb{R}^{2k_2-1}) \cap \mathbb{C}^{k_1}_{\mathfrak{c}}$*;*
(b): $\sigma_{\mathbb{R}} : \mathcal{A} \to \{\pm I_{2k_1}\} \subset SO(2k_1)$ *on* $S(\mathbb{R}^{2k_1} \oplus \mathbb{R}^{2k_2-1}) \cap \mathbb{R}^{2k_1}$*;*

respectively, such that if $(1,\ \cdots,\ 1,\ -I_{2k_2-1}) \in \mathcal{A} \subset \prod_{j=1}^{b} \mathbb{Z}_2$ then

$$\sigma(1,\ \cdots,\ 1,\ -I_{2k_2-1}) = -1$$

for $\sigma = \sigma_{\mathbb{C}}$ and $\sigma_{\mathbb{R}}$.

Here, G-actions on M are as follows: $\prod SO(2m_j+1)$ and $\prod SU(l_i+1)$ act naturally on $\prod S^{2m_j}$ and $\prod S^{2l_i+1}$, respectively; and $U(k_1)$, $SO(2k_1)$ and $SO(2k_2-1)$ act naturally on $\mathbb{C}^{k_1}_{\mathfrak{c}}$, $\mathbb{R}^{2k_1}$ and $\mathbb{R}^{2k_2-1}$, respectively.

Note that the following facts: if $(1,\ \cdots,\ 1,\ -I_{2k_2-1}) \notin \mathcal{A}$ then $\mathcal{A}$ acts on $\prod_{j=1}^{b-1} S^{2m_j} \times \mathbb{R}^{2k_2-1}$ freely; if $(1,\ \cdots,\ 1,\ -I_{2k_2-1}) \in \mathcal{A}$ then $\mathcal{A} = \mathcal{A}' \times \{\pm I_{2k_2-1}\}$ and $\mathcal{A}'$ acts on $\prod_{j=1}^{b-1} S^{2m_j}$ freely and $\{\pm I_{2k_2-1}\}$ acts on $S(\mathbb{C}^{k_1}_{\mathfrak{c}} \oplus \mathbb{R}^{2k_2-1})$ or $S(\mathbb{R}^{2k_1} \oplus \mathbb{R}^{2k_2-1})$ freely because of the properties of σ described in Theorem 8.1. Therefore, M in Theorem 8.1 is a manifold. Moreover, there is the case that $\mathcal{A} \subset \prod_{j=1}^{b-1} \mathbb{Z}_2$; hence, we do not write manifolds in Theorem 8.1 as manifolds in Theorem 5.1, i.e., manifolds divided by $\mathcal{A} \times \mathbb{Z}_2$ where $\mathbb{Z}_2$ acts on $S(\mathbb{C}^{k_1}_{\mathfrak{c}} \oplus \mathbb{R}) \cap \mathbb{R}$ (or $S(\mathbb{R}^{2k_1} \oplus \mathbb{R}) \cap \mathbb{R}$).

In order to prove the above Theorem 8.1, we will use the following notations.

- Natural projections: $p_i : G \to SU(l_i+1)$, $p : G \to SU(k_1)$, $q : G \to T^1$ and $r_j : G \to SO(2m_j+1)$, where $i = 1,\ \cdots,\ a$ and $j = 1,\ \cdots,\ b$.
- Inclusions: $\iota : K \to K_2$ or $\iota : K^o \to K_2^o$, and $\iota_2 : K_2 \to G$ or $\iota_2 : K_2^o \to G$.

8.2. Structure of K_2. Before we start to prove Theorem 8.1, in this subsection, we will prove the following Lemma 8.2.

Let $N_1 \circ N_2$ be $(N_1 \times N_2)/F$ for some finite, normal subgroup $F \subset N_1 \times N_2$, where N_1 and N_2 are connected Lie groups. Then, the following lemma holds.

LEMMA 8.2. *For the cases (3)-(a) and (3)-(b), the pair $(K_2^o,\ K^o)$ is isomorphic to*

$$(Spin(2k_2-1) \circ K_2',\ Spin(2k_2-2) \circ K_2')$$

for some Lie group K_2' and $k_2 \geq 2$.

PROOF. Let $\widetilde{K_2^o}$ be the covering of K_2^o such that it is a product of simply connected, simple Lie groups and tori (see [**12**, Section 2.3]). Since a connected component K_2^o acts on $S^{2k_2-2} \cong K_2/K$ transitively through σ_2 (see (8.3)), there is a factor H in the product group $\widetilde{K_2^o}$ (i.e., $\widetilde{K_2^o} = H \times \widetilde{K_2'}$ for some product group $\widetilde{K_2'}$) such that $H = Spin(2k_2 - 1)$ or $H = G_2$ for $k_2 = 4$ by the classification result of transitive actions on even dimensional spheres (see [**13**, Theorem 5.1, 5.2]), where here G_2 is the exceptional Lie group. Therefore, there is a subgroup K_2' such that $(K_2^o,\ K^o)$ can be denoted as follows:

$$(Spin(2k_2 - 1) \circ K_2',\ Spin(2k_2 - 2) \circ K_2');$$
$$(G_2 \circ K_2',\ SU(3) \circ K_2').$$

In order to prove this lemma, we assume

$$(K_2^o,\ K^o) = (G_2 \circ K_2',\ SU(3) \circ K_2'). \tag{8.4}$$

We will prove that this case does not occur.

Taking some covering of K^o in (8.4), we can put

$$\widetilde{K^o} = SU(3) \times \widetilde{K_2'}, \tag{8.5}$$

where $\widetilde{K_2'}$ is a product of simply connected, simple Lie groups and tori. On the other hand, taking a covering of K^o in the cases (3)-(a) and (3)-(b) (see Section 4.1.1, 4.1.2 or (9.1) in Section 9, (10.1) in Section 10), we can put

$$\widetilde{K^o} = \prod_{j=1}^{b} Spin(2m_j) \times \prod_{i=1}^{a} SU(l_i) \times L \times T, \tag{8.6}$$

where T is a torus and

- $L = SU(k_1 - 1)$ for the case (3)-(a),
- $L = Spin(2k_1 - 1)$ for the case (3)-(b).

Because $\dim Spin(x) \neq \dim SU(3)$ for all $x \in \mathbb{N}$, there are the following two cases by (8.5) and (8.6):

(1) $l_a = 3$ and $\widetilde{K_2'} = \prod_{j=1}^{b} Spin(2m_j) \times \prod_{i=1}^{a-1} SU(l_i) \times L \times T$, in the case (3)-(a) or (3)-(b);
(2) $k_1 = 4$ and $\widetilde{K_2'} = \prod_{j=1}^{b} Spin(2m_j) \times \prod_{i=1}^{a} SU(l_i) \times T$, in the case (3)-(a).

Suppose $l_a = 3$ and $\widetilde{K_2'} = \prod_{j=1}^{b} Spin(2m_j) \times \prod_{i=1}^{a-1} SU(l_i) \times L \times T$. Let $p_a : G \to SU(l_a + 1)$ be the natural projection (see notations in Section 8.1). Then we have that

$$p_a(SU(3) \circ \{e\}) \subset p_a(G_2 \circ \{e\}) \subset p_a(K_2^o) \subset p_a(G) = SU(4)$$

because $SU(3) \circ \{e\} \subset G_2 \circ \{e\} \subset K_2^o \subset G$, where $\{e\} \subset K_2'$ is the identity element in K_2'. Since $SU(3) \circ \{e\} \subset K^o$ and $p_a(K^o) = S(U(3) \times U(1))$ by Section 4.1, we also have that

$$p_a(SU(3) \circ \{e\}) = SU(3).$$

Therefore, $p_a(G_2 \circ \{e\})$ is a non-trivial subgroup in $SU(4)$. Since the restricted representation $p_a|_{G_2 \circ \{e\}}$ is a non-trivial representation and G_2 is a simple Lie group, we also have that

$$\dim p_a(G_2 \circ \{e\}) = \dim G_2 = 14.$$

It follows that there is a subgroup $H \subset SU(4)$ such that $\dim H = 14$ and $SU(4)/H \cong S^1$, because $SU(4)$ is compact and $\dim SU(4) = 15$. However, this gives a contradiction because $SU(4)$-action on S^1 is trivial (see [**13**, Theorem 5.2, 5.3]).

Suppose $k_1 = 4$ and $\widetilde{K_2'} = \prod_{j=1}^b Spin(2m_j) \times \prod_{i=1}^a SU(l_i) \times T$. In this case, the above argument for $l_a = 3$ can also work for the natural projection $p : G \to SU(k_1)$. It follows that the case $(K_2^o,\ K^o) = (G_2 \circ K_2',\ SU(3) \circ K_2')$ does not occur. Hence, we have Lemma 8.2. $\square$

In the next two sections, we study the cases (3)-(a) and (3)-(b).

9. The case (3)-(a)

In this section, we study the case (3)-(a). From Section 4.1, we have

$$G = \prod_{j=1}^{b} SO(2m_j+1) \times \prod_{i=1}^{a} SU(l_i+1) \times SU(k_1) \times T^1,$$

$$K_1 = \mathcal{S}_1 \times \prod_{i=1}^{a} S(U(l_i) \times U(1)) \times SU(k_1) \times T^1,$$

$$K = \left\{ \left(A,\ (t_1,\ \cdots,\ t_a),\ \begin{pmatrix} X & 0 \\ 0 & x \end{pmatrix},\ t \right) \,\middle|\, \sigma_1(A) t_1^{\alpha_1} \cdots t_a^{\alpha_a} x t^{\alpha} = 1 \right\}. \tag{9.1}$$

where $A \in \mathcal{S}_1 \subset \prod_{j=1}^b S(O(2m_j) \times O(1))$, $(t_1,\ \cdots,\ t_a) \in \prod_{i=1}^a S(U(l_i) \times U(1))$, $t \in T^1$ and $X \in U(k_1 - 1)$ such that $x \det X = 1$. If $k_1 = 1$, then we can take $\alpha = 1$. Moreover, we have that the finite covering of K^o is as follows by (9.1):

$$\widetilde{K^o} = \prod_{j=1}^{b} Spin(2m_j) \times \prod_{i=1}^{a} SU(l_i) \times T^a \times SU(k_1 - 1) \times T^1 \quad \text{if } k_1 \geq 2;$$

$$\widetilde{K^o} = \prod_{j=1}^{b} Spin(2m_j) \times \prod_{i=1}^{a} SU(l_i) \times T^a \quad \text{if } k_1 = 1.$$

Because of Lemma 8.2, we also have

$$K^o = Spin(2k_2 - 2) \circ K_2' \quad \text{and} \quad \widetilde{K^o} = Spin(2k_2 - 2) \times \widetilde{K_2'}, \tag{9.2}$$

$$K_2^o = Spin(2k_2 - 1) \circ K_2' \quad \text{and} \quad \widetilde{K_2^o} = Spin(2k_2 - 1) \times \widetilde{K_2'}. \tag{9.3}$$

In order to classify the case (3)-(a), we will divide this case into the following two cases:

- $k_2 \geq 3$ (we will discuss in Section 9.1);
- $k_2 = 2$ (we will discuss in Section 9.2).

9.1. The case $k_2 \geq 3$. Assume $k_2 \geq 3$. Comparing coverings $\widetilde{K^o}$ of the above K^o's in (9.1) and (9.2), and using the fact that $Spin(4) \simeq SU(2) \times SU(2)$ and $Spin(6) \simeq SU(4)$, there are the following five cases:

(i): $Spin(2k_2 - 2) = Spin(2m_b)$, and $k_2 = m_b + 1 \geq 3$;
(ii): $Spin(2k_2 - 2) = SU(l_{a-1}) \times SU(l_a)$, and $k_2 = 3$, $l_{a-1} = l_a = 2$;
(iii): $Spin(2k_2 - 2) = SU(l_a) \times SU(k_1 - 1)$, and $k_2 = 3 = k_1$, $l_a = 2$;
(iv): $Spin(2k_2 - 2) = SU(l_a)$ and $k_2 = 4$, $l_a = 4$;
(v): $Spin(2k_2 - 2) = SU(k_1 - 1)$ and $k_2 = 4$, $k_1 = 5$,

First we prove the following lemma.

LEMMA 9.1. *In the above cases, the cases from (ii) to (v) do not occur.*

PROOF. If the case (ii) occurs, then $Spin(2k_2-2) = SU(l_{a-1}) \times SU(l_a)$ and $k_2 = 3$, $l_{a-1} = l_a = 2$. Let $\varphi : \widetilde{K_2^o} \to K_2^o$ be the finite covering. By using (9.1), (9.2) and (9.3), we have

$$\begin{aligned} K_2' &= \prod_{j=1}^{b} SO(2m_j) \times \left\{ \left(\left(\begin{pmatrix} A_1 & 0 \\ 0 & t_1 \end{pmatrix}, \cdots, \begin{pmatrix} A_{a-2} & 0 \\ 0 & t_{a-2} \end{pmatrix}, \right.\right.\right. \\ &\left.\left.\left. \begin{pmatrix} r_{a-1}I_2 & 0 \\ 0 & t_{a-1} \end{pmatrix}, \begin{pmatrix} r_aI_2 & 0 \\ 0 & t_a \end{pmatrix}, \begin{pmatrix} X & 0 \\ 0 & x \end{pmatrix}, t \right) \right| t_1^{\alpha_1} \cdots t_a^{\alpha_a} x t^{\alpha} = 1 \right\} \\ &= \varphi(\widetilde{K_2''} \times T^2), \end{aligned}$$

where $r_{a-1}^{-2} = t_{a-1}$, $r_a^{-2} = t_a$, $\widetilde{K_2''}$ is the product of factors in $\widetilde{K^o}$ except $Spin(2k_2-2)$ and $T^2 (\subset T^a)$. By (9.2), (9.3), [**12**, Section 3.1] (i.e., for the factor $H \subset \widetilde{K^o}$ in $H' \subset \widetilde{K_2^o}$ such that $H \subset H'$, if $\widetilde{K^o}$ and $\widetilde{K_2^o}$ are same rank then H and H' are same rank) and the assumption of the case (ii), we have the following commutative diagram:

$$\begin{array}{ccc} (T^2 \times SU(2) \times SU(2)) \times \widetilde{K_2''} & \xrightarrow{\widetilde{\iota}} & (T^2 \times Spin(5)) \times \widetilde{K_2''} \\ \varphi \downarrow & & \downarrow \varphi \\ K^o & \xrightarrow{\iota} & K_2^o, \end{array}$$

where $\widetilde{\iota}$ is an inclusion map. By using the above diagram and the definition of K and G, we have the following sequence (also see Section 8.1 about the definitions of notations):

$$\begin{aligned} & S(U(2) \times U(1)) \times S(U(2) \times U(1)) \\ = \ & (p_{a-1} \times p_a) \circ \iota_2 \circ \iota \circ \varphi(T^2 \times SU(2) \times SU(2)) \\ = \ & (p_{a-1} \times p_a) \circ \iota_2 \circ \varphi \circ \widetilde{\iota}(T^2 \times SU(2) \times SU(2)) \\ \subset \ & (p_{a-1} \times p_a) \circ \iota_2 \circ \varphi(T^2 \times Spin(5)) \\ \subset \ & p_{a-1} \times p_a(G) \\ = \ & SU(3) \times SU(3). \end{aligned}$$

This sequence implies that there is a non-trivial representation from $T^2 \times Spin(5)$ to $SU(3) \times SU(3)$. Since $Spin(5)$ is a simple Lie group and rank $(T^2 \times Spin(5)) =$ rank $(SU(3) \times SU(3))$, there is some subgroup $H \subset SU(3)$ such that $Spin(5) \approx H$ (because of [**12**, Section 3.1]), where $Spin(5) \approx H$ means that $Spin(5)$ and H have the same Lie algebra. This gives a contradiction, because $\dim Spin(5) = \dim H = 10 > 8 = \dim SU(3)$. Hence, the case (ii) does not occur.

With an argument similar to the above for the case (ii), we can also prove that the cases (iii) does not occur.

If the case (iv) occurs, then $Spin(2k_2-2) = SU(l_a)$ and $k_2 = 4$, $l_a = 4$. With a method similar to that demonstrated in the proof of the case (ii), we have the following commutative diagram:

$$\begin{array}{ccc} (T^1 \times SU(4)) \times \widetilde{K_2''} & \xrightarrow{\widetilde{\iota}} & (T^1 \times Spin(7)) \times \widetilde{K_2''} \\ \varphi \downarrow & & \downarrow \varphi \\ K^o & \xrightarrow{\iota} & K_2^o. \end{array}$$

We have the following sequence by using the above diagram and the definitions of K and G:

$$\begin{aligned} & S(U(4)\times U(1)) \\ = & \ p_a\circ\iota_2\circ\iota\circ\varphi(T^1\times SU(4)) \\ = & \ p_a\circ\iota_2\circ\varphi\circ\widetilde{\iota}(T^1\times SU(4)) \\ \subset & \ p_a\circ\iota_2\circ\varphi(T^1\times Spin(7)) \\ = & \ H \\ \subset & \ p_a(G)=SU(5). \end{aligned}$$

Because $S(U(4)\times U(1))$, $SU(5)$ and $T^1\times Spin(7)$ are the same rank Lie groups and $Spin(7)$ is a simple Lie group, we have $\dim H=\dim(T^1\times Spin(7))=22$. Then we see $SU(5)/H$ is a 2-dimensional manifold by $\dim SU(5)=24$. Moreover, $H=p_a\circ\iota_2\circ\varphi(T^1\times Spin(7))$ is connected. Therefore, we have $SU(5)/H$ is a simply connected, compact manifold, because of the homotopy exact sequence of $H\to SU(5)\to SU(5)/H$. Hence, $SU(5)/H\cong S^2$. However, the $SU(5)$-action on S^2 must be trivial (see [**13**, Theorem 5.2]). This gives a contradiction. Hence, the case (iv) does not occur.

With a method similar to that demonstrated in the above for the case (iv), we can also prove that the case (v) does not occur. □

Because of the above Lemma 9.1, we have that

(9.4) $$Spin(2k_2-2)=Spin(2m_b) \quad\text{and}\quad k_2=m_b+1\geq 3.$$

Now we set

$$\left\{\left((t_1,\ \cdots,\ t_a),\ \begin{pmatrix} X & 0\\ 0 & x\end{pmatrix},\ t\right)\ \middle|\ t_1^{\alpha_1}\cdots t_a^{\alpha_a}xt^{\alpha}=1\right\}=P(\alpha_1,\ \cdots,\ \alpha_a,\ x,\ \alpha).$$

If $k_1=1$, then $x=1$ and we can assume $\alpha=1$ up to essential isomorphism. Therefore, $P(\alpha_1,\ \cdots,\ \alpha_a,\ x,\ \alpha)$ is connected. Then, the following relation holds by using (9.1), (9.2) and (9.4):

$$\begin{aligned} K^o &= Spin(2k_2-2)\circ K_2'=Spin(2m_b)\circ K_2' \\ &= \prod_{j=1}^{b} SO(2m_j)\times P(\alpha_1,\ \cdots,\ \alpha_a,\ x,\ \alpha). \end{aligned}$$

Therefore, we can put $K^o=SO(2m_b)\times K_2'=SO(2k_2-2)\times K_2'$ and

$$\begin{aligned} K_2' &= \prod_{j=1}^{b-1} SO(2m_j)\times P(\alpha_1,\ \cdots,\ \alpha_a,\ x,\ \alpha) \\ &\subset \prod_{j=1}^{b-1} SO(2m_j+1)\times\prod_{i=1}^{a} SU(l_i+1)\times SU(k_1)\times T^1 \\ &= G/SO(2m_b+1)=G/SO(2k_2-1). \end{aligned}$$

By using $K_2^o=Spin(2k_2-1)\circ K_2'$ (by (9.3)) and the above K^o, we have the following covering map φ:

$$\begin{array}{ccc} Spin(2k_2-2)\times K_2' & \longrightarrow & Spin(2k_2-1)\times K_2' \\ \varphi\downarrow & & \downarrow\varphi \\ SO(2k_2-2)\times K_2' & \longrightarrow & K_2^o \end{array}$$

where the top and bottom maps are inclusions. Because the restricted representation $\varphi|_{K_2'}$ is the identity representation, there is the K_2' factor in K_2^o. Therefore, we have that

$$\begin{aligned} K_2^o &= SO(2k_2-1)\times K_2' \\ &= \prod_{j=1}^{b-1} SO(2m_j)\times SO(2k_2-1)\times P(\alpha_1,\ \cdots,\ \alpha_a,\ x,\ \alpha), \end{aligned}$$

and there is some inclusion $SO(2k_2-1)\to K_2$ whose image is the factor $SO(2m_b+1)\subset G$. Because $K_2^o\subset K_2\subset G$, we can put K_2 is as follows:

$$\begin{aligned} & K_2 = SO(2k_2-1)\times K_2'' \\ \subset\ & G = SO(2k_2-1)\times\left(\prod_{j=1}^{b-1} SO(2m_j+1)\times\prod_{i=1}^{a} SU(l_i+1)\times SU(k_1)\times T^1\right), \end{aligned}$$

where $2m_b+1=2k_2-1$ and K_2'' is a subgroup of $\prod_{j=1}^{b-1} SO(2m_j+1)\times\prod_{i=1}^{a} SU(l_i+1)\times SU(k_1)\times T^1$ whose connected component is K_2'. By using the argument of Section 8.2, the $SO(2k_2-1)$-factor in K_2 acts transitively on $K_2/K\cong S^{2k_2-2}$. Therefore, for the natural projection $\Phi: K_2=SO(2k_2-1)\times K_2''\to K_2''$, we have the following relation (see [**11**, Lemma 8.0.2]):

$$\Phi(K_2)=K_2''=\Phi(K).$$

Hence, we have

$$K_2=SO(2k_2-1)\times K_2''=SO(2k_2-1)\times\Phi(K)\subset G.$$

It follows that the inclusion $K_2\subset G$ is completely determined by K (more precisely the projection $\Phi(K)$).

Next, we consider the slice representation $\sigma_2: K_2\to O(2k_2-1)$. Since the $SO(2k_2-1)$-factor in K_2 acts transitively on $K_2/K\cong S^{2k_2-2}$, the restricted representation $\sigma_2|_{SO(2k_2-1)}$ is the natural isomorphism to $SO(2k_2-1)\subset O(2k_2-1)$. Hence, we have that

$$\sigma_2(K_2'')\subset Z(SO(2k_2-1))=\{\pm I_{2k_2-1}\}\subset O(2k_2-1).$$

Moreover, by (9.1), we have the following formula for K:

$$\begin{aligned} K &= \sigma_2^{-1}(O(2k_2-2)) \\ &= \left\{\left(A,\ (t_1,\ \cdots,\ t_a),\ \begin{pmatrix} X & 0 \\ 0 & x\end{pmatrix},\ t\right)\ \middle|\ \sigma_1(A)t_1^{\alpha_1}\cdots t_a^{\alpha_a}xt^{\alpha}=1\right\} \\ &= \left\{\left(\begin{pmatrix} A_b & 0 \\ 0 & a_b\end{pmatrix}, Y\right)\ \middle|\ a_b=\det A_b^{-1}=\det\sigma_2(Y)\right\}. \\ &\subset S(O(2k_2-2)\times O(1))\times K_2''. \end{aligned}\tag{9.5}$$

Therefore, we can easily show that the following lemma by using (9.5).

Lemma 9.2. *The following two statements are equivalent:*

(1): $\sigma_2(K_2'')=\{I_{2k_2-1}\}$ *(resp.* $\sigma_2(K_2'')=\{\pm I_{2k_2-1}\}$*);*

(2): $K=SO(2k_2-2)\times K_2''$ *(resp.* $r_b(K)=S(O(2m_b)\times O(1))$*).*

Moreover, the following statement holds:

(3): *if* $\sigma_2(K_2'')=\{\pm I_{2k_2-1}\}$*, then we have* $K\neq S(O(2m_b)\times O(1))\times K_2''$*.*

It follows from (9.5) and Lemma 9.2 (1), (2) that the slice representation $\sigma_2 : K_2 \to O(2k_2-2)$ is also completely determined by K. Therefore, the tubular neighborhood $X_2 = G \times_{K_2} D^{2k_2-1}$ of G/K_2 is completely determined by K, and equivariantly diffeomorphic to the following manifold:

$$\prod_{i=1}^{a} S^{2l_i+1} \times_{T^a} \left(\left(\prod_{j=1}^{b-1} S^{2m_j} \times D(\mathbb{R}^{2k_2-1}) \right) \times_{\mathcal{A}} S(\mathbb{C}_{\mathfrak{c}}^{k_1}) \right),$$

where T^a-quotient is defined similarly to that in the previous cases (e.g. the case (2)-(a)), $\mathcal{A} \simeq \mathcal{S}_1/\mathcal{S}_1^o$-quotient is defined by the following actions:

- on $\prod_{j=1}^{b-1} S^{2m_j} \times D(\mathbb{R}^{2k_2-1})$ as the subgroup $\mathcal{A} \subset \prod_{j=1}^{b-1} \mathbb{Z}_2 \times \{\pm I_{2k_2-1}\}$;
- on $S(\mathbb{C}_{\mathfrak{c}}^{k_1}) \subset \mathbb{C}_{\mathfrak{c}}^{k_1}$ by the representation $\sigma_{\mathbb{C}} : \mathcal{A} \to \{\pm 1\}$ (induced by σ_1).

Moreover, by using (9.5) and Lemma 9.2 (3), if $(1, \cdots, 1, -I_{2k_2-1}) \in \mathcal{A} \subset \prod_{j=1}^{b-1} \mathbb{Z}_2 \times \{\pm I_{2k_2-1}\}$, then $\sigma_{\mathbb{C}}(1, \cdots, 1, -I_{2k_2-1}) = -1$ (because if not so the principal isotropy subgroup is $K = S(O(2m_b) \times O(1)) \times K_2''$). It follows that $\mathcal{A}$ acts on $\prod_{j=1}^{b-1} S^{2m_j} \times D(\mathbb{R}^{2k_2-1}) \times S(\mathbb{C}_{\mathfrak{c}}^{k_1})$ freely; therefore, X_2 is a manifold.

By using Remark 4.2, we can easily check that the pair (M, G) of the case (3)-(a) and $k_2 \geq 3$ is as follows (up to essential isomorphism):

$$M = \prod_{i=1}^{a} S^{2l_i+1} \times_{T^a} \left(\prod_{j=1}^{b-1} S^{2m_j} \times_{\mathcal{A}} S(\mathbb{R}^{2k_2-1} \oplus \mathbb{C}_{\mathfrak{c}}^{k_1}) \right);$$

$$G = \prod_{i=1}^{a} SU(l_i+1) \times \prod_{j=1}^{b-1} SO(2m_j+1) \times SO(2k_2-1) \times U(k_1),$$

where $\mathcal{A}$ acts on $\prod_{j=1}^{b-1} S^{2m_j} \times S(\mathbb{R}^{2k_2-1} \oplus \mathbb{C}_{\mathfrak{c}}^{k_1})$ as follows: on $\mathbb{C}_{\mathfrak{c}}^{k_1}$ by $\sigma_{\mathbb{C}} : \mathcal{A} \to \{\pm 1\}$; on $\prod_{j=1}^{b-1} S^{2m_j} \times \mathbb{R}^{2k_2-1}$ as the subgroup $\prod_{j=1}^{b-1} \mathbb{Z}_2 \times \{\pm I_{2k_2-1}\}$ such that if $(1, \cdots, 1, -I_{2k_2-1}) \in \mathcal{A}$ then $\sigma_{\mathbb{C}}(1, \cdots, 1, -I_{2k_2-1}) = -1$.

This corresponds with the first manifold in Theorem 8.1 for $k_2 \geq 3$.

9.2. The case $k_2 = 2$. Assume $k_2 = 2$. By (9.1) and (9.2), the covering of K^o is as follows:

$$\begin{aligned} \widetilde{K^o} &= Spin(2k_2-2) \times \widetilde{K_2'} \\ &= \prod_{j=1}^{b} Spin(2m_j) \times \prod_{i=1}^{a} SU(l_i) \times T^a \times SU(k_1-1) \times T^1 \quad \text{for } k_1 \geq 2 \quad \text{or} \\ &= \prod_{j=1}^{b} Spin(2m_j) \times \prod_{i=1}^{a} SU(l_i) \times T^a \quad \text{for } k_1 = 1, \end{aligned}$$

where $l_i \geq 1$ for all $i = 1, \cdots, a$. Comparing the above coverings of K^o's, there are the following three cases:

- **(i):** $Spin(2k_2-2) = Spin(2m_b)$, and $k_2 = m_b + 1 = 2$;
- **(ii):** $Spin(2k_2-2) = T_a$, where T_a is the a-th factor of $T^a = T_1 \times \cdots \times T_a$ ($T_i \simeq T^1$);
- **(iii):** $Spin(2k_2-2) = T^1$, and $k_1 \geq 2$.

Remark 9.3. The above case (i) is the same as the case (i) in Section 9.1. By the same argument of Section 9.1, we have that $(M,\ G)$ for the above case (i) is as follows ($k_2 = 2$):

$$M = \prod_{i=1}^{a} S^{2l_i+1} \times_{T^a} \left(\prod_{j=1}^{b-1} S^{2m_j} \times_{\mathcal{A}} S(\mathbb{R}^{2k_2-1} \oplus \mathbb{C}_{\mathfrak{c}}^{k_1}) \right);$$

$$G = \prod_{i=1}^{a} SU(l_i+1) \times \prod_{j=1}^{b-1} SO(2m_j+1) \times SO(2k_2-1) \times U(k_1),$$

where T^a and $\mathcal{A}$ quotients are similarly defined as that of M in Section 9.1. This corresponds with the first manifold in Theorem 8.1 for $k_2 = 2$. Hence, in this section, we may only discuss with the other cases: the case (ii) and (iii).

First we prove the following lemma.

Lemma 9.4. *In the above cases, the case (iii) does not occur.*

Proof. By the definition of the case (iii),

$$\widetilde{K_2'} = \prod_{j=1}^{b} Spin(2m_j) \times \prod_{i=1}^{a} SU(l_i) \times T^a \times SU(k_1-1).$$

By (9.2), (9.3), we can put $\widetilde{K_2^o} = Spin(2k_2-1) \times \widetilde{K_2'}$. Therefore, with a method similar to that demonstrated in the proof of Lemma 9.1, there is the following commutative diagram ($k_2 = 2$):

$$\begin{array}{ccc} Spin(2k_2-2) \times \widetilde{K_2'} & \xrightarrow{\widetilde{\iota}} & Spin(2k_2-1) \times \widetilde{K_2'} \\ \varphi \downarrow & & \downarrow \varphi \\ K^o & \xrightarrow{\iota} & K_2^o, \end{array}$$

where φ is the finite covering, and $\widetilde{\iota}$ and ι are inclusion maps. Hence, the following sequence holds by the commutativity of the above diagram and the definition of G (also see notations in Section 8.1):

$$\begin{array}{rcl} q \circ \iota_2 \circ \iota \circ \varphi(Spin(2)) & = & q \circ \iota_2 \circ \varphi \circ \widetilde{\iota}(Spin(2)) \\ & \subset & q \circ \iota_2 \circ \varphi(Spin(3)) \\ & \subset & q(G) = T^1, \end{array}$$

On the other hand, by (9.1), we have

$$q \circ \iota_2 \circ \iota \circ \varphi(Spin(2)) = T^1 = q(G) = q \circ \iota_2 \circ \iota(K^o).$$

Consequently, we have $q \circ \iota_2 \circ \phi(Spin(3)) = T^1$. However, this gives a contradiction, because there is no non-trivial representation from $Spin(3)$ to T^1. It follows that the case (iii) does not occur. □

By Remark 9.3 and Lemma 9.4, we may only study the case (ii) in this subsection.

Assume the case (ii) occurs, that is, $Spin(2k_2-2) = T_a$, where T_a is the a-th factor of $T^a = T_1 \times \cdots \times T_a$ ($T_i \simeq T^1$). Let $\pi_j : Spin(2m_j) \to SO(2m_j)$ be the double covering. In order to study this case, we divide this case into two parts:

- the case $k_1 = 1$;
- the case $k_1 \geq 2$.

9.2.1. *The case* $k_1 = 1$. First, we assume $k_1 = 1$. Then, we have the following finite covering:

$$\pi : \prod_{j=1}^{b} Spin(2m_j) \times \prod_{i=1}^{a} SU(l_i) \times T^a = \widetilde{K_2'} \times T_a \longrightarrow K^o$$

such that

$$\begin{array}{rcl} \pi(A_j,\ B_i,\ t_i) & = & \left(\pi_j(A_j), \begin{pmatrix} B_i t_i^{-1/l_i} & 0 \\ 0 & t_i \end{pmatrix}, t_1^{-\alpha_1} \cdots t_a^{-\alpha_a}\right) \\ & \in & K^o \subset \prod_{j=1}^{b} SO(2m_j) \times \prod_{i=1}^{a} S(U(l_i) \times U(1)) \times T^1, \end{array}$$

for the element $(A_j,\ B_i,\ t_i) \in \prod_{j=1}^{b} Spin(2m_j) \times \prod_{i=1}^{a} SU(l_i) \times T^a$.

If $\alpha_a \neq 0$, then K^o has the following subgroup:

$$\pi(T_a) = \pi(Spin(2k_2-2)) = \left\{ \left(\begin{pmatrix} I_{l_a} t_a^{-1/l_a} & 0 \\ 0 & t_a \end{pmatrix}, t_a^{-\alpha_a} \right) \middle| \ t_a \in T_a \right\}$$
$$\subset \quad (S(U(l_a) \times U(1)) \times T^1) \cap K^o,$$

where I_{l_a} is the identity element in $U(l_a)$. Therefore, for the following commutative diagram (by (9.2) and (9.3)):

$$\begin{array}{ccc} T_a \times \widetilde{K_2'} & \stackrel{\widetilde{\iota}}{\longrightarrow} & Spin(3) \times \widetilde{K_2'} \\ \pi \downarrow & & \downarrow \widetilde{\pi} \\ K^o & \stackrel{\iota}{\longrightarrow} & K_2^o, \end{array}$$

we have the following sequence (also see notations in Section 8.1):

$$\begin{array}{rcl} T^1 = q \circ \iota_2 \circ \iota \circ \pi(T_a) & = & q \circ \iota_2 \circ \iota \circ \pi(Spin(2k_2-2)) \\ & = & q \circ \iota_2 \circ \widetilde{\pi} \circ \widetilde{\iota}(Spin(2k_2-2)) \\ & \subset & q \circ \iota_2 \circ \widetilde{\pi}(Spin(2k_2-1)) \\ & \subset & q(G) = T^1. \end{array}$$

This gives a contradiction with a method similar to that demonstrated in the proof of Lemma 9.4. Hence, we have $\alpha_a = 0$.

Because $\alpha_a = 0$, we have that $\pi(T_a \times SU(l_a)) = S(U(l_a) \times U(1))$. Therefore, there is the following sequence:

$$\begin{array}{rcl} S(U(l_a) \times U(1)) & = & p_a \circ \iota_2 \circ \iota \circ \pi(T_a \times SU(l_a)) \\ & = & p_a \circ \iota_2 \circ \widetilde{\pi} \circ \widetilde{\iota}(T_a \times SU(l_a)) \\ & \subset & p_a \circ \iota_2 \circ \widetilde{\pi}(Spin(3) \times SU(l_a)) \\ & \subset & p_a(G) = SU(l_a+1). \end{array}$$

Because $Spin(3)$ and $SU(l_a)$ are simple and $\dim(SU(l_a) \times Spin(3)) = l_a^2 + 2$ and $\dim S(U(l_a) \times U(1)) = l_a^2$, we have that $p_a \circ \iota_2 \circ \widetilde{\pi}(SU(l_a) \times Spin(3)) \neq S(U(l_a) \times U(1))$. Since $S(U(l_a) \times U(1))$ is a maximal rank, maximal subgroup in $SU(l_a+1)$ (see [**15**]), we have

$$p_a \circ \iota_2 \circ \widetilde{\pi}(SU(l_a) \times Spin(3)) = SU(l_a+1).$$

Comparing their dimension ($\dim(SU(l_a)\times Spin(3)) = l_a^2+2$ and $\dim SU(l_a+1) = l_a^2+2l_a$), we have

$$l_a = 1$$

(remark $Spin(3)\simeq SU(2)$). Hence, in the case $k_1 = 1$, we can regard

$$\begin{aligned} & (SU(l_a+1),\ S(U(l_a)\times U(1))) \\ = & (Spin(2k_2-1),\ Spin(2k_2-2)) \\ = & (Spin(2m_{b+1}+1),\ Spin(2m_{b+1})) \end{aligned}$$

where $m_{b+1} = 1$. Therefore, by regarding $b+1$ as b and $a-1$ as a, we can regard the case (ii) with $k_1 = 1$ as the case (i). Hence, this case corresponds with the first case of Theorem 8.1 which satisfies $k_1 = 1$ and $k_2 = 2$ by using Remark 9.3.

9.2.2. *The case $k_1 \geq 2$.* Next, we study the other case: the case $k_1 \geq 2$.

If $k_1 \geq 2$, then we have the following finite coverings:

$$\pi : \prod_{j=1}^{b} Spin(2m_j)\times \prod_{i=1}^{a} SU(l_i)\times T^a \times SU(k_1-1)\times T^1 \to K^o$$

such that,

$$\begin{aligned} \pi(A_j, B_i, t_i, Y, t) &= \left(\pi_j(A_j), \begin{pmatrix} B_i t_i^{-1/l_i} & 0 \\ 0 & t_i \end{pmatrix}, \begin{pmatrix} Y x^{-1/(k_1-1)} & 0 \\ 0 & x \end{pmatrix}, t\right) \\ \in & \prod_{j=1}^{b} SO(2m_j)\times \prod_{i=1}^{a} S(U(l_i\times U(1))\times S(U(k_1-1)\times U(1))\times T^1, \end{aligned}$$

where $x = t_1^{-\alpha_1}\cdots t_a^{-\alpha_a} t^{-\alpha}$.

If $\alpha_a = 0$, then we can easily show that $l_a = 1$ and this case corresponds with the first case of Theorem 8.1 which satisfies $k_1 \geq 2$ and $k_2 = 2$ with the method similar to that demonstrated in the previous section (Section 9.2.1). Therefore, we may only discuss with the case $\alpha_a \neq 0$.

Assume $\alpha_a \neq 0$. We will prove this case does not occur. First, we prove the following lemma.

LEMMA 9.5. *If $k_1 \geq 2$ and $\alpha_a \neq 0$, then we can put $l_a = 1$, $k_1 = 2$ and $\alpha_a = \pm 1$.*

PROOF. If $k_1 \geq 2$ and $\alpha_a \neq 0$, then K^o has the following subgroups ($k_2 = 2$):

$$\begin{aligned} (9.6) \qquad & \pi(SU(l_a)\times Spin(2k_2-2)) \\ = & \left\{\left(\begin{pmatrix} B_a t_a^{-1/l_a} & 0 \\ 0 & t_a \end{pmatrix}, \begin{pmatrix} I_{k_1-1} t_a^{\alpha_a/(k_1-1)} & 0 \\ 0 & t_a^{-\alpha_a} \end{pmatrix}\right)\right\} \\ (9.7) \qquad & \pi(Spin(2k_2-2)\times SU(k_1-1)) \\ = & \left\{\left(\begin{pmatrix} I_{l_a} t_a^{-1/l_a} & 0 \\ 0 & t_a \end{pmatrix}, \begin{pmatrix} Y t_a^{\alpha_a/(k_1-1)} & 0 \\ 0 & t_a^{-\alpha_a} \end{pmatrix}\right)\right\} \end{aligned}$$

where $t_a \in T_a = Spin(2k_2-2)$, $B_a \in SU(l_a)$, $Y \in SU(k_1-1)$, and I_{k_1-1}, I_{l_a} are identity elements in $U(k_1-1)$ and $U(l_a)$, respectively. Moreover, there is the following commutative diagram by (9.2) and (9.3):

$$\begin{array}{ccc} (SU(l_a)\times Spin(2)\times SU(k_1-1))\times \widetilde{K_2''} & \stackrel{\widetilde{\iota}}{\longrightarrow} & (SU(l_a)\times Spin(3)\times SU(k_1-1))\times \widetilde{K_2''} \\ \pi\downarrow & & \downarrow\widetilde{\pi} \\ K^o & \stackrel{\iota}{\longrightarrow} & K_2^o, \end{array}$$

where $\widetilde{K_2''} = \prod_{j=1}^{b} Spin(2m_j)\times\prod_{i=1}^{a-1} SU(l_i)\times T^{a-1}\times T^1$. Then, we have the following sequence by (9.6):

$$\begin{aligned} S(U(l_a)\times U(1)) &= p_a\circ\iota_2\circ\iota\circ\pi(SU(l_a)\times Spin(2)) \\ &= p_a\circ\iota_2\circ\widetilde{\pi}\circ\widetilde{\iota}(SU(l_a)\times Spin(2)) \\ &\subset p_a\circ\iota_2\circ\widetilde{\pi}(SU(l_a)\times Spin(3)) \\ &\subset p_a(G) = SU(l_a+1). \end{aligned}$$

With a method similar to that demonstrated in the proof of $l_a = 1$ in Section 9.2.1, we have $l_a = 1$. Moreover, by (9.7), we have the following sequence:

$$\begin{aligned} S(U(k_1-1)\times U(1)) &= p\circ\iota_2\circ\iota\circ\pi(Spin(2)\times SU(k_1-1)) \\ &= p\circ\iota_2\circ\widetilde{\pi}\circ\widetilde{\iota}(Spin(2)\times SU(k_1-1)) \\ &\subset p\circ\iota_2\circ\widetilde{\pi}(Spin(3)\times SU(k_1-1)) \\ &\subset p(G) = SU(k_1). \end{aligned}$$

Similarly, we have $k_1 = 2$.

Moreover, we can easily show that $p\circ\iota_2\circ\widetilde{\pi} : Spin(3)\to SU(k_1) = SU(2)$ is an isomorphic map. By considering the restricted representation to $Spin(2)$ of this isomorphic map and using (9.7), we also have $\alpha_a = \pm 1$. □

By Lemma 9.5, we have $l_a = 1$, $k_1 = 2$ and $|\alpha_a| = 1$; moreover, we have the following commutative diagram (see the proof of Lemma 9.5):

$$\begin{array}{ccc} Spin(2)\times\widetilde{K_2''} & \stackrel{\widetilde{\iota}}{\longrightarrow} & Spin(3)\times\widetilde{K_2''} \\ \pi\downarrow & & \downarrow\widetilde{\pi} \\ K^o & \stackrel{\iota}{\longrightarrow} & K_2^o. \end{array}$$

Then, we have the following sequence:

$$\begin{aligned} H &= (p_a\times p)\circ\iota_2\circ\iota\circ\pi(Spin(2)) \\ &= (p_a\times p)\circ\iota_2\circ\widetilde{\pi}\circ\widetilde{\iota}(Spin(2)) \\ &\subset (p_a\times p)\circ\iota_2\circ\widetilde{\pi}(Spin(3)) \\ &\subset p_a\times p(G) = SU(l_a+1)\times SU(k_1) = SU(2)\times SU(2). \end{aligned}$$

Here, H is one of the followings, because of (9.6) and (9.7) in the proof of Lemma 9.5 and $|\alpha_a| = 1$:

$$\Delta = \left\{\left(\begin{array}{cc} t_a^{-1} & 0 \\ 0 & t_a \end{array}\right), \left(\begin{array}{cc} t_a^{-1} & 0 \\ 0 & t_a \end{array}\right)\right\} \subset SU(2)\times SU(2) \text{ for } \alpha_a = -1;$$

$$\Delta' = \left\{\left(\begin{array}{cc} t_a^{-1} & 0 \\ 0 & t_a \end{array}\right), \left(\begin{array}{cc} t_a & 0 \\ 0 & t_a^{-1} \end{array}\right)\right\} \subset SU(2)\times SU(2) \text{ for } \alpha_a = 1.$$

Since Δ and Δ' are conjugate in G, we can take $\alpha_a = -1$ and $H = \Delta$. Because $Spin(3)\simeq SU(2)$ and $H = \Delta$, we can easily show that

$$(p_a\times p)\circ\iota_2\circ\widetilde{\pi}(Spin(3)) = \{(X,\ X) \mid X\in SU(2)\}\subset SU(2)\times SU(2),$$

i.e., the diagonal subgroup. By the definition of π and the above argument, we see the followings:

$$\begin{aligned} & \iota\circ\pi(Spin(2)\times T^1) \\ = & \left\{ \left(\begin{pmatrix} t_a^{-1} & 0 \\ 0 & t_a \end{pmatrix},\ \begin{pmatrix} t_a^{-1}t^{\alpha} & 0 \\ 0 & t_a t^{-\alpha} \end{pmatrix},\ t \right) \in SU(2)\times SU(2)\times T^1 \right\} \\ \subset & \ \widetilde{\pi}(Spin(3)\times T^1) \\ = & \left\{ (X,\ X,\ t) \in SU(2)\times SU(2)\times T^1 \right\}. \end{aligned}$$

It follows that $\alpha = 0$. However, this gives a contradiction because $\alpha \in \mathbb{N}$ (see Section 4.1.1). Therefore, this case (the case $k_1 \geq 2$ and $\alpha_a \neq 0$) does not occur.

10. The case (3)-(b)

In this section, we study the case (3)-(b). From Section 4.1, we have

$$G = \prod_{j=1}^{b} SO(2m_j+1) \times \prod_{i=1}^{a} SU(l_i+1) \times SO(2k_1),$$

$$K_1 = \mathcal{S}_1 \times \prod_{i=1}^{a} S(U(l_i)\times U(1)) \times SO(2k_1),$$

$$(10.1)\quad K = \prod_{i=1}^{a} S(U(l_i)\times U(1)) \times \left\{ \left(A,\ \begin{pmatrix} X & 0 \\ 0 & x \end{pmatrix} \right) \in \mathcal{S}_1 \times S(O(2k_1-1)\times O(1)) \ \middle|\ \sigma_1(A)x = 1 \right\},$$

where $k_1 \geq 2$. From Lemma 8.2, we also have

$$K^o = Spin(2k_2-2)\circ K_2', \tag{10.2}$$

$$K_2^o = Spin(2k_2-1)\circ K_2' \tag{10.3}$$

Therefore, we have the covering of K^o as follows by (10.1) and (10.2):

$$\begin{aligned} \widetilde{K^o} &= Spin(2k_2-2)\times \widetilde{K_2'} \\ &= \prod_{j=1}^{b} Spin(2m_j) \times \prod_{i=1}^{a} SU(l_i) \times T^a \times Spin(2k_1-1). \end{aligned}$$

Comparing the above covering of K^o's and using the fact that $Spin(2) \simeq T^1$, $Spin(3) \simeq SU(2)$, $Spin(4) \simeq SU(2)\times SU(2)$ and $Spin(6) \simeq SU(4)$, there are the following five cases:

- **(i):** $Spin(2k_2-2) = Spin(2m_b)$ and $k_2 = m_b + 1 \geq 2$;
- **(ii):** $Spin(2k_2-2) = T_a$ and $k_2 = 2$, where T_a is the a-th factor of $T^a = T_1 \times \cdots \times T_a$ ($T_i \simeq T^1$);
- **(iii):** $Spin(2k_2-2) = SU(l_{a-1}) \times SU(l_a)$ and $l_a = l_{a-1} = 2$, $k_2 = 3$;
- **(iv):** $Spin(2k_2-2) = SU(l_a) \times Spin(2k_1-1)$ and $l_a = 2$, $k_1 = 2$, $k_2 = 3$;
- **(v):** $Spin(2k_2-2) = SU(l_a)$ and $l_a = 4$, $k_2 = 4$.

Similarly to Lemma 9.1, we can show the following lemma.

LEMMA 10.1. *In the above cases, the cases from (iii) to (v) do not occur.*

Proof. For the cases (iii) and (iv), we can apply a method similar to the proof of that the case (ii) does not occur in Lemma 9.1. For the case (v), we can apply a method similar to the proof of that the case (iv) does not occur in Lemma 9.1. So we may omit the detail of the proof. □

From the next section, we study the cases (i) and (ii): we call them (3)-(b)-(i) and (3)-(b)-(ii), respectively.

10.1. The case (3)-(b)-(i). Suppose the case (3)-(b)-(i) occurs, that is,

$$Spin(2k_2-2) = Spin(2m_b) \quad \text{and} \quad k_2 = m_b + 1 \geq 2.$$

It follows from (10.1) and (10.2) that the following relation holds:

$$\begin{aligned} Spin(2k_2-2) \circ K_2' &= Spin(2m_b) \circ K_2' \\ &= \prod_{j=1}^{b} SO(2m_j) \times \prod_{i=1}^{a} S(U(l_i) \times U(1)) \times SO(2k_1-1) = K^o. \end{aligned}$$

Therefore, we can put $K^o = SO(2m_b) \times K_2' = SO(2k_2-2) \times K_2'$ and

$$K_2' = \prod_{j=1}^{b-1} SO(2m_j) \times \prod_{i=1}^{a} S(U(l_i) \times U(1)) \times SO(2k_1-1).$$

Because $K_2^o = Spin(2k_2-1) \circ K_2'$ (by (10.3)), by using the same argument in Section 9.1, we have

$$K_2^o = \prod_{i=1}^{a} S(U(l_i) \times U(1)) \times \prod_{j=1}^{b-1} SO(2m_b) \times SO(2k_2-1) \times SO(2k_1-1),$$

and there is the inclusion $SO(2k_2-1) \to K_2$ such that its image is $SO(2m_b+1) \subset G$. Because $K_2^o \subset K_2 \subset G$, we can put K_2 is as follows:

$$\begin{aligned} & K_2 = SO(2k_2-1) \times K_2'' \\ \subset \quad & G = SO(2k_2-1) \times \left(\prod_{j=1}^{b-1} SO(2m_j+1) \times \prod_{i=1}^{a} SU(l_i+1) \times SO(2k_1) \right), \end{aligned}$$

where $2m_b+1 = 2k_2-1$ and K_2'' is a subgroup of $\prod_{j=1}^{b-1} SO(2m_j+1) \times \prod_{i=1}^{a} SU(l_i+1) \times SO(2k_1)$. By using the same argument of Section 9.1, we have that

$$K_2 = SO(2k_2-1) \times \Phi(K) \subset G \quad \text{such that} \quad \Phi(K) = \Phi(K_2) = K_2'',$$

where $\Phi : G \to G/SO(2k_2-1) = G/SO(2m_b+1)$ is the natural projection. It follows that the inclusion $K_2 \subset G$ is completely determined by K.

Next, we consider the slice representation $\sigma_2 : K_2 \to O(2k_2-1)$. By the same reason demonstrated in Section 9.1, the restricted slice representation $\sigma_2|_{SO(2k_2-1)} : SO(2k_2-1) \to O(2k_2-1)$ is the natural inclusion. Hence, we have that

$$\sigma_2(K_2'') \subset Z(SO(2k_2-1)) = \{\pm I_{2k_2-1}\} \subset O(2k_2-1).$$

Moreover, by (10.1), we have the following formula for K:

$$\begin{aligned} & K = \sigma_2^{-1}(O(2k_2-2)) \\ = & \left\{ \left(\begin{pmatrix} B & 0 \\ 0 & b \end{pmatrix}, Y \right) \in S(O(2k_2-2)\times O(1)) \times K_2'' \;\middle|\; b = \det B^{-1} = \det \sigma_2(Y) \right\} \\ = & \prod_{i=1}^{a} S(U(l_i)\times U(1)) \times \left\{ \left(A, \begin{pmatrix} X & 0 \\ 0 & x \end{pmatrix} \right) \in \mathcal{S}_1 \times S(O(2k_1-1)\times O(1)) \;\middle|\; \sigma_1(A)x = 1 \right\}. \end{aligned}$$

Therefore, we can easily show that the following lemma by using the above formula for K.

Lemma 10.2. *The following two statements are equivalent:*

(1): $\sigma_2(K_2'') = \{I_{2k_2-1}\}$ *(resp.* $\sigma_2(K_2'') = \{\pm I_{2k_2-1}\}$*);*
(2): $K = SO(2k_2-2)\times K_2''$ *(resp.* $r_b(K) = S(O(2m_b)\times O(1))$*).*

Moreover, the following statement holds:

(3): *if* $\sigma_2(K_2'') = \{\pm I_{2k_2-1}\}$ *then* $K \neq S(O(2m_b)\times O(1))\times K_2''$.

It follows from this Lemma 10.2 and the above formula for K that the slice representation $\sigma_2 : K_2 \to O(2k_2-2)$ is also completely determined by K. Therefore, the tubular neighborhood of G/K_2 is completely determined by K and equivariantly diffeomorphic to the following manifold:

$$\prod_{i=1}^{a} \mathbb{C}P(l_i) \times \left(\prod_{j=1}^{b-1} S^{2m_j} \times D(\mathbb{R}^{2k_2-1}) \right) \times_{\mathcal{A}} S(\mathbb{R}^{2k_1}),$$

where $\mathcal{A} \simeq \mathcal{S}_1/\mathcal{S}_1^o$-quotient is defined by the following actions: on $\prod_{j=1}^{b-1} S^{2m_j} \times D(\mathbb{R}^{2k_2-1})$ as the subgroup $\mathcal{A} \subset \prod_{j=1}^{b-1}\mathbb{Z}_2 \times \{\pm I_{2k_2-1}\}$; and on $S(\mathbb{R}^{2k_1}) \subset \mathbb{R}^{2k_1}$ by the representation $\sigma_{\mathbb{R}} : \mathcal{A} \to \{\pm I_{2k_1}\}$ (induced by σ_1). Moreover, by using Lemma 10.2 (3), if $(1, \cdots, 1, -I_{2k_2-1}) \in \mathcal{A} \subset \prod_{j=1}^{b-1}\mathbb{Z}_2 \times \{\pm I_{2k_2-1}\}$, then $\sigma_{\mathbb{R}}(1, \cdots, 1, -I_{2k_2-1}) = -I_{2k_1}$. It follows that $\mathcal{A}$ acts on $\prod_{j=1}^{b-1} S^{2m_j} \times D(\mathbb{R}^{2k_2-1}) \times S(\mathbb{R}^{2k_1})$ freely; therefore, X_2 is a manifold.

From Remark 4.2, we can easily check that the pair of $(M,\ G)$ of the case (3)-(b)-(i) is as follows:

$$M = \prod_{i=1}^{a} \mathbb{C}P(l_i) \times \prod_{j=1}^{b-1} S^{2m_j} \times_{\mathcal{A}} S(\mathbb{R}^{2k_2-1} \oplus \mathbb{R}^{2k_1});$$
$$G = \prod_{i=1}^{a} SU(l_i+1) \times \prod_{j=1}^{b-1} SO(2m_j+1) \times SO(2k_2-1) \times SO(2k_1),$$

where $\mathcal{A}$ acts on $\prod_{j=1}^{b-1} S^{2m_j} \times S(\mathbb{R}^{2k_2-1} \oplus \mathbb{R}^{2k_1})$ as follows:

- on $\mathbb{R}^{2k_1}$ by $\sigma_{\mathbb{R}} : \mathcal{A} \to \{\pm I_{2k_1}\}$;
- on $\prod_{j=1}^{b-1} S^{2m_j} \times \mathbb{R}^{2k_2-1}$ as a subgroup $\prod_{j=1}^{b-1}\mathbb{Z}_2 \times \{\pm I_{2k_2-1}\}$ such that if $(1, \cdots, 1, -I_{2k_2-1}) \in \mathcal{A}$, then $\sigma_{\mathbb{R}}(1, \cdots, 1, -I_{2k_2-1}) = -I_{2k_1}$.

This corresponds with the second manifold in Theorem 8.1.

10.2. The case (3)-(b)-(ii). Suppose the case (3)-(b)-(ii) occurs, that is,

$$Spin(2k_2-2)=T_a \quad \text{and} \quad k_2=2,$$

where T_a is the a-th factor of $T^a=T_1\times\cdots\times T_a$ ($T_i\simeq T^1$). Let $\psi:\widetilde{K^o}\to K^o$ be the finite covering projection, where $\widetilde{K^o}$ is a product of Lie groups. By (10.1), we can easily show that

$$\psi(Spin(2k_2-2))=\left\{\left(\begin{array}{cc} I_{l_a}t_a^{-1/l_a} & 0\\ 0 & t_a\end{array}\right)\ \middle|\ t_a\in T_a=Spin(2k_2-2)\right\}.$$

Moreover, there is the following commutative diagram by (10.2) and (10.3):

$$\begin{array}{ccc} (SU(l_a)\times Spin(2k_2-2))\times\widetilde{K_2''} & \stackrel{\widetilde{\iota}}{\longrightarrow} & (SU(l_a)\times Spin(2k_2-1))\times\widetilde{K_2''}\\ \psi\downarrow & & \downarrow\widetilde{\psi}\\ K^o & \stackrel{\iota}{\longrightarrow} & K_2^o, \end{array}$$

where $\widetilde{K_2''}=\prod_{j=1}^b Spin(2m_j)\times\prod_{i=1}^{a-1}SU(l_i)\times T^{a-1}\times Spin(2k_1-1)$. Because $p_a(K^o)=S(U(l_a)\times U(1))$ (by (10.1)), we have the following sequence ($k_2=2$):

$$\begin{aligned} & S(U(l_a)\times U(1))\\ = & \ p_a\circ\iota_2\circ\iota\circ\psi(SU(l_a)\times Spin(2k_2-2))\\ = & \ p_a\circ\iota_2\circ\widetilde{\psi}\circ\widetilde{\iota}(SU(l_a)\times Spin(2k_2-2))\\ \subset & \ p_a\circ\iota_2\circ\widetilde{\psi}(SU(l_a)\times Spin(2k_2-1))\\ \subset & \ p_a(G)=SU(l_a+1). \end{aligned}$$

Therefore, with a method similar to the proof of $l_a=1$ in Section 9.2.1, we also have $l_a=1$ (remark $Spin(3)\simeq SU(2)$). Hence, we have

$$\begin{aligned} K_2^o &= \prod_{j=1}^b SO(2m_b)\times\prod_{i=1}^{a-1}S(U(l_i)\times U(1))\times SU(2)\times SO(2k_1-1)\\ &= \prod_{j=1}^b SO(2m_b)\times\prod_{i=1}^{a-1}S(U(l_i)\times U(1))\times Spin(3)\times SO(2k_1-1). \end{aligned}$$

By the similar argument of Section 10.1 and using (10.1), we have that

$$\begin{aligned} K_2 &= \prod_{i=1}^{a-1}S(U(l_i)\times U(1))\times Spin(3)\\ &\quad\times\left\{\left(A,\left(\begin{array}{cc} X & 0\\ 0 & x\end{array}\right)\right)\in\mathcal{S}_1\times S(O(2k_1-1)\times O(1))\ \middle|\ \sigma_1(A)x=1\right\}. \end{aligned}$$

Similarly, the slice representation $\sigma_2: K_2\to O(3)$ is the natural representation from the $Spin(3)$-factor in K_2 to $SO(3)\subset O(3)$, and from the other factors K_2'' in K_2 to $ZSO(3)=\{\pm I_3\}$. However, by $\sigma_2^{-1}(O(2))=K\supset K_2''=\psi(\widetilde{K_2''})$ (see (10.1)) and $O(2)\cap\{\pm I_3\}=\{I_3\}$, we have that $\sigma_2(K_2'')=\{I_3\}$. Therefore, $K_2\subset G$ and σ_2 are completely determined by K.

In this case, by constructing the G-manifold, we can regard $Spin(3)$ as $SO(3)=SO(2m_{b+1}+1)$ in G up to essential isomorphism. By regarding $b+1$ as b and $a-1$ as a, we can easily show that this case is the same as the case (3)-(b)-(i) with $k_2=2$

and $\sigma_2(K_2'') = \{I_3\}$. Hence, this corresponds with the second manifold in Theorem 8.1 such that $k_2 = 2$ and $\sigma_2 : K_2 \to SO(3) \subset O(3)$.

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