Operations on three dimensional small covers

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Abstract. The purpose of this paper is to study relations among equivariant operations on three-dimensional small covers. We get three formulas for these operations (Theorem 1). As an application, we improve the Nishimura's theorem on the construction of oriented 3-dimensional small covers (Corollary 2) and the Lü-Yu's theorem on the construction of all 3-dimensional small covers (Corollary 3). Moreover, for a construction of three-dimensional 2-torus manifolds, we show that all operations can be obtained by using the equivariant surgeries (Theorem 4).

1. Introduction

Small covers were introduced by Davis and Januszkiewicz as a real version of quasitoric manifolds in 1991. In their paper [**2**], they showed that there are strong links between a small cover M with the orbit projection map $\pi : M \to P$ and a combinatorial structure of its orbit polytope P. For example, some topological invariants (e.g., equivariant cohomologies or $\mathbb{Z}_2$-Betti numbers) of small covers $\pi : M \to P$ are decided by the combinatorial invariants of their orbit polytopes P (e.g., Stanley-Reisner rings or h-vectors).

Not only topological invariants but also topological operations on M (e.g., equivariant connected sums or equivariant surgeries) correspond with combinatorial operations on the orbit polytope P (see Section 3). Making use of this correspondence, constructions of 3-dimensional small covers M^3 from basic small covers have been studied by Izmestiev, Nishimura, Lü and Yu. In [**3**] Izmestiev studies the class of small covers M^3 called a *linear model*, i.e., small covers over 3-colored polytopes. He proves that 3-dimensional linear models are constructed from one basic small cover (the 3-dimensional torus T^3) by using two operations $\sharp$ and $\natural$ (see Section 3 and Theorem 4.2). In [**11**] Nishimura generalizes Izmestiev's result to *oriented* small covers M^3, i.e., small covers over 3 or 4-colored polytopes. He proves that 3-dimensional oriented small covers are constructed from two basic small covers (T^3 and the real projective space $\mathbb{R}P(3)$) by using three operations $\sharp$, $\natural$ and $\flat$ (see Section 3 and Theorem 4.3). In [**7**] Lü and Yu prove that all 3-dimensional small

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covers are constructed from five basic small covers by using six operations $\sharp$, $\sharp^e$, $\sharp^{eve}$, $\natural$, $\sharp^{\Delta}$ and $\sharp^C$ (see Section 3 and Theorem 4.5).

The operation $\flat$ appearing in the Nishimura's theorem (Theorem 4.3) is not used in the Lu- Yu's theorem (Theorem 4.5). On the other hand, the operations $\sharp^e$, $\sharp^{eve}$, $\sharp^{\Delta}$ and $\sharp^C$ appearing in the Lu- Yu's theorem are not used in the Nishimura's theorem. So we can naturally ask what relations exist between the Nishimura's theorem and the Lü-Yu's theorem. Motivated by this question, in this paper, we prove the following theorem (Theorem 4.1).

Theorem 1. The operations $\flat$, $\sharp^e$, $\sharp^{eve}$ can be obtained by using $\sharp$, $\natural$ as follows:
(1) $\flat = \natural \circ (\sharp\Delta^3)$;
(2) $\sharp^e = \natural \circ (\sharp P^3(3))$;
(3) $\sharp^{eve} = \natural^2 \circ (\sharp P^3_-(3))$,
where $\natural^2$ denotes two times Dehn surgery $\natural \circ \natural$.

Using the above Theorem 1 and the Nishimura's theorem (Theorem 4.3), we have the following corollary (Corollary 4.4).

Corollary 2. Each 3-dimensional oriented small cover can be (equivariantly) constructed from the real projective space $\mathbb{R}P(3)$ and the 3-dimensional torus T^3 by using finite times following two operations: the equivariant connected sum $\sharp$; and the equivariant Dehn surgery $\natural$.

Moreover, we prove the following corollary (Corollary 4.8) by making use of Theorem 1 and the Lü-Yu's theorem (Theorem 4.5).

Corollary 3. Each 3-dimensional small cover can be (equivariantly) constructed from $\mathbb{R}P(3)$, $M(P^3(3),\ \lambda_2)$ and $M(P^3(3),\ \lambda_3)$ (up to weakly $\mathbb{Z}_2^3$-equivariant diffeomorphism) by using finite times following four operations: the equivariant connected sum $\sharp$; the equivariant Dehn surgery $\natural$; the operation $\sharp^{\Delta}$ and the coloring change $\sharp^C$, where $(M(P^3(3)),\ \lambda_i)$ denotes $S^1 \times \mathbb{R}P(2)$ with the $\mathbb{Z}_2^3$-action induced by λ_i for $i = 2,\ 3$.

For 2-torus manifolds, the following theorem (Theorem 6.1) holds.

Theorem 4. The operations $\sharp^{\Delta}$, $\sharp^C$ can be obtained by using $\natural$, $\natural_0$ and $\sharp$ as follows:
(1) $\sharp^{\Delta} = \natural_0 \circ \natural \circ \sharp$;
(2) $\sharp^C P^3(l) = \natural_0 \circ \natural^{l-2} \circ (\sharp P^3(l))$,
where $P^3(l)$ is the l-sided prism, $l \geq 3$ and $\natural^{l-2}$ denotes $(l-2)$-times $\natural$.

The organization of this paper is as follows. In Section 2, we recall the basic facts for small covers. In Section 3, we introduce the seven operations on 3-dimensional small covers. In Section 4, we prove the main theorem (Theorem 4.1). As an application of Theorem 4.1, we also prove Corollary 4.4 and 4.8 which improve the Nishimura's theorem (Theorem 4.3) and the Lü-Yu's theorem (Theorem 4.5). In Section 5, we remark a relation between the Nishimura's theorem and Lü-Yu's theorem. In Section 6, we introduce a new operation and prove Theorem 6.1 for 2-torus manifolds.

2. Basics of small covers

In this section, we recall some basic facts for small covers (see [**2**] for detail). We describe the quotient additive group $\mathbb{Z}/2\mathbb{Z}$ as $\mathbb{Z}_2$ throughout this paper.

2.1. Definition of a small cover. First we shall recall the terminology: 2-torus manifolds in [**5, 6**]. A 2*-torus manifold* M^n is an n-dimensional, closed smooth manifold with a non-free effective smooth $\mathbb{Z}_2^n$-action.

Let P^n be a *simple* convex n-polytope, i.e., precisely n facets of P^n meet at each vertex. A *small cover* is a 2-torus manifold M^n which satisfies the following two conditions:

(a): the $\mathbb{Z}_2^n$-action is *locally standard*, i.e., locally same as the $\mathbb{Z}_2^n$-action on $\mathbb{R}^n$; and

(b): there is the orbit projection map $\pi : M^n \to P^n$ constant on $\mathbb{Z}_2^n$-orbits which maps every rank k orbit (i.e., orbits which isomorphic to $\mathbb{Z}_2^k$) to a point in the interior of k-dimensional face of P^n, $k = 0, \cdots, n$.

We can easily show that π sends $\mathbb{Z}_2^n$-fixed points in M^n to vertices of P^n by using the above condition (b). We often call P^n an *orbit polytope* of M.

2.2. Characteristic function and constructions of small covers. On the other hand, for given P^n, small covers M^n with orbit projection $\pi : M^n \to P^n$ can be reconstructed by using a characteristic function $\lambda : \mathcal{F} \to \mathbb{Z}_2^n$, where $\mathcal{F}$ is the set of facets in P. In this subsection, we shall recall the characteristic function λ and the construction of small covers by using P and λ.

Due to the definition of a small cover $\pi : M \to P$, we have that $\pi^{-1}(int\ F^{n-1})$ consists of $(n-1)$-rank orbits, in other words, the isotropy subgroup at $x \in \pi^{-1}(int\ F^{n-1})$ is $K \subset \mathbb{Z}_2^n$ such that $K \simeq \mathbb{Z}_2$, where $int\ F^{n-1}$ is an interior of a facet F^{n-1}. Hence, the isotropy subgroup at x is determined by a primitive vector $v \in \mathbb{Z}_2^n$ generating a subgroup K. In this way we obtain a function λ from the set of facets of P, denoted by $\mathcal{F}$, to primitive vectors in $\mathbb{Z}_2^n$. We call such $\lambda : \mathcal{F} \to \mathbb{Z}_2^n$ a *characteristic function* or *coloring* on P. By the locally standard property, a characteristic function satisfies the following property (called *the property* $(\star)$):

$(\star)$**:** if $F_1 \cap \cdots \cap F_n \neq \emptyset$ for $F_i \in \mathcal{F}$ $(i = 1, \cdots, n)$, then $\{\lambda(F_1), \cdots, \lambda(F_n)\}$ spans $\mathbb{Z}_2^n$.

Next we mention the construction of small covers by using P^n and λ. Let P^n be a simple convex polytope. Suppose that the characteristic function $\lambda : \mathcal{F} \to \mathbb{Z}_2^n$ which satisfies the above property $(\star)$ is defined on P^n. Small covers can be constructed from P and λ as follows:

$$\mathbb{Z}_2^n \times P/\sim,$$

where $(t, x) \sim (t', y)$ is defined as $x = y \in P$ and

$$\begin{aligned} &t = t' \quad \text{if } x \in intP; \\ &t^{-1}t' \in \langle \lambda(F_1), \cdots, \lambda(F_l) \rangle \simeq \mathbb{Z}_2^l \quad \text{if } x \in int(F_1 \cap \cdots \cap F_l), \end{aligned}$$

where $\langle \lambda(F_1), \cdots, \lambda(F_l) \rangle \subset \mathbb{Z}_2^n$ denotes the subgroup generated by $\lambda(F_i)$ for $i = 1, \cdots, l$. We describe such small cover as $M(P, \lambda)$.

Before we show examples of small covers, we define the equivalence relation on small covers. Let $(M_1, \mathbb{Z}_2^n)$ and $(M_2, \mathbb{Z}_2^n)$ be small covers. We denote their $\mathbb{Z}_2^n$-actions as φ_1 and φ_2, respectively. We call $(M_1, \mathbb{Z}_2^n)$ and $(M_2, \mathbb{Z}_2^n)$ are *weakly equivariantly homeomorphic*, if there is a homeomorphism $f : M_1 \to M_2$ such that $f(\varphi_1(t, x)) = \varphi_2(g(t), f(x))$, where $t \in \mathbb{Z}_2^n$, $x \in M_1$ and $g : \mathbb{Z}_2^n \to \mathbb{Z}_2^n$ is an isomorphism. If g is the identity map, we call $(M_1,\ \mathbb{Z}_2^n)$ and $(M_2,\ \mathbb{Z}_2^n)$ are *equivariantly homeomorphic*.

2.3. Examples. Let $\{e_1,\ \cdots,\ e_n\}$ be the standard basis in $\mathbb{Z}_2^n$. We call a pair of a polytope and its characteristic funcation $(P^n,\ \lambda)$ a *polytope with m-coloring* or an *m-coloring polytope* if the image of λ is a set of m-independent vectors in $\mathbb{Z}_2^n$, i.e., $\lambda(\mathcal{F}) = \{f_1,\ \cdots,\ f_m\}$ where f_j $(j = 1,\ \cdots,\ m)$ is a linear combination of $e_1,\ \cdots,\ e_n$. Now Figure 1 shows two examples of characteristic functions on polytopes. The polytope of the left example is the 3-simplex Δ^3 with 4-coloring, i.e., $\lambda_0(\mathcal{F}) = \{e_1,\ e_2,\ e_3,\ e_1 + e_2 + e_3\}$. The polytope of the right example is the 3-cube I^3 with 3-coloring, i.e., $\lambda_0^I(\mathcal{F}) = \{e_1,\ e_2,\ e_3\}$.

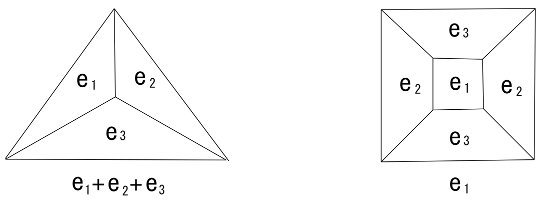

FIGURE 1. The left figure is $(\Delta^3,\ \lambda_0)$ and the right figure is $(I^3,\ \lambda_0^I)$, where the bottom $e_1 + e_2 + e_3$ and e_1 are the colors of facets on the back. We see that $M(\Delta^3,\ \lambda_0) = \mathbb{R}P(3)$ and $M(I^3,\ \lambda_0^I) = T^3$, where $\mathbb{R}P(3)$ and T^3 have the standard $\mathbb{Z}_2^3$-actions.

We can easily show that Δ^3 has the unique characteristic function λ_0 up to $GL(3,\ \mathbb{Z}_2)$, i.e., for all characteristic function λ on Δ^3, there is an element $\sigma \in GL(3,\ \mathbb{Z}_2)$ such that $\sigma \circ \lambda = \lambda_0$. In other words, small covers over Δ^3 are unique up to weakly equivariant diffeomorphisms. We can assume that there is at least one vertex $v = F_1 \cap \cdots \cap F_n$ in P such that $\lambda(F_i) = e_i$ for all $i = 1,\ \cdots,\ n$ up to $GL(3,\ \mathbb{Z}_2)$ because of the property $(\star)$.

A small cover over P^n with n-coloring, i.e., $\lambda(\mathcal{F}) = \{e_1,\ \cdots,\ e_n\}$ (up to $GL(n,\ \mathbb{Z}_2)$) is called a *linear model*. So the right example in Figure 1 is a linear model. Nakayama and Nishimura show that a 3-dimensional small cover is orientable if and only if the characteristic function on the orbit polytope P^3 is 3 or 4 colored in [**10**]. Therefore, small covers which are constructed from the two examples in Figure 1 are orientable.

2.4. $\mathbb{Z}_2^3$-invariant normal bundle over an invariant S^1. Let $\pi : M \to P$ be a 3-dimensional small cover. In this subsection, we study the equivariant normal bundle of the inverse of edges in P.

Fix $\{e_1,\ e_2,\ e_3\}$ as a basis (not necessary standard) of $\mathbb{Z}_2^3$. We can easily show, by using the property $(\star)$, that the characteristic functions on neighboring facets around one edge in a 3-dimensional polytope have only the four cases in Figure 2 (in the next page).

The small cover over a 1-simplex $\Delta^1 (= I^1)$ is identified with $\mathbb{R}P(1) (= S^1)$. Hence, for the small cover $\pi : M \to P$, the inverse $\pi^{-1}(I^1)$ of any edge I^1 in P is an invariant submanifold diffeomorphic to S^1. Moreover, we know an equivariant normal bundle of $\pi^{-1}(I^1)$ from a characteristic function around I^1. Let $(t_1, t_2, t_3) \in \mathbb{Z}_2^3$ act on $((x_1, x_2), (y_1, y_2)) \in S^1 \times D^2$ by $((x_1, x_2), (y_1, y_2)) \mapsto ((t_1x_1, x_2), (t_2y_1, t_3y_2))$.

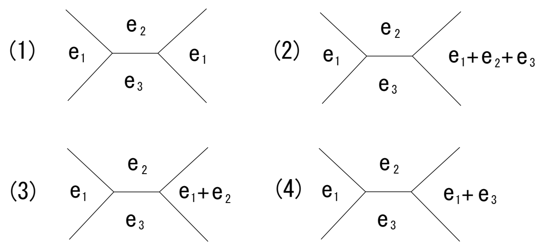

FIGURE 2. The characteristic functions around edges.

We can easily show the following proposition by computing a characteristic function of $S^1 \times_{\mathbb{Z}_2} D^2$'s and using Figure 2 and the construction of small covers in Section 2.2.

PROPOSITION 2.1. Let I be an edge in P and N a small neighborhood of I. For an invariant submanifold $S^1 = \pi^{-1}(I)$ in the 3-dimensional small cover, the normal bundles $\pi^{-1}(N)$ are weakly $\mathbb{Z}_2^3$-equivariantly isomorphic to one of the following four disk bundles:

(1) if N satisfies (1) in Figure 2, then $\pi^{-1}(N) \simeq S^1 \times_{\mathbb{Z}_2} D(\underline{\mathbb{R}} \oplus \underline{\mathbb{R}})$;
(2) if N satisfies (2) in Figure 2, then $\pi^{-1}(N) \simeq S^1 \times_{\mathbb{Z}_2} D(\mathbb{R} \oplus \mathbb{R})$;
(3) if N satisfies (3) in Figure 2, then $\pi^{-1}(N) \simeq S^1 \times_{\mathbb{Z}_2} D(\mathbb{R} \oplus \underline{\mathbb{R}})$;
(4) if N satisfies (4) in Figure 2, then $\pi^{-1}(N) \simeq S^1 \times_{\mathbb{Z}_2} D(\underline{\mathbb{R}} \oplus \mathbb{R})$,

where the non-trivial element in $\mathbb{Z}_2$ acts on S^1 by the antipodal involution, and $D(V \oplus V') = D^2$ denotes a closed disk in $V \oplus V'$ (V, V' are 1-dimensional real vector spaces), $\mathbb{Z}_2$ acts on $\mathbb{R}$ canonically and on $\underline{\mathbb{R}}$ trivially.

REMARK 2.2. In the above Proposition 2.1, we have that $S^1 \times_{\mathbb{Z}_2} (\underline{\mathbb{R}} \oplus \underline{\mathbb{R}}) \cong \epsilon^1 \oplus \epsilon^1$, $S^1 \times_{\mathbb{Z}_2} (\mathbb{R} \oplus \mathbb{R}) \cong \gamma_1^1 \oplus \gamma_1^1$, $S^1 \times_{\mathbb{Z}_2} (\mathbb{R} \oplus \underline{\mathbb{R}}) \cong \gamma_1^1 \oplus \epsilon^1$ and $S^1 \times_{\mathbb{Z}_2} (\underline{\mathbb{R}} \oplus \mathbb{R}) \cong \epsilon^1 \oplus \gamma_1^1$, where ϵ^1 is the trivial bundle and γ_1^1 is the canonical bundle over $\mathbb{R}P(1)(= S^1)$. Hence, we have that the bundle (1) (resp. (3)) is isomorphic to the bundle (2) (resp. (4)) in Proposition 2.1, by making use of the basic facts of the vector bundle over $\mathbb{R}P(1)$ (see [**9**]). However, these four bundles are different as the $\mathbb{Z}_2^3$-equivariant bundle because their characteristic functions are different (see Figure 2). We also remark that, up to weakly $\mathbb{Z}_2^3$-equivariant diffeomorphism, (3) and (4) are same.

3. Operations for 3-dimensional small covers

Henceforth, we assume that $M(P,\ \lambda)$ is a 3-dimensional small cover over a 3-dimensional simple convex polytope with coloring $(P,\ \lambda)$, and $\{e_1,\ e_2,\ e_3\}$ is a basis (not necessary standard) of $\mathbb{Z}_2^3$. In this section, we introduce operations on small covers and orbit polytopes (also see [**3, 7, 11**]). From this section, we often call the set of colorings included in $\{f_1,\ f_2,\ f_1+f_2\}$ a 2-*independent coloring*, where f_i $(i = 1,\ 2)$ is a linear combination of $\{e_1,\ e_2,\ e_3\}$.

3.1. The equivariant connected sum $\sharp$. The operation in Figure 3 is called the *equivariant connected sum* $\sharp$ (from left to right) and its inverse $\sharp^{-1}$ (from right to left). The left figure shows two neighborhoods of two same colored vertices in $(P_1,\ \lambda_1)$ and $(P_2,\ \lambda_2)$. We can do the connected sum of the two polytopes at these vertices (see [**3**, Definition 3]), then we get a new polytope with coloring $(P_1 \sharp P_2,\ \lambda)$, vice versa. Remark that $P_1 \sharp P_2$ is a combinatorial simple convex polytope by using *the Steinitz' theorem*: the graph Γ is a graph of the 3-dimensional polytope P if and only if Γ is 3-connected and planer (see [**13**, Chapter 4]).

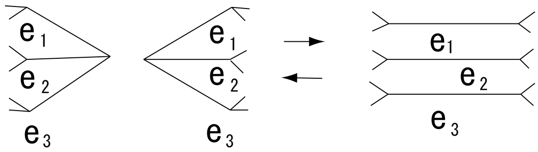

FIGURE 3. The equivariant connected sum $\sharp$ (from left to right) and its inverse $\sharp^{-1}$ (from right to left). Here, e_3 is the coloring of the facet on the back.

From the geometric point of view, this operation $\sharp$ corresponds with the equivariant connected sum $M(P_1,\ \lambda_1)\sharp M(P_2,\ \lambda_2)$ for two fixed points in $M(P_1,\ \lambda_1)$ and $M(P_2,\ \lambda_2)$. We can easily check $M(P_1,\ \lambda_1)\sharp M(P_2,\ \lambda_2)$ is a small cover and its orbit polytope with coloring is $(P_1\sharp P_2,\ \lambda)$, i.e., $M(P_1,\ \lambda_1)\sharp M(P_2,\ \lambda_2) = M(P_1\sharp P_2,\ \lambda)$ (also see [**3**, Lemma 2]).

3.2. The cutting edge operation $\sharp^e$. Before we mention the cutting edge operation, we introduce the connected sum along edges. Let P_1 and P_2 be 3-dimensional, simple, convex polytopes. Suppose that the edges $I_1 \subset P_1$ and $I_2 \subset P_2$ are chosen, and a one-to-one correspondence $F_i \mapsto F_i'$ $(i = 1,\ \cdots,\ 4)$ is established, where this correspondence is from the facets $\{F_1,\ F_2,\ F_3,\ F_4\}$ containing I_1 as $I_1 = F_2 \cap F_3$ to the facets $\{F_1',\ F_2',\ F_3',\ F_4'\}$ containing I_2 as $I_2 = F_2' \cap F_3'$. The *connected sum along edges* with respect to these data is a polytope combinatorially equivalent to the result of the gluing P_1 and P_2 with small neighborhoods of I_1 and I_2 removed. The corresponding facets must be glued together. The operation in Figure 4 is the special case of this operation.

The operation in Figure 4 is called the *cutting edge operation* $\sharp^e$ (from left to right) and its inverse $(\sharp^e)^{-1}$ (from right to left). The left figure shows two neighborhoods of two edges whose neighboring facets have a same coloring in $(P_1,\ \lambda_1)$ and $(P_2,\ \lambda_2)$, where P_2 is the 3-sided prism, i.e., $P_2 = P^3(3) = I^1 \times \Delta^2$. We can do the connected sum along these edges, then we get a new polytope (remark that this is combinatorially equivalent to a simple polytope because of the Steinitz' theorem) with coloring $(P_1\sharp^e P_2,\ \lambda)$, vice versa (also see [**7**, Section 2]).

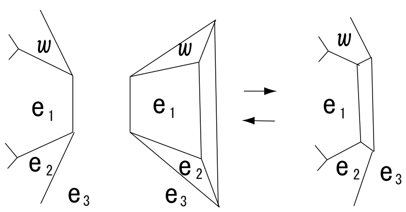

FIGURE 4. The cutting edge operation $\sharp^e$ (from left to right) and its inverse $(\sharp^e)^{-1}$ (from right to left). Here, e_3 is the coloring of the facet on the back, and w is an element in $\mathbb{Z}_2^3$ such that the property $(\star)$ holds around vertices (see Figure 2).

From the geometric point of view, the operation $\sharp^e$ corresponds with the following operation. Let $\pi_i : M_i \to P_i$ be a small cover and I_i an edge in P_i for $i = 1,\ 2$. Suppose that $P_2 = P^3(3)$ and colorings of facets around I_1 and I_2 are same as in Figure 4. Then we see that a closed invariant tubular neighborhood N_1 of $\pi_1^{-1}(I_1)$ is equivariantly isomorphic to a closed invariant tubular neighborhood N_2 of $\pi_2^{-1}(I_2)$ (see Section 2.4). Therefore, two boundaries of $M(P_1,\ \lambda_1)\backslash intN_1$ and $M(P_2,\ \lambda_2)\backslash intN_2$ are equivariantly diffeomorphic, where $intN_i$ is the interior of N_i for $i = 1,\ 2$. Hence, we can glue equivariantly these two boundaries, and get the $\mathbb{Z}_2^3$-manifold $M(P_1,\ \lambda_1)\backslash intN_1 \cup_\partial M(P_2,\ \lambda_2)\backslash intN_2$; we denote it as $M(P_1,\ \lambda_1)\sharp^e M(P_2,\ \lambda_2)$. We can easily show that $M(P_1, \lambda_1)\sharp^e M(P_2,\ \lambda_2)$ is small cover and its orbit polytope with coloring is $(P_1\sharp^e P_2,\ \lambda)$ (also see [**7**, Section 5.2]).

3.3. The cutting edge-vertex-edge operation $\sharp^{eve}$. In this paper, the operation in Figure 5 is called the *cutting e-v-e operation* $\sharp^{eve}$ (from left to right) and its inverse $(\sharp^{eve})^{-1}$ (from right to left), where e-v-e means edge-vertex-edge. Let 2-*edges* in a simple polytope P be an union of two different edges with the common vertex. The left figure shows two neighborhoods of two same colored 2-edges in $(P_1,\ \lambda_1)$ and $(P_2,\ \lambda_2)$, where P_2 is the *truncated prism*, i.e., P_2 is the polytope constructed by the connected sum of the 3-simplex Δ^3 and the 3-sided prism P_3^3: $P_2 = P_-^3(3) = P^3(3)\sharp\Delta^3$. We denote the 2-edges in $(P_1,\ \lambda_1)$ as $I_1 \vee I_2$ and that in $(P_2,\ \lambda_2)$ as $I_1' \vee I_2'$. Then we can establish a one-to-one correspondence $F_i \mapsto F_i'$ $(i = 1,\ \cdots,\ 5)$, from the facets $\{F_1,\ \cdots,\ F_5\}$ containing $I_1 \vee I_2$ as $I_1 = F_2 \cap F_5$ and $I_2 = F_3 \cap F_5$ to the facets $\{F_1',\ \cdots,\ F_5'\}$ containing $I_1' \vee I_2'$ as $I_1' = F_2' \cap F_5'$ and $I_2' = F_3' \cap F_5'$, such that $\lambda_1(F_i) = \lambda_2(F_i')$. The *cutting e-v-e operation* with respect to these data is a polytope combinatorially equivalent to the result of the gluing P_1 and P_2 with small neighborhoods of $I_1 \vee I_2$ and $I_1' \vee I_2'$ removed. The corresponding facets must be glued together. Remark that the cutting e-v-e operation of P_1 and P_2 is combinatorially equivalent to a simple polytope because of the Steinitz' theorem. Therefore, we have the new polytope with coloring $(P_1\sharp^{eve}P_2,\ \lambda)$ (also see [**7**, Section 2]).

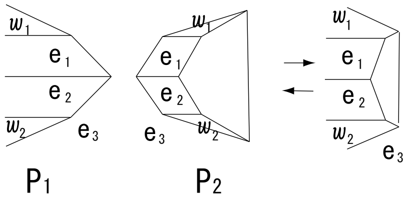

FIGURE 5. The cutting e-v-e operation $\sharp^{eve}$ (from left to right) and its inverse $(\sharp^{eve})^{-1}$ (from right to left). Here, e_3 is the coloring of the facet on the back, and $w_1,\ w_2$ are elements in $\mathbb{Z}_2^3$ such that the property $(\star)$ holds around vertices.

From the geometric point of view, this operation $\sharp^{eve}$ corresponds with the following operation. Let $\pi_i : M_i \to P_i$ be a small cover and I_i an edge in P_i for $i = 1,\ 2$. Assume that $P_2 = P_-^3(3)$ and colorings of facets around $I_1 \vee I_2$ in P_1 and $I_1' \vee I_2'$ in P_2 are same as in Figure 5. Now we see that $\pi_1^{-1}(I_1 \vee I_2) \cong S^1 \vee S^1$ and $\pi_2^{-1}(I_1' \vee I_2') \cong S^1 \vee S^1$. Here, $S^1 \vee S^1$ is the $\mathbb{Z}_2^3$-invariant bouquet of two S^1's on the fixed points, i.e., $S^1 \vee S^1 = (S^1 \amalg S^1)/v_1 \sim v_2$, where $S^1 \amalg S^1$ is a disjoint union of two S^1's and v_i is a Z_2^3-fixed point in the i-th S^1 for $i = 1,\ 2$. Moreover, by our assumption, there are $\mathbb{Z}_2^3$-invariant closed neighborhoods N_1 and N_2 of $\pi_1^{-1}(I_1 \vee I_2)$ and $\pi_2^{-1}(I_1' \vee I_2')$, respectively, and they are equivariantly diffeomorphic. Therefore, with a method similar to that demonstrated in the previous section (Section 3.3), we have the manifold $M(P_1,\ \lambda_1)\backslash intN_1 \cup_\partial M(P_2,\ \lambda_2)\backslash intN_2$ with a $\mathbb{Z}_2^3$-action; we denote it as $M(P_1,\ \lambda_1)\sharp^{eve}M(P_2,\ \lambda_2)$. We can easily show that $M(P_1, \lambda_1)\sharp^{eve}M(P_2,\ \lambda_2)$ is a small cover and its orbit polytope with coloring is $(P_1\sharp^{eve}P_2,\ \lambda)$ (see [**7**, Section 5.3] for detail).

3.4. The equivariant Dehn surgery $\natural$. The operation described in Figure 6 is called the *equivariant Dehn surgery* $\natural$ (from left to right) and its inverse $\natural^{-1}$ (from right to left). If a coloring around an edge is 3-coloring (i.e., from geometric point of view, an equivariant normal bundle which is weakly equivariantly isomorphic to $S^1 \times_{\mathbb{Z}_2} D(\underline{\mathbb{R}} \oplus \underline{\mathbb{R}})$ by Proposition 2.1), then we can do this operation $\natural$ for $(P,\ \lambda)$. Remark that the obtained object by $\natural$ might not be a convex polytope (also see [**3**]); however, we get the coloring on this object which satisfies the property $(\star)$ around a vertex. We denote the object obtained by the operation $\natural$ as $\natural(P,\ \lambda)$.

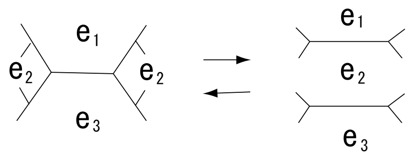

FIGURE 6. The equivariant Dehn surgery $\natural$ (from left to right) and its inverse $\natural^{-1}$ (from right to left).

From the geometric point of view, this operation $\natural$ corresponds with the following operation. Let $\mathbb{Z}_2^3$ act on the last three coordinate in $S^3 \subset \mathbb{R} \oplus \mathbb{R}^3$, naturally. Then there is the $\mathbb{Z}_2^3$-invariant submanifold $S^1 \subset \mathbb{R}^2 \oplus \{0\} \oplus \{0\}$ (remark that there are two fixed points in this submanifold), and we can take its tubular neighborhood $S^1 \times D^2 \cong S^1 \times_{\mathbb{Z}_2} D(\underline{\mathbb{R}} \oplus \underline{\mathbb{R}})$. We next consider $S^3 = S^1 \times D^2 \cup D^2 \times S^1$, and prepare the $\mathbb{Z}_2^3$-invariant part $S^3 \backslash S^1 \times D^2 = D^2 \times S^1$ (remark that there is no fixed points in this manifold). Then we remove the invariant neighborhood $S^1 \times D^2 (\simeq S^1 \times_{\mathbb{Z}_2} D(\underline{\mathbb{R}} \oplus \underline{\mathbb{R}}))$ around S^1 from $M(P,\ \lambda)$, i.e., we take $M(P,\ \lambda)\backslash S^1 \times D^2$. Finally we glue these two invariant manifold $M(P,\ \lambda)\backslash S^1 \times D^2 \cup_\partial S^3 \backslash S^1 \times D^2$. This operation is identically the *equivariant Dehn surgery* of three dimensional manifold (also see [**3**, Definition 5] or [**7**, Section 5.4]). We denote the manifold obtained by this equivariant surgery as $\natural(M(P,\ \lambda))$. If $\natural(P,\ \lambda)$ is a convex polytope, then $\natural(M(P,\ \lambda)) = M(\natural(P,\ \lambda))$.

3.5. The equivariant Dehn surgery $\flat$. The operation described in Figure 7 is called the *equivariant Dehn surgery* $\flat$ (from left to right) and its inverse $\flat^{-1}$ (from right to left). This move is also called the *bistellar* 1*-move* (see [**1**, Chapter 2]: remark that the operation in Figure 7 corresponds with the dual of the bistellar 1-move in [**1**]). If a coloring around an edge is 4-coloring as that in Figure 7 (i.e., from geometric point of view, an equivariant normal bundle which is weakly equivariantly isomorphic to $S^1 \times_{\mathbb{Z}_2} D(\mathbb{R} \oplus \mathbb{R})$ by Proposition 2.1) , then we can do this operation $\flat$. We describe the object with coloring obtained by this operation as $\flat(P,\ \lambda)$. Remark that if the polytope P is prime, i.e, P is not decomposed into connected sum of two different polytopes, then this operation $\flat$ does not destroy the convex property (see [**11**]). We also remark that this move is the same as the connected sum along edges between $(P,\ \lambda)$ and $(\Delta^3,\ \sigma\circ\lambda_0)$ for some $\sigma \in GL(3;\mathbb{Z}_2)$.

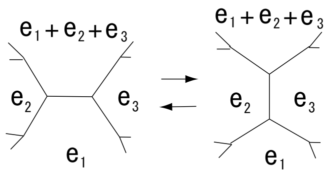

FIGURE 7. The equivariant surgery $\flat$ (from left to right) and its inverse $\flat^{-1}$ (from right to left).

Now we may explain what happens from the geometric point of view. We can regard $\mathbb{R}P(3)$ as $S^3/\mathbb{Z}_2$ by the antipodal involution $\mathbb{Z}_2$. Then we can consider $\mathbb{R}P(3) = S^1 \times_{\mathbb{Z}_2} D^2 \cup D^2 \times_{\mathbb{Z}_2} S^1$, where $S^1 \times_{\mathbb{Z}_2} D^2 \simeq S^1 \times_{\mathbb{Z}_2} D(\mathbb{R}\oplus\mathbb{R})$. We first prepare the $\mathbb{Z}_2^3$-invariant part $\mathbb{R}P(3)\backslash S^1 \times_{\mathbb{Z}_2} D^2 = D^2 \times_{\mathbb{Z}_2} S^1$ (remark that there are two fixed points in this manifold). Next, we remove the invariant neighborhood $S^1 \times_{\mathbb{Z}_2} D^2$ around S^1 from $M(P,\ \lambda)$, i.e., we take $M(P,\ \lambda)\backslash S^1 \times_{\mathbb{Z}_2} D^2$. Finally, we glue these two invariant manifold $M(P,\ \lambda)\backslash S^1 \times_{\mathbb{Z}_2} D^2 \cup_\partial \mathbb{R}P(3)\backslash S^1 \times_{\mathbb{Z}_2} D^2$. We denote the manifold obtained by this operation as $\flat(M(P,\ \lambda))$. If $\flat(P,\ \lambda)$ is a convex polytope, then $\flat(M(P,\ \lambda)) = M(\flat(P,\ \lambda))$.

3.6. The operation $\sharp^\Delta$. The operation described in Figure 8 is called the *operation* $\sharp^\Delta$ (from left to right) and its inverse $(\sharp^\Delta)^{-1}$ (from right to left). If a coloring around a triangle facet is 2-independent coloring (i.e., their colorings can be choose from one of $\{e_1,\ e_2,\ e_1+e_2\}$), then we can do this operation $\sharp^\Delta$. Using the Steinitz' theorem, $P_1\sharp^\Delta P_2$ is a convex, simple polytope (also see [**7**, Section 2.2]).

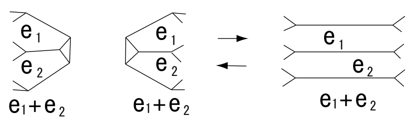

FIGURE 8. The operation $\sharp^\Delta$ (from left to right) and its inverse $(\sharp^\Delta)^{-1}$ (from right to left).

From the geometric point of view, a neighborhood of a triangle facet whose neighboring facets are 2-independent coloring corresponds with an invariant normal bundle which is weakly equivariantly isomorphic to $\mathbb{R}P(2)\times I^1$ with the standard $\mathbb{Z}_2^3$-action (i.e., the first $\mathbb{Z}_2^2$ acts on $\mathbb{R}P(2)$ and the last $\mathbb{Z}_2$ acts on $I^1 = [-1,1] \subset \mathbb{R}$ naturally), by computing its characteristic function. Therefore, we have that the operation $\sharp^\Delta$ corresponds with the following operation. We first remove an open invariant neighborhood $\mathbb{R}P(2)\times intI^1$ from $M(P_i,\ \lambda_i)$, i.e., $M(P_i,\ \lambda_i)\backslash\mathbb{R}P(2)\times intI^1$ for $i = 1,\ 2$. Next we glue these two manifolds along boundaries, i.e., $M(P_1,\ \lambda_1)\backslash\mathbb{R}P(2)\times intI^1 \cup_\partial M(P_2,\ \lambda_2)\backslash\mathbb{R}P(2)\times intI^1$; we denote it as $M(P_1,\ \lambda_1)\sharp^\Delta M(P_2,\ \lambda_2)$. We can easily show $M(P_1,\ \lambda_1)\sharp^\Delta M(P_2,\ \lambda_2)$ is a small cover and its orbit polytope with coloring is $(P_1\sharp^\Delta P_2,\ \lambda)$ (also see [**7**, Section 5.5]).

3.7. The coloring change $\sharp^C$. We explain the operation in Figure 9. Let F be an l-gon facet whose neighboring facets are 2-independent coloring in $(P,\ \lambda)$ (see the left bottom polytope in Figure 9). Then we can construct a l-sided prism $P^3(l) = F \times I^1$ (see the left above polytope in Figure 9), which naturally admits a coloring such that the coloring of the neighboring facets around the bottom facet (or top facet) is the same as that of F in $(P,\ \lambda)$. Next we glue these two polytopes along the facet F, and we have the new polytope; we denote this new polytope as $P\sharp^C P^3(l)$. Remark that two polytopes P and $P\sharp^C P^3(l)$ are combinatorially equivalent; however, the colorings of the l-gon facet F in P and $P\sharp^C P^3(l)$ are different (also see [**7**, Section 2.3]).

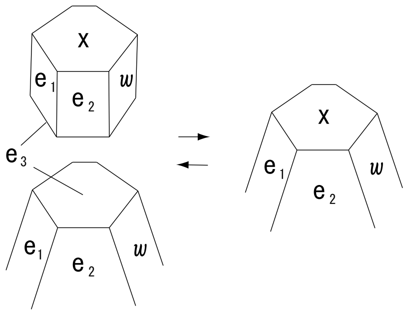

FIGURE 9. The coloring change $\sharp^C$ (from left to right) and its inverse $(\sharp^C)^{-1}$ (from right to left), where $w = e_1$ or $e_1 + e_2$ and the left e_3 means the coloring of the bottom of prism and the top of polytope. We also have $x = e_1 + e_3,\ e_2 + e_3$ or $e_1 + e_2 + e_3$.

From the geometric point of view, this operation corresponds with a (geometric) operation similar to that described in Section 3.6 (see [**7**, Section 5.6] for detail).

4. Main theorem and corollaries

In this section, we shall prove the main theorem of this paper.

4.1. Main theorem. First we prove the following main theorem.

THEOREM 4.1. *The operations $\flat$, $\sharp^e$, $\sharp^{eve}$ can be obtained by using $\sharp$, $\natural$ as follows:*

(1) $\flat = \natural \circ (\sharp\Delta^3)$, *i.e.,* $\flat(P,\ \lambda) = \natural((P,\ \lambda)\sharp(\Delta^3,\ \lambda'))$;
(2) $\sharp^e = \natural \circ (\sharp P^3(3))$, *i.e.,* $(P,\ \lambda)\sharp^e(P^3(3),\ \lambda') = \natural((P,\ \lambda)\sharp(P^3(3),\ \lambda'))$;
(3) $\sharp^{eve} = \natural^2 \circ (\sharp P^3_-(3))$, *i.e.,* $(P,\ \lambda)\sharp^{eve}(P^3_-(3),\ \lambda') = \natural^2((P,\ \lambda)\sharp(P^3_-(3),\ \lambda'))$,

where $\natural^2$ denotes $\natural \circ \natural$.

PROOF. These relations (1), (2) and (3) are shown by the Figures 10, 11 and 12, respectively.

In these Figures 10, 11 and 12, the upper map from left to right means the connected sum $\sharp$ on two black vertices with Δ, $P^3(3)$ and $P^3_-(3)$ respectively, and the right map from top to bottom means the equivariant Dehn surgery $\natural$ along the black edge. Remark that in Figure 12, we do the operation $\natural$ two times along two black edges. As the result, we get the formulas (1), (2), (3) in the statement. $\square$

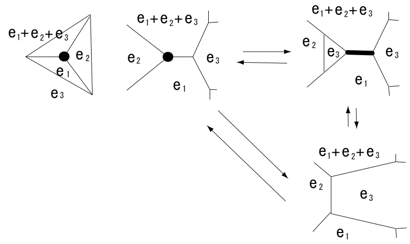

FIGURE 10. $\flat = \natural \circ (\sharp\Delta)$.

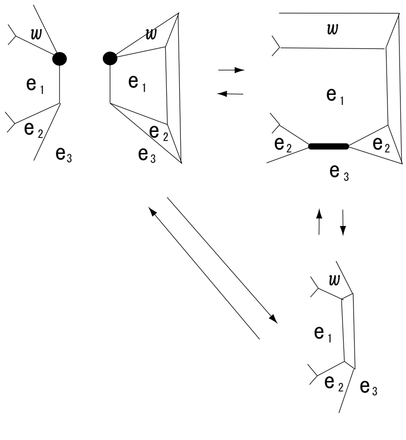

FIGURE 11. $\sharp^e = \natural \circ (\sharp P^3(3))$.

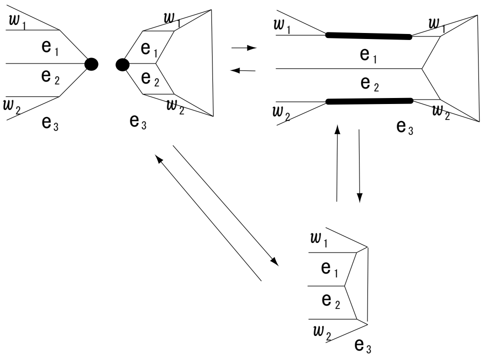

FIGURE 12. $\sharp^{eve} = \natural^2 \circ (\sharp P^3_-(3))$.

4.2. Constructions of oriented small covers. In this and next subsection, we apply our main theorem (Theorem 4.1) to constructions of the three-dimensional small covers. First, we recall the following Izmestiev's theorem ([**3**, Theorem 3]).

THEOREM 4.2 (Izmestiev). *Suppose that the characteristic function λ of a 3-dimensional small cover $M(P,\ \lambda)$ satisfies that $\lambda(\mathcal{F}) = \{e_1,\ e_2,\ e_3\}$, i.e., $M(P,\ \lambda)$ is a 3-dimensional linear model. Then $M(P,\ \lambda)$ can be (equivariantly) constructed* *from the* 3*-dimensional torus* T^3 *by using finite times following two operations: the equivariant connected sum* $\sharp$*; and the equivariant Dehn surgery* $\natural$*.*

Nishimura generalizes the above theorem to the following theorem ([**11**, Theorem 1.10]).

THEOREM 4.3 (Nishimura). *Suppose that the characteristic function* λ *of a* 3*-dimensional small cover* $M(P,\ \lambda)$ *satisfies that* $\lambda(\mathcal{F}) \subset \{e_1,\ e_2,\ e_3,\ e_1+e_2+e_3\}$*, i.e.,* $M(P,\ \lambda)$ *is an oriented small cover. Then* $M(P,\ \lambda)$ *can be (equivariantly) constructed from* T^3 *and the* 3*-dimensional real projective space* $\mathbb{R}P(3)$ *by using finite times following three operations: the equivariant connected sum* $\sharp$*; and the equivariant Dehn surgeries* $\natural$ *and* $\flat$*.*

By Section 2.3, we see that the small cover over Δ^3 is $\mathbb{R}P(3)$. Therefore we can prove the following corollary by applying Theorem 4.1 (1) to the above Nishimura's theorem.

COROLLARY 4.4. Each 3-dimensional oriented small cover can be (equivariantly) constructed from T^3 and $\mathbb{R}P(3)$ by using finite times following two operations: the equivariant connected sum $\sharp$; and the equivariant Dehn surgery $\natural$.

4.3. Constructions of all small covers. For all 3-dimensional small covers, the following Lü-Yu's theorem are known ([**7**, Theorem 1.1, 1.2]).

THEOREM 4.5 (Lü-Yu). *Each* 3*-dimensional small cover can be (equivariantly) constructed from* $\mathbb{R}P(3)$ *and* $S^1 \times \mathbb{R}P(2)$ *with certain four type* $\mathbb{Z}_2^3$*-actions (up to weakly equivariant diffeomorphism) by using the following six operations:* $\sharp$*;* $\sharp^e$*;* $\sharp^{eve}$*;* $\natural$*;* $\sharp^\Delta$ *and* $\sharp^C$*.*

REMARK 4.6. In this paper, we abuse the notations of the operations on small covers and on polytopes.

We shall explain the four types $\mathbb{Z}_2^3$-actions on $S^1 \times \mathbb{R}P(2)$ in Theorem 4.5. The manifold $S^1 \times \mathbb{R}P(2)$ is the small cover over the three sided prism $P^3(3) = I^1 \times \Delta^2$. The four types $\mathbb{Z}_2^3$-actions are defined by using the coloring in Figure 13. We call them (from left) $M(P^3(3),\ \lambda_1)$, $M(P^3(3),\ \lambda_2)$, $M(P^3(3),\ \lambda_3)$, $M(P^3(3),\ \lambda_4)$, respectively.

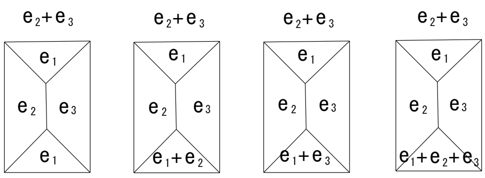

FIGURE 13. Basic four $\mathbb{Z}_2^3$-actions on $S^1 \times \mathbb{R}P(2)$ in Theorem 4.5; $M(P^3(3),\ \lambda_1)$, $M(P^3(3),\ \lambda_2)$, $M(P^3(3),\ \lambda_3)$, $M(P^3(3),\ \lambda_4)$. Here, e_2+e_3 is the coloring of facets on the back.

Remark that colorings of neighboring facets of triangle facets in Figure 13 are 2-independent. Therefore, we can do $\sharp^\Delta$ (see Section 3.6) for these manifolds as Figure 14.

In Figure 14, we remark that the middle two $P^3(3)$'s are equivariantly diffeomorphic to $M(P,\ \lambda_2)$ because there is the combinatorially equivalent to the

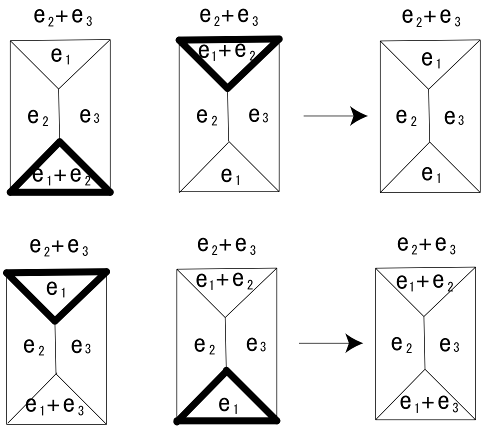

FIGURE 14. $M(P,\ \lambda_1)$ and $M(P,\ \sigma\circ\lambda_4)$ can be constructed from $M(P,\ \lambda_2)$ and $M(P,\ \lambda_3)$ by using $\sharp^{\Delta}$. This figure shows that we do $\sharp^{\Delta}$ along black facets.

left-second figure (i.e., $M(P^3(3),\ \lambda_2)$) in Figure 13 which preserves the colorings (from the geometric point of view, this equivalence on polytopes corresponds with the equivariant diffeomorphisms on small covers). We can easily show that the coloring of the right bottom figure in Figure 14 is same as the right figure (i.e., $M(P^3(3),\ \lambda_4)$) in Figure 13 up to $GL(3,\ \mathbb{Z}_2)$-equivariant isomorphism, by using the following $\sigma\in GL(3,\ \mathbb{Z}_2)$:

$$\sigma=\begin{pmatrix}1&0&0\\1&1&0\\0&0&1\end{pmatrix}.$$

By the above argument, we have the following lemma.

LEMMA 4.7. *For the basic small covers in Theorem 4.5 (Figure 13), the following relations hold:*

$$(P^3(3),\ \lambda_1)=(P^3(3),\ \lambda_2)\sharp^{\Delta}(P^3(3),\ \lambda_2);$$
$$(P^3(3),\ \sigma\circ\lambda_4)=(P^3(3),\ \lambda_2)\sharp^{\Delta}(P^3(3),\ \lambda_3).$$

Next we remark that the operation $\sharp^e$ (resp. $\sharp^{eve}$) itself decompsed as the connected sum of $P^3(3)$ (resp. $P^3_-(3)$) with Dehn surgery. However, the 3-sided prism with coloring $(P^3(3),\ \lambda)$ which is not included in Figure 13 can be constructed as $(\Delta,\ \lambda_0)\sharp(\Delta,\ \lambda_0)$ using the argument in [**7**, Section 3]. Moreover, all truncated prisms with colorings $(P^3_-(3),\ \lambda)$ can be constructed as $(\Delta,\ \lambda_0)\sharp(P^3(3),\ \lambda)$ using the argument in [**7**, Section 3 (a)]. Hence, we have the following corollary (Corollary 4.8) by applying Theorem 4.1 (2), (3) and Lemma 4.7 to the Lü-Yu's theorem (Theorem 4.5).

COROLLARY 4.8. Each 3-dimensional small cover can be (equivariantly) constructed from $\mathbb{R}P(3)$, $M(P^3(3),\ \lambda_2)$ and $M(P^3(3),\ \lambda_3)$ (up to weakly equivariant diffeomorphism) by using the following four operations: $\sharp$; $\natural$; $\sharp^{\Delta}$ and $\sharp^{C}$.

5. Relation between the Nishimura's theorem and the Lü-Yu's theorem

In this section, we give the relations among the Izmestiev's, Nishimura's and Lü-Yu's theorem.

5.1. Relation of Corollary 4.4 and the operation $\sharp^C$. In this subsection, we apply the operation $\sharp^C$ to Corollary 4.4 for oriented small covers.

The torus with standard $\mathbb{Z}_2^3$-action $T^3 = M(I^3,\ \lambda_0^I)$ can be constructed from $(\Delta^3,\ \lambda_0)$ by using $\sharp$, $\natural$ and $\sharp^C$ as Figure 15.

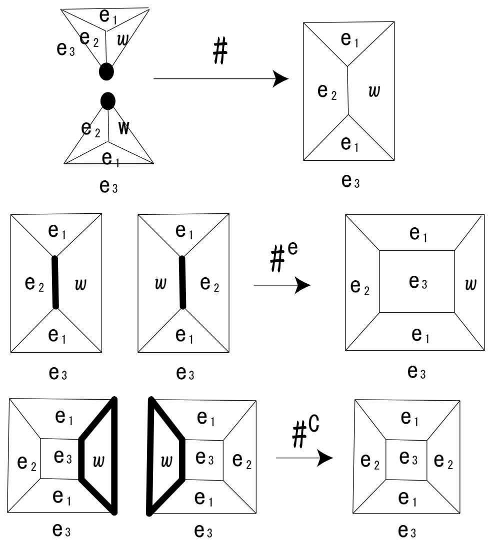

FIGURE 15. $(I^3,\ \lambda_0^I)$ can be constructed from $(\Delta^3,\ \lambda_0)$ by using $\sharp$, $\natural$ and $\sharp^C$, where $w = e_1 + e_2 + e_3$.

In the Figure 15, the first figure shows that the connected sum on the black vertices; then, we get $((\Delta^3,\ \lambda_0))\sharp(\Delta^3,\ \lambda_0) = (P^3(3),\ \lambda)$. The second figure shows that the connected sum along the black edges: this operation is identical with $\sharp^e = \natural \circ (\sharp P^3(3))$, and we get $(P^3(3),\ \lambda)\sharp^e(P^3(3),\ \lambda) = (I^3,\ \lambda')$. In $(I^3,\ \lambda')$ (the second right figure I^3), the colorings around the square facets with coloring w are 2-independent; therefore, we can do $\sharp^C$ along this facets. The third figure shows this coloring change, i.e., the coloring change along the black square facets and we get $(I^3,\ \lambda')\sharp^C(I^3,\ \lambda') = (I^3,\ \lambda_0^I)$.

Therefore, we have the following proposition by applying the above argument to Corollary 4.4.

PROPOSITION 5.1. Each 3-dimensional oriented small cover can be (equivariantly) constructed from $\mathbb{R}P(3)$ by using finite times following three operations: the equivariant connected sum $\sharp$; the equivariant Dehn surgery $\natural$; and the coloring change $\sharp^C$.

5.2. Problem. The above Proposition 5.1 shows that Corollary 4.8 is not the (direct) generalization of Corollary 4.4. Now we may give the relations among Theorem 4.2, Corollary 4.4, 4.8 and Proposition 5.1 as the following list:

linear model	oriented	3-dimensional small covers
Izmestiev (Theorem 4.2)	Nishimura (Corollary 4.4)	??
	Proposition 5.1	Lü-Yu (Corollary 4.8)

Here, the column in the list means the category of 3-dimensional small covers, and this list means that Corollary 4.4 is the generalization of Theorem 4.2 and Corollary 4.8 is the generalization of Proposition 5.1. So we can ask the following problem:

PROBLEM 5.2. What is the generalization of Corollary 4.4 for 3-dimensional small covers? In other wards, what are basic small covers which construct all 3-dimensional small covers using operations $\sharp$, $\natural$ (or $\sharp^{\Delta}$)?

6. On 2-torus manifolds

In this final section, we shall give some remark for 2-torus manifolds. A *2-torus manifold* M^n is an n-dimensional, closed smooth manifold with a non-free effective smooth $\mathbb{Z}_2^n$-action (see Section 2.1). In this paper, we are interested in the $n = 3$ case. First we give some basic facts for 2-torus manifolds (see [**5, 6**] for detail).

6.1. Basics of 2-torus manifolds. Let $\pi : M^3 \to M^3/\mathbb{Z}_2^3$ be the orbit projection of the 2-torus manifold M. We see that the orbit space $M^3/\mathbb{Z}_2^3$ is a three dimensional closed space. If $\mathbb{Z}_2^3$-action is locally standard then the orbit space is a 3-dimensional manifold with corner. Moreover, we can define the cell decomposition on $\partial M^3/\mathbb{Z}_2^3$ (we call it the *orbit cell decomposition*) induced from the information of $\mathbb{Z}_2^3$-orbits as follows: the information of fixed points as 0-cells (vertices); the information of rank one-orbits (i.e., orbits $\mathbb{Z}_2^3/K \simeq \mathbb{Z}_2$) as 1-cells (edges); the information of rank two-orbits (i.e., orbits $\mathbb{Z}_2^3/K \simeq \mathbb{Z}_2^2$) as 2-cells (facets); we remark that the free orbits correspond with the interior of $M^3/\mathbb{Z}_2^3$. With a method similar to that defines the characteristic function on the small cover, we can define the characteristic function from facets in $\partial M^3/\mathbb{Z}_2^3$ to $\mathbb{Z}_2^3$. For example, $\mathbb{Z}_2^3$ acts canonically on the last three coordinates of $S^3 \subset \mathbb{R} \oplus \mathbb{R}^3$. Then S^3 is a 2-torus manifold (not a small cover), and its orbit space with characteristic function is as that in Figure 16 by computing isotropy subgroups on rank two-orbits.

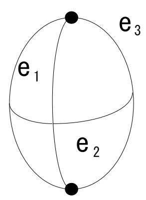

FIGURE 16. The orbit cell decomposition on $S^3/\mathbb{Z}_2^3$. We describe this orbit cell decomposition with coloring as $(D^3,\ \rho)$

Figure 16 shows that $S^3/\mathbb{Z}_2^3$ becomes the 3-disk D^3, and its orbit cell decomposition is as follows: two vertices; three edges; and three facets. We remark that, in Figure 16, e_3 is the coloring of the facet on the back.

We denote the orbit cell decomposition with coloring in Figure 16 as $(D^3,\ \rho)$

6.2. The equivariant surgery $\natural_0$. For the 2-torus manifold, we can define the new operation $\natural_0$ introduced from the (geometric) equivariant surgery. First, we explain this operation $\natural_0$.

The *equivariant surgery* $\natural_0$ is the operation described in Figure 17. As we see in Figure 16, generally in the orbit cell decomposition of 2-torus manifolds, there is the multi-edge (i.e., two vertices connected by more than two edges, also see the left figures in Figure 17). The left figures in Figure 17 show the neighborhood around the multi-edge with coloring in the orbit cell decomposition $(Q,\ \lambda)$ and $(D^3,\ \rho)$. First we take two facets $D_1 \subset Q$ and $D_2 \subset D^3$ as the facets surrounded by black edges in Figure 17. The equivariant surgery $\natural_0$ is the gluing Q and D^3 with small neighborhoods of D_1 and D_2 removed. As a result, we get the new orbit cell decomposition with coloring $(Q',\ \lambda')$ as in the right figure in Figure 17, vice versa. $(Q',\ \lambda')$ is denoted by $\natural_0(Q,\ \lambda)$.

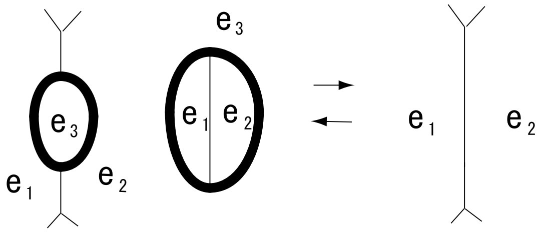

FIGURE 17. The equivariant surgery $\natural_0$ (from left to right) and its inverse $(\natural_0)^{-1}$ (from right to left).

Let D be a facet in the orbit cell decomposition Q whose boundary consists of 2 vertices and 2 edges, and $\pi : M \to Q$ be an orbit projection. We can easily show that this facet D corresponds with the invariant submanifold diffeomorphic to S^2 in M, that is, $\pi^{-1}(D) = S^2$. So we can understand the geometric meaning of the operation $\natural_0$ as follows. Because $\mathbb{Z}_2^3$ acts on the last three coordinates of $S^3 \subset \mathbb{R} \oplus \mathbb{R}^3$, there is the $\mathbb{Z}_2^3$-invariant submanifold $S^2 \subset \mathbb{R}^3 \oplus \{0\}$ (remark that there are two fixed points in this submanifold). Then its invariant, closed, tubular neighborhood is equivariantly isomorphic to $S^2 \times D^1$ with the standard $\mathbb{Z}_2^3$-action, where $D^1 = I^1$. We next consider $S^3 = S^2 \times D^1 \cup D^3 \times S^0$, and prepare the $\mathbb{Z}_2^3$-invariant part $S^3 \backslash S^2 \times D^1 = D^3 \times S^0$ (remark that there is no fixed points in this disconnected manifold). Next we remove the invariant neighborhood $S^2 \times D^1$ around $S^2 = \pi^{-1}(D)$ from the 2-torus manifold M, i.e., we take $M \backslash S^2 \times D^1$. Finally we glue these two invariant manifold $M \backslash S^2 \times D^1 \cup_\partial S^3 \backslash S^2 \times D^1$. This operation is identically the *equivariant surgery* of three dimensional manifold which is different form the equivariant surgery explained in Section 3.4.

6.3. Remark on the operations $\sharp^\Delta$ and $\sharp^C$. Because we can easily regard the equivariant connected sum as the equivariant surgery, the three type different operations $\sharp$, $\natural$ and $\natural_0$ in this paper are introduced by the (geometric) equivariant surgeries. Finally in this paper, we prove that $\sharp^\Delta$ and $\sharp^C$ can be constructed by these equivariant surgeries, i.e., we prove the following theorem.

THEOREM 6.1. *The operations $\sharp^\Delta$, $\sharp^C$ can be obtained by using $\natural$, $\natural_0$ and $\sharp$ as follows:*

(1) $\sharp^{\Delta} = \natural_0 \circ \natural \circ \sharp$, *i.e.*, $(P_1,\ \lambda)\sharp^{\Delta}(P_2,\ \lambda') = \natural_0 \circ \natural((P_1,\ \lambda)\sharp(P_2,\ \lambda'))$;
(2) $\sharp^{C} P^3(l) = \natural_0 \circ \natural^{l-2} \circ \sharp$, *i.e.*, $(P,\ \lambda)\sharp^{C}(P^3(l),\ \lambda') = \natural_0 \circ \natural \circ \cdots \circ \natural((P,\ \lambda)\sharp(P^3(l),\ \lambda'))$,
where $P^3(l)$ *is the* l*-sided prism,* $l \geq 3$ *and* $\natural^{l-2}$ *denotes* $(l-2)$*-times* $\natural \circ \cdots \circ \natural$.

PROOF. These relations (1) and (2) are shown by the Figures 18 and 19, respectively.

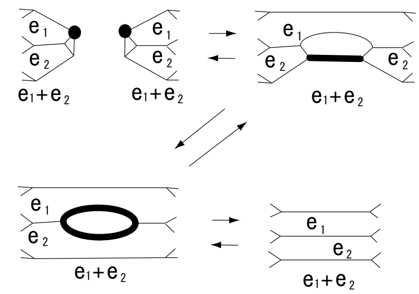

FIGURE 18. $\sharp^{\Delta} = \natural_0 \circ \natural \circ \sharp$.

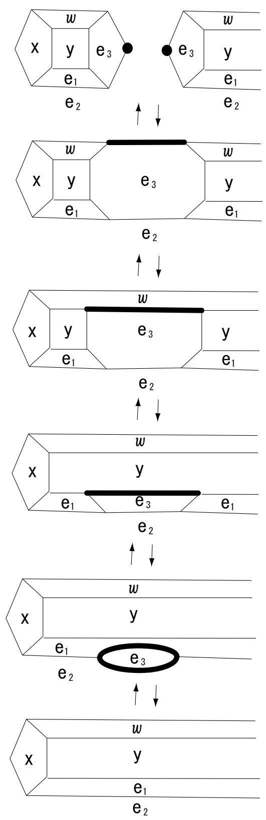

FIGURE 19. $\sharp^{C} P^3(l) = \natural_0 \circ \natural^{l-2} \circ \sharp$.

In Figures 18, the picture on the top from left to right means the connected sum $\sharp$ on two black vertices. Next we can do $\natural$ along the black edge; then we have the 2-gon facet as in the left bottom figure. Finally we can do $\natural_0$ for this 2-gon facet. As the result, we have the operation $\sharp^{\Delta} = \natural_0 \circ \natural \circ \sharp$.

In Figure 19, we explain the $l = 5$ case only. However, we can easily apply the same argument for all $l \geq 3$. In Figure 19, the first map means the connected sum $\sharp$ on two black vertices. Then we have the $(2l-2)$-gon facet (in Figure 19, this facet is the 8-gon). Next we can do $\natural$ along the black edge as in the second figure; then, this facet becomes the $(2l-4)$-gon facet. Iterating this argument ($(l-2)$-times), finally we have the 2-gon facet. Then we can do $\natural_0$ along this 2-gon facet. As the result, we have the operation $\sharp^C P^3(l) = \natural_0 \circ \natural^{l-2} \circ \sharp$. □

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