Singular positive solutions for a fourth order elliptic problem in $\mathbb{R}^N$

Tokushi Sato[1]
Mathematical Institute, Tohoku University, 980-8578 Sendai, Japan

and

Tatsuya Watanabe[2]
Osaka City University Advanced Mathematical Institute, 3-3-138, Sugimoto, Sumiyoshi-ku, Osaka 558-8585, Japan

Abstract

In this paper, we consider the following fourth order elliptic problem:

$$\Delta^2 u - c_1 \Delta u + c_2 u = u^p + \kappa \sum_{i=1}^{m} \alpha_i \delta_{a_i} \text{ in } \mathcal{D}'(\mathbb{R}^N),$$

$$u(x) > 0,\ u(x) \to 0 \text{ as } |x| \to \infty.$$

We will prove if $0 < \kappa < \kappa^*$ for some $\kappa^* \in (0, \infty)$, then this problem has at least two singular positive solutions.

Keywords: Fourth order elliptic problem, Singular solutions, Minimal solutions, Mountain Pass methods

1 Introduction

In this paper, we consider the following fourth order elliptic problem:

$$\begin{cases} \Delta^2 u - c_1 \Delta u + c_2 u = u^p + \kappa \sum_{i=1}^{m} \alpha_i \delta_{a_i} \text{ in } \mathcal{D}'(\mathbb{R}^N), \\ u(x) > 0,\ u(x) \to 0 \text{ as } |x| \to \infty, \end{cases} \tag{1.1}$$

where $N \geq 5$, $\kappa > 0$, $m \in \mathbb{N}$, $\alpha_i > 0$ and c_1, $c_2 > 0$. Here we denote by δ_{a_i} the delta function supported at $a_i \in \mathbb{R}^N$. We assume further that

$$1 < p < \frac{N}{N-4} \text{ and } c_1^2 - 4c_2 \geq 0.$$

We observe that if $u \in L^p_{loc}(\mathbb{R}^N)$ satisfies $\Delta^2 u - c_1 \Delta u + c_2 u = u^p$ in $\mathcal{D}'(\mathbb{R}^N \setminus \bigcup_{i=1}^m \{a_i\})$, then $u \in C^4(\mathbb{R}^N \setminus \bigcup_{i=1}^m \{a_i\})$ (see Proposition 3.1 below). In this paper, we say that u is a solution of (1.1) when $u \in L^p_{loc}(\mathbb{R}^N)$ and it satisfies (1.1).

For simplicity, we write $L = \Delta^2 - c_1 \Delta + c_2 I$ and $f = \sum_{i=1}^m \alpha_i \delta_{a_i}$. Then problem (1.1) can be written by $Lu = u^p + \kappa f$. Moreover the condition $c_1^2 - 4c_2 \geq 0$ enables us to rewrite (1.1) by the following elliptic system:

$$-\Delta u + t_1 u = v, -\Delta v + t_2 v = u^p + \kappa f. \tag{1.2}$$

Here $t_1 = \dfrac{c_1 - \sqrt{c_1^2 - 4c_2}}{2}$ and $t_2 = \dfrac{c_1 + \sqrt{c_1^2 - 4c_2}}{2}$. We note that $0 < t_1 \leq t_2$. We also put $L_1 = -\Delta + t_1 I$ and $L_2 = -\Delta + t_2 I$. In this situation, we obtain the following result.

[1]E-mail address:tokushi@math.tohoku.ac.jp
[2]Email address:watanabe@sci.osaka-cu.ac.jp

Theorem 1.1. *There exists $\kappa^* \in (0, \infty)$ satisfying the following properties:*
(i) Problem (1.1) has no solution for all $\kappa > \kappa^*$.
(ii) For any $\kappa \in (0, \kappa^*)$, there exist two solutions u_κ and u^κ of (1.1) such that
(a) u_κ, $u^\kappa \in C^4(\mathbb{R}^N \setminus \bigcup_{i=1}^m \{a_i\})$ and $u_\kappa(x) < u^\kappa(x)$ for $x \in \mathbb{R}^N \setminus \bigcup_{i=1}^m \{a_i\}$.
(b) $u_\kappa(x)$, $u^\kappa(x) \approx \frac{1}{|x-a_i|^{N-4}}$ as $|x - a_i| \to 0, (i = 1, 2, \cdots, m)$.
(c)

$$u_\kappa(x), u^\kappa(x) \approx \begin{cases} |x|^{-\frac{N-3}{2}} e^{-\sqrt{t_1}|x|} \text{ as } |x| \to \infty & \text{if } t_1 = t_2 \\ |x|^{-\frac{N-1}{2}} e^{-\sqrt{t_1}|x|} \text{ as } |x| \to \infty & \text{if } t_1 < t_2. \end{cases}$$

Here in this paper, $\phi(x) \approx \psi(x)$ as $|x| \to \infty$ (as $|x| \to 0$) means there exist c, c' and $R_0 > 0$ such that $c' \leq \frac{\phi(x)}{\psi(x)} \leq c$ for all $|x| \geq R_0$ ($|x| \leq R_0$ respectively).

Let us summarize known results for the second order problem:

$$-\Delta u + u = u^p + \kappa f \text{ in } \mathbb{R}^N, \ u(x) > 0. \tag{1.3}$$

This problem has been studied widely when the perturbation term f is regular (see [1], [12], [15], [16], [17], [28], [43], [46]). When $f \in H^{-1}(\mathbb{R}^N)$, we can consider the energy functional which corresponds to (1.3). We can easily see that if κ is small, then there exists a local minimizer. Then we usually apply the Mountain Pass Theorem to find a second solution. However to estimate the mountain pass energy of the functional, we need that f decays exponentially faster than the ground state solution $-\Delta u + u = u^p$ in $\mathbb{R}^N$ (see [46]).

On the other hand, we can see that the convexity of the nonlinear term enables us to prove the local minimizer is non-degenerate. Then we rewrite (1.3) (see (5.1) below) and consider an another energy functional. We can see that the non-degeneracy guarantees this functional has the mountain pass geometry. In this case, the decay rate of f is allowed to be algebraic (see [16], [17]). In [43], the second author studied problem (1.3) with general nonlinear terms which may not satisfy neither the convexity nor the positivity.

Singular solutions for second order elliptic problems on bounded domains also have been studied widely. See [2], [9], [10], [14], [31], [42]. For the problem in $\mathbb{R}^N$, we refer [30], [32], [35], [36]. In particular, problem (1.3) with $f = \sum_{i=1}^m \alpha_i \delta_{a_i}$ is studied in [32]. They used an iteration technique to find a first positive solution for small κ. Next they applied the bifurcation and comparison arguments and proved the existence of positive minimal solutions as long as positive solutions exist. Next they showed that positive minimal solutions are non-degenerate. Finally they used the Mountain Pass method to obtain the second positive solution. In [36], the first author studied problem (1.3) with general convex nonlinear terms and general perturbation terms which contain non-negative Radon measures.

In this paper, we use similar arguments as in [32]. Thus problem (1.1) can be regarded as the fourth order version of [32]. In second order case, we can use the maximum principle and the comparison principle to get the positivity or the decay rate of solutions. However in the fourth order case, the maximum principle does not hold in general. This makes our problem difficult although system (1.2) enables us to treat (1.1) as the second order problem.

Here we briefly summarize known results for the maximum principle of higher order elliptic operators. Let Ω be a bounded domain. In the second order problem: $-\Delta u = \phi$ in Ω and $u = 0$ on $\partial\Omega$, it is well-known that $\phi \geq 0$ in Ω implies $u \geq 0$ in Ω. This result holds for general domains and dimensions. However this positivity preserving property becomes very sensitive to the dimension, topology of domains and boundary conditions in higher order elliptic problems. Actually there are some counter-examples (see [33], [37]). Moreover we don't know whether the comparison principle near infinity holds for the fourth order operator. For the maximum principle and related results for higher order elliptic operators, we refer [6], [8], [23], [24], [25], [26], [39], [41].

Finally we introduce known results for fourth order nonlinear elliptic problems. Most papers are devoted to the critical and biharmonic case, that is, $\Delta^2 u = u^{\frac{N+4}{N-4}}$ (see [5], [7], [13], [21]). For the subcritical case, we refer [34] or [38]. They studied the existence, multiplicity and bifurcation of solutions. However fourth order problems with $-\Delta$ are less studied. In [18], [19], [22], [27], they studied the following fourth order nonlinear elliptic problem with the critical nonlinearity on the conformally flat manifold (M, g):

$$P_g u := \Delta_g^2 u + b\Delta_g u + cu = u^{\frac{N+4}{N-4}},\ \Delta_g = -\mathrm{div}_g \nabla,\ N \geq 5.$$

They also imposed the condition $c \leq \frac{b^2}{4}$, which is same as ours. The operator P_g is related to the Paneitz-Branson operator, which is conformally invariant and can be seen as a natural extension of the Laplace-Beltrami operator.

Let us introduce the organization of this paper, ideas and difficulties. In section 2, we study properties of the fundamental solution of L. In section 3, we study the characterization of positive solutions of (1.1). We will also prove properties of positive solutions. Basic ideas are similar to those in [32]. As noted earlier, nonlinear elliptic problems with singular perturbation terms f are less familiar. However there is one advantage in our fourth order problem. To prove the decay rate at infinity of positive solutions, we can not use general comparison principles. However we will see that if $u(x)$ is a positive solution of (1.1), then it follows $u(x) > cG(x)$ where $G(x)$ is the fundamental solution of the operator L. In other words, we can automatically get a positive subsolution. This enables us to estimate positive solutions from below. Moreover by the standard argument on the convolution, we can estimate positive solutions from above and obtain the precise decay rate at infinity (see Lemma 3.8 below).

In section 4, we will prove the existence of a positive minimal solution and its non-degeneracy if κ belongs to some range. First we prove the existence of a positive solution for small κ via the Implicit Function Theorem. Next we apply the continuation argument to obtain positive minimal solutions. To prove the non-degeneracy, we need to show that the first eigenfunction of the linearized eigenvalue problem:

$$L\phi(x) = \lambda p u_\kappa^{p-1} \phi(x),\ x \in \mathbb{R}^N,\ \phi \in H^2(\mathbb{R}^N)$$

has the constant sign. In the second order case, this fact is well-known. However in the fourth order case, we can not argue as in the second order case. This is one of the difficult parts when we consider higher order elliptic problems. To this aim, we use the fractional operator $(-\Delta + t_2 I)^{\frac{1}{2}}$ (see Lemma 4.5 and Remark 4.6). In the end of section 4, we will also give the non-existence result.

Finally in section 5, we will obtain the second positive solution by using the Mountain Pass Theorem.

In this paper, we use the following notation. We define the inner product:

$$(u,v) := \int_{\mathbb{R}^N} (\Delta u \Delta v + c_1 \nabla u \cdot \nabla v + c_2 uv) dx$$

and the norm:

$$\|u\|^2 := \int_{\mathbb{R}^N} (|\Delta u|^2 + c_1 |\nabla u|^2 + c_2 u^2) dx \text{ for } u, v \in H^2(\mathbb{R}^N).$$

Since c_1 and c_2 are all positive, $\|\cdot\|$ is equivalent to the usual H^2-norm.

2 Properties of the fundamental solution

In this section, we study properties of the fundamental solution of L. As we will see later, the fundamental solution of the operator L plays an important role to control the singularities of solutions for which we are looking.

Now we denote by G the fundamental solution of L, that is, $LG = \delta_0$. We put $B_r := \{x \in \mathbb{R}^N ; |x| \le r\}$ for $r > 0$. Then we have the following.

Proposition 2.1. *$G \in C^\infty(\mathbb{R}^N \setminus \{0\})$ and*
(i) $G(x) \approx \frac{1}{|x|^{N-4}}$ as $|x| \to 0$.
(ii)

$$G(x) \approx \begin{cases} |x|^{-\frac{N-3}{2}} e^{-\sqrt{t_1}|x|} \text{ as } |x| \to \infty & \text{if } t_1 = t_2 \\ |x|^{-\frac{N-1}{2}} e^{-\sqrt{t_1}|x|} \text{ as } |x| \to \infty & \text{if } t_1 < t_2. \end{cases}$$

(iii) There exists $c > 0$ such that for all $x \in \mathbb{R}^N \setminus \{0\}$,

$$0 < G(x) \le \frac{c}{|x|^{N-4}}, \ |\nabla G(x)| \le \frac{c}{|x|^{N-3}}, \ |\Delta G(x)| \le \frac{c}{|x|^{N-2}}.$$

(iv) $G \in L^q(\mathbb{R}^N)$ for all $q \in [1, \frac{N}{N-4})$.

If $c_1^2 - 4c_2 = 0$, then $L = (-\Delta + t_1 I)^2$. In this case, Proposition 2.1 is rather well-known (see [3] section 4 or [40]). Thus it suffices to consider the case $t_1 < t_2$. To prove Proposition 2.1, let G_1 and G_2 be fundamental solutions for L_1 and L_2 respectively. Then it follows $G = G_1 * G_2$ and $G \in C^\infty(\mathbb{R}^N \setminus \{0\})$. Now we use the following properties of G_1.

Lemma 2.2. *([3], [40]) G_1 satisfies the following properties.*
(i) $G_1(x) \approx \frac{1}{|x|^{N-2}}$ as $|x| \to 0$.
(ii) $G_1(x) \approx |x|^{-\frac{N-1}{2}} e^{-\sqrt{t_1}|x|}$ as $|x| \to \infty$.
(iii) $0 < G_1(x) \le \frac{c}{|x|^{N-2}}$, $|\nabla G_1(x)| \le \frac{c}{|x|^{N-1}}$, for all $x \in \mathbb{R}^N \setminus \{0\}$.
(iv) $G_1 \in L^q(\mathbb{R}^N)$ for all $q \in [1, \frac{N}{N-2})$.

Next we prove the following lemma.

Lemma 2.3. *(i)* $G_2(x) = (\frac{t_2}{t_1})^{\frac{N-2}{2}} G_1((\frac{t_2}{t_1})^{\frac{1}{2}} x)$ *for all* $x \in \mathbb{R}^N \setminus \{0\}$.
(ii) If $t_1 < t_2$*, then*

$$G(x) = G_1 * G_2(x) = \frac{1}{t_2 - t_1}(G_1(x) - G_2(x)) \text{ for all } x \in \mathbb{R}^N \setminus \{0\}.$$

Proof. (i) For simplicity, we write $\tilde{G}_2(x) := (\frac{t_2}{t_1})^{\frac{N-2}{2}} G_1((\frac{t_2}{t_1})^{\frac{1}{2}} x)$. It is sufficient to show that

$$(-\Delta + t_2 I)\tilde{G}_2 = \delta_0 \text{ in } \mathcal{S}'(\mathbb{R}^N).$$

We choose $\phi \in \mathcal{S}(\mathbb{R}^N)$ arbitrary and put $\tilde{\phi}(y) = \phi((\frac{t_1}{t_2})^{\frac{1}{2}} y)$ for $y \in \mathbb{R}^N$. Then it follows $\tilde{\phi}(0) = \phi(0)$ and

$$\Delta\tilde{\phi}(y) = \frac{t_1}{t_2}[\Delta\phi]((\frac{t_1}{t_2})^{\frac{1}{2}} y) \text{ for } y \in \mathbb{R}^N.$$

Thus we obtain

$$\int_{\mathbb{R}^N} \tilde{G}_2(x)[-\Delta\phi + t_2\phi](x)dx = \int_{\mathbb{R}^N} (\frac{t_2}{t_1})^{\frac{N-2}{2}} G_1((\frac{t_2}{t_1})^{\frac{1}{2}} x)[-\frac{t_2}{t_1}\Delta\tilde{\phi} + t_2\tilde{\phi}]((\frac{t_2}{t_1})^{\frac{1}{2}} x)dx$$

$$= \int_{\mathbb{R}^N} G_1(y)[-\Delta\tilde{\phi} + t_1\tilde{\phi}](y)dy = \tilde{\phi}(0) = \phi(0).$$

(ii) We can see that $(t_2 - t_1)G = G_1 - G_2$ because $-\Delta G + t_1 G = G_2$ and $-\Delta G + t_2 G = G_1$. □

To prove Proposition 2.1, we use the following fact: For α, $\beta > 0$ with $\alpha + \beta < N$, there exists $c_{\alpha\beta} > 0$ such that

$$[\frac{1}{|\cdot|^{N-\alpha}}] * [\frac{1}{|\cdot|^{N-\beta}}](x) = \frac{c_{\alpha\beta}}{|x|^{N-\alpha-\beta}} \text{ for } x \in \mathbb{R}^N \setminus \{0\}. \tag{2.1}$$

Proof of Proposition 2.1. (i): By Lemma 2.2 (i) and from (2.1), we have

$$\frac{1}{c}\frac{1}{|x|^{N-4}} \le G_1 * G_1(x) \le \frac{c}{|x|^{N-4}} \text{ for all } x \in B_1 \setminus \{0\}$$

for some $c > 1$. By Lemma 2.3 (ii), it follows $0 < G_2 < G_1$ in $\mathbb{R}^N \setminus \{0\}$. Thus we obtain

$$G(x) = G_1 * G_2(x) \le G_1 * G_1(x) \le \frac{c}{|x|^{N-4}} \text{ for all } x \in B_1 \setminus \{0\}.$$

On the other hand by Lemma 2.3 (i), it follows

$$G_2 * G_2(x) = (\frac{t_2}{t_1})^{\frac{N-4}{2}} G_1 * G_1((\frac{t_2}{t_1})^{\frac{1}{2}} x) \text{ for all } x \in B_1 \setminus \{0\}.$$

Then we have

$$G(x) = G_1 * G_2(x) \ge G_2 * G_2(x) = (\frac{t_2}{t_1})^{\frac{N-4}{2}} G_1 * G_1((\frac{t_2}{t_1})^{\frac{1}{2}} x)$$

$$\geq \frac{1}{c}(\frac{t_2}{t_1})^{\frac{N-4}{2}}\frac{1}{|(t_2/t_1)^{\frac{1}{2}}x|^{N-4}} = \frac{1}{c}\frac{1}{|x|^{N-4}} \text{ for all } x \in B_1 \setminus \{0\}.$$

(ii): Now we observe that $0 < G_2 < G_1$ in $\mathbb{R}^N \setminus \{0\}$ and $\frac{G_2(x)}{G_1(x)} \to 0$ as $|x| \to \infty$. Then the claim follows from Lemma 2.2 (ii).

(iii): By Lemma 2.2 (iii) and Lemma 2.3 (i), we have

$$G_2(x) = (\frac{t_2}{t_1})^{\frac{N-2}{2}}G_1((\frac{t_2}{t_1})^{\frac{1}{2}}x) \leq \frac{c_1}{|x|^{N-2}},$$

$$|\nabla G_2(x)| = |(\frac{t_2}{t_1})^{\frac{N-1}{2}}[\nabla G_1]((\frac{t_2}{t_1})^{\frac{1}{2}}x)| \leq \frac{c_1}{|x|^{N-1}} \text{ for all } x \in \mathbb{R}^N \setminus \{0\}.$$

The claim of (iii) follows from (2.1) and the facts:

$$\nabla G = [\nabla G_1]*G_2,\ \Delta G = \sum_{j=1}^{N}\frac{\partial G_1}{\partial x_j}*\frac{\partial G_2}{\partial x_j}.$$

Finally we can easily see that (iv) follows from (i)-(iii).

□

Remark 2.4. *(i) The operators L_1, $L_2 : \mathcal{S}'(\mathbb{R}^N) \to \mathcal{S}'(\mathbb{R}^N)$ are invertible and the inverse operators are given by*

$$L_1^{-1}v = G_1*v,\ L_2^{-1}v = G_2*v \text{ for } v \in \mathcal{S}'(\mathbb{R}^N).$$

Thus the operator $L : \mathcal{S}'(\mathbb{R}^N) \to \mathcal{S}'(\mathbb{R}^N)$ is also invertible and $L^{-1}v = G*v$ for $v \in \mathcal{S}'(\mathbb{R}^N)$.
(ii) For $q \in (1,\infty)$, the operator $L : W^{4,q}(\mathbb{R}^N) \to L^q(\mathbb{R}^N)$ is isomorphic. In particular, the inverse operator $L^{-1} : L^q(\mathbb{R}^N) \to W^{4,q}(\mathbb{R}^N)$ is bounded.
(iii) For $s \in \mathbb{R}$, the operator $L : H^{s+4}(\mathbb{R}^N) \to H^s(\mathbb{R}^N)$ is isomorphic and especially its inverse $L^{-1} : H^s(\mathbb{R}^N) \to H^{s+4}(\mathbb{R}^N)$ is bounded.

Remark 2.5. *(i) If $v \in L^q(\mathbb{R}^N)$ for $q \in (\frac{N}{4},\infty)$, then $G*v \in C(\mathbb{R}^N)$ and $G*v(x) \to 0$ as $|x| \to \infty$ (see [32] Lemma A.3).*
(ii) If $v \in L^\infty(\mathbb{R}^N)$ and $v(x) = O(e^{-\mu|x|})$ as $|x| \to \infty$ for some $\mu > \sqrt{t_1}$, then $G*v$, $G_1*v \in C(\mathbb{R}^N)$ and

$$G*v(x) \approx G(x),\ G_1*v(x) \approx G_1(x) \text{ as } |x| \to \infty. \tag{2.2}$$

Moreover if $v \in L^1(\mathbb{R}^N)$ and supp *$v \subset B_R$, then $G*v$, $G_1*v \in C(\mathbb{R}^N \setminus B_R)$ and (2.2) holds.*

3 Characterization of positive solutions of (1.1) and their properties

In this section, we characterize positive solutions of (1.1). We will also give some basic properties of positive solutions. First we show the following property.

Proposition 3.1. *Let $K \subset \mathbb{R}^N$ be a closed subset. If $u \in L^p_{loc}(\mathbb{R}^N \setminus K)$ satisfies $u \geq 0$ a.e. in $\mathbb{R}^N \setminus K$ and*

$$Lu = u^p \text{ in } \mathcal{D}'(\mathbb{R}^N \setminus K), \tag{3.1}$$

then $u \in C^4(\mathbb{R}^N \setminus K)$ and $u(x) \to 0$ as $|x| \to \infty$.

Proof. We choose $p < s_0 < \min\{\frac{N}{N-4}, \frac{Np}{4}\}$ arbitrary. We define a sequence $\{s_j\}$ by $\frac{1}{s_j} = \frac{p}{s_{j-1}} - \frac{4}{N}$, $j \in \mathbb{N}$. This implies

$$\frac{1}{s_j} = \frac{4}{N(p-1)} - (\frac{4}{N(p-1)} - \frac{1}{s_0})p^j.$$

Since $s_0 > p$ and $p < \frac{N}{N-4}$, we have $\frac{4}{N(p-1)} - \frac{1}{s_0} > 0$. Moreover from $s_0 < \frac{Np}{4}$, it follows $\frac{1}{s_1} = \frac{p}{s_0} - \frac{4}{N} > 0$. Thus there exists $j_1 \in \mathbb{N}$ such that $\frac{1}{s_{j_1}} < 0 < \frac{1}{s_{j_1-1}}$. Then by the definition of s_j, we have $\frac{s_{j_1-1}}{p} > \frac{N}{4}$.

We claim that $u \in L^{s_{j_1-1}}_{loc}(\mathbb{R}^N \setminus K)$. For $j = 0$, we know that $u^p \in L^1_{loc}(\mathbb{R}^N \setminus K)$ and hence $Lu \in L^1_{loc}(\mathbb{R}^N \setminus K)$. We put $v = L_1 u$. Then $L_2 v \in L^1_{loc}(\mathbb{R}^N \setminus K)$. By the result due to Brezis and Strauss [11], we have $v \in W^{1,r}_{loc}(\mathbb{R}^N \setminus K)$ for all $r \in [1, \frac{N}{N-1})$. By Sobolev's inequality, we obtain $L_1 u = v \in L^q_{loc}$ for all $q \in [1, \frac{N}{N-2})$. Applying L^p-estimate, it follows $u \in W^{2,q}_{loc}$. Using Sobolev's inequality again, we obtain $u \in L^s(\mathbb{R}^N \setminus K)$ for all $s \in [1, \frac{N}{N-4})$. Especially this implies the claim above holds for $j = 0$.

Next we suppose by induction that $u \in L^{s_j-1}_{loc}$. Then it follows $u^p \in L^{\frac{s_{j-1}}{p}}_{loc}$. By using L^q-estimate in (3.1), we have $u \in W^{4,\frac{s_{j-1}}{p}}_{loc}$. By the definition of s_j and the Sobolev's embedding, we obtain $u \in L^{s_j}_{loc}$. Thus it follows $u \in L^{s_{j_1-1}}_{loc}(\mathbb{R}^N \setminus K)$. Using L^q-estimate in (3.1) again, we have $u \in W^{4,\frac{s_{j_1-1}}{p}}_{loc}$. Since $\frac{s_{j_1-1}}{p} > \frac{N}{4}$, it follows $u \in C^\alpha_{loc}$ for some $\alpha \in (0,1)$. Applying the Schauder estimate in (3.1), we have $u \in C^{4,\alpha}_{loc}$. Thus we obtain $u \in C^4(\mathbb{R}^N \setminus K)$ and $u(x) \to 0$ as $|x| \to \infty$. □

Now we write $F(x) = \sum_{i=1}^m \alpha_i G(x - a_i)$ for simplicity. Then F satisfies $LF = f = \sum_{i=1}^m \alpha_i \delta_{a_i}$. To control the singularities of solutions of (1.1), we define a sequence of functions inductively. For $\kappa \in \mathbb{R}$ and $j \in \mathbb{N} \cup \{0\}$, we define $U_j(x) = U_j^\kappa(x)$ by

$$U_0 = \kappa F, \ U_j = G*[(U_{j-1})_+^p] + \kappa F, \ j \geq 1.$$

Then $\{U_j\}$ is an approximate sequence for a solution of (1.1). Actually in section 4, we will see that the limit is a minimal positive solution of (1.1) for $\kappa > 0$. We also note that $U_j^\kappa = \kappa F$ for $\kappa < 0$ and $j \in \mathbb{N}$ because $F > 0$. First we introduce basic properties of U_j.

Lemma 3.2. *For every $j \in \mathbb{N} \cup \{0\}$ and $\kappa > 0$, it follows*
(i) $U_j \in L^q(\mathbb{R}^N)$ for all $q \in [1, \frac{N}{N-4})$ and $0 < U_j(x) < U_{j+1}(x)$ a.e. $x \in \mathbb{R}^N$.
(ii) $U_j \in C^4(\mathbb{R}^N \setminus \bigcup_{i=1}^m \{a_i\})$, $U_j(x) \approx G(x)$ as $|x| \to \infty$ and $\frac{U_j(x)}{G(x-a_i)} \to \kappa\alpha_i$ as $x \to a_i$, $1 \leq i \leq m$.

Proof. (i) The proof is similar to that in [32]. Thus we omit the proof here.

(ii) We can easily observe that the claims follow for $j = 0$. We suppose by induction that the claims hold for some $j > 1$. By the definition of U_j, it follows $LU_j = (U_{j-1})^p$ in $\mathcal{D}'(\mathbb{R}^N \setminus \bigcup_{i=1}^m \{a_i\})$. Then we can see that $U_{j+1} \in C^4(\mathbb{R}^N \setminus \bigcup_{i=1}^m \{a_i\})$.

Next by the assumption, we have $U_j(x)^p = O(e^{-p\sqrt{t_1}|x|})$ as $|x| \to \infty$. Then by Remark 2.5 (ii), we obtain

$$U_{j+1}(x) = G*[(U_j)^p](x) + U_0(x) \approx G(x) \text{ as } |x| \to \infty.$$

Finally by Proposition 2.1, we have

$$U_j(x)^p \le c \sum_{i=1}^m G(x-a_i)^p \le c \sum_{i=1}^m \frac{1}{|x-a_i|^{(N-4)p}} \text{ for } x \in \mathbb{R}^N \setminus \{0\}.$$

Moreover for $\max\{0, 4-(N-4)p\} < \mu < N-(N-4)p$, there exists $c > 0$ such that

$$G(x) \le \frac{c}{|x|^{N-4+\mu}} \text{ for } x \in \mathbb{R}^N \setminus \{0\}.$$

Since $\mu > 4-(N-4)p$, we can apply (2.1). Then we obtain

$$0 \le G*[(U_j)^p](x) \le c \sum_{i=1}^m [\frac{1}{|\cdot|^{N-(4-\mu)}} * \frac{1}{|\cdot - a_i|^{N-(N-(N-4)p)}}](x)$$

$$= c \sum_{i=1}^m \frac{1}{|x-a_i|^{\mu+(N-4)p-4}} \text{ for } x \in \mathbb{R}^N \setminus \bigcup_{i=1}^m \{a_i\}.$$

Since $\mu < N-(N-4)p$, it follows $G*[(U_j)^p](x) = o(G(x-a_i))$ as $x \to a_i$ and hence

$$\frac{U_{j+1}(x)}{G(x-a_i)} = \frac{G*[(U_j)^p](x)}{G(x-a_i)} + \frac{U_0(x)}{G(x-a_i)} \to \kappa\alpha_i \text{ as } x \to a_i.$$

□

Next we define a sequence of numbers. Let q_0 be a number such that $p < q_0 < \min\{\frac{N}{N-4}, \frac{Np}{4}\}$. We define a sequence $\{q_j\}$ as follows.

$$\frac{1}{q_j} := \frac{1}{q_0} - (\frac{4}{N} - \frac{p-1}{q_0})j = \frac{1}{q_{j-1}} - (\frac{4}{N} - \frac{p-1}{q_0}).$$

We note that $\frac{4}{N} - \frac{p-1}{q_0} > \frac{4-N}{N} + \frac{1}{p} > 0$ because $q_0 > p$. Moreover we have $q_1 > 0$ because $q_1 = \frac{p}{q_0} - \frac{4}{N}$. Then there exists $j_0 \in \mathbb{N}$ such that $\frac{1}{q_{j_0-1}} > 0 > \frac{1}{q_{j_0}}$. Finally we put

$$\frac{1}{r_j} := \frac{1}{q_j} + \frac{4}{N} = \frac{p-1}{q_0} + \frac{1}{q_{j-1}}.$$

Then it follows $r_{j_0} > \frac{N}{4}$. Lemma 3.2 implies that the functions U_j themselves never become regular by the inductive step. However we will see that their differences become smooth by the j_0-th step. To this aim, we define a new sequence of functions $V_j = V_j^\kappa$ by $V_j = U_j - U_{j-1}$ for $j \in \mathbb{N}$. We observe that $V_j \equiv 0$ for $\kappa \le 0$. By the definitions, we also have $V_0 = U_0$, $V_1 = G*[(U_0)^p]$ and

$$V_j = G*[(U_{j-1})^p - (U_{j-2})^p] = G*[(U_{j-2} + V_{j-1})^p - (U_{j-2})^p] \text{ for } \kappa > 0.$$

Then we have the following.

Lemma 3.3. *(i)* $V_j \in L^{q_j}(\mathbb{R}^N)$ *for all* $0 \le j \le j_0 - 1$.
(ii) $V_j \in H^2 \cap L^1 \cap C(\mathbb{R}^N)$ *and* $V_j(x) \to 0$ *as* $|x| \to \infty$ *for* $j \ge j_0$.

Proof. (i): First we observe that $V_{j_0} \in L^1(\mathbb{R}^N)$ because $U_j \in L^1(\mathbb{R}^N)$ for all $j \in \mathbb{N}$ by Lemma 3.2.

By the choice of q_0, we have $U_0 \in L^{q_0}(\mathbb{R}^N)$. This implies $U_0^p \in L^{\frac{q_0}{p}}(\mathbb{R}^N)$. Now we use the following inequality: For any $p_1 \in (1, \frac{N}{4})$, there exists $c > 0$ such that if $\phi \in L^{p_1}(\mathbb{R}^N)$, then

$$\|G*\phi\|_{L^{p_2}(\mathbb{R}^N)} \le c\|\phi\|_{L^{p_1}(\mathbb{R}^N)}, \text{ where } \frac{1}{p_2} = \frac{1}{p_1} - \frac{4}{N}. \tag{3.2}$$

Actually by Proposition 2.1 (iii), it follows $G(x) \le \frac{c}{|x|^{N-4}}$. Then by the Hardy-Littlewood-Sobolev inequality ([20] Theorem 4.19), we obtain (3.2).

Now we take $p_1 = \frac{q_0}{p} < \frac{N}{4}$. Then by the definition of q_j, it follows $\frac{1}{p_2} := \frac{p}{q_0} - \frac{4}{N} = \frac{1}{q_1}$. From (3.2), we obtain $V_1 \in L^{q_1}(\mathbb{R}^N)$. Next we assume by induction that $V_{j-1} \in L^{q_{j-1}}(\mathbb{R}^N)$. By Mean Value Theorem, we have

$$(U_{j-1})^p - (U_{j-2})^p = (V_{j-1} + U_{j-2})^p - (U_{j-2})^p$$

$$\le p(V_{j-1} + U_{j-2})^{p-1} V_{j-1} = p(U_{j-1})^{p-1} V_{j-1}.$$

By Lemma 3.2, it follows $U_{j-1} \in L^{q_0}(\mathbb{R}^N)$ and hence $(U_{j-1})^{p-1} \in L^{\frac{q_0}{p-1}}(\mathbb{R}^N)$. We recall that $\frac{1}{q_{j-1}} + \frac{p-1}{q_0} = \frac{1}{r_j}$. Then by the Hölder inequality, we have $(U_{j-1})^{p-1} V_{j-1} \in L^{r_j}(\mathbb{R}^N)$. Since $\frac{1}{q_j} = \frac{1}{r_j} - \frac{4}{N}$, we can use (3.2) provided $p_1 = r_j$ and $p_2 = q_j$. Then we obtain $V_j \in L^{q_j}(\mathbb{R}^N)$.

(ii): First we show that if $V_{j_0-1} \in L^{q_{j_0-1}}(\mathbb{R}^N)$, then $V_{j_0} \in C(\mathbb{R}^N)$ and $V_{j_0}(x) \to 0$ as $|x| \to \infty$.

Now we recall that $r_{j_0} > \frac{N}{4}$ and $\frac{1}{q_{j_0-1}} + \frac{p-1}{q_0} = \frac{1}{r_{j_0}}$. From (i) and by Hölder's inequality, we have

$$(U_{j_0-1})^p - (U_{j_0-2})^p \le p(U_{j_0-1})^{p-1} V_{j_0-1} \in L^{r_{j_0}}(\mathbb{R}^N).$$

Then by Remark 2.5 (i), it follows $V_{j_0} \in C(\mathbb{R}^N)$ and $V_{j_0}(x) \to 0$ as $|x| \to \infty$.

Next by the Hardy-Littlewood-Sobolev inequality, we have

$$L_1 V_{j_0} = G_2*[(U_{j_0-1})^p - (U_{j_0-2})^p] \in L^1 \cap L^r(\mathbb{R}^N)$$

for some $r > \frac{N}{2}$. Since $N \ge 5$, it follows $L_1 V_{j_0} \in L^2(\mathbb{R}^N)$ and hence $V_{j_0} \in H^2(\mathbb{R}^N)$. Finally by induction, the same statements hold for $j > j_0$. □

Using the functions U_j and V_j, we have the following result.

Proposition 3.4. *For $\kappa > 0$, these three equations are equivalent to each other.*
(i) $u = w + U_{j_0} \in L^p_{loc}(\mathbb{R}^N)$ is a solution of (1.1).
(ii) $w \in C(\mathbb{R}^N)$ satisfies $w(x) \to 0$ as $|x| \to \infty$ and

$$w = G*[(w + U_{j_0})^p - (U_{j_0-1})^p] \geq 0 \text{ in } \mathbb{R}^N. \tag{3.3}$$

(iii) $w \in H^2(\mathbb{R}^N)$ is a positive weak solution to the problem:

$$Lw = (w + U_{j_0})^p - (U_{j_0-1})^p \text{ in } \mathbb{R}^N, \tag{3.4}$$

that is, $w > 0$ a.e. in $\mathbb{R}^N$ and it satisfies

$$\int_{\mathbb{R}^N} \Delta w \Delta \psi + c_1 \nabla w \cdot \nabla \psi + c_2 w \psi dx = \int_{\mathbb{R}^N} ((w + U_{j_0})^p - (U_{j_0-1})^p)\psi dx$$

for all $\psi \in H^2(\mathbb{R}^N)$.

It is rather useful to consider problem (3.4) to find a solution of (1.1). Moreover to prove the ordering property of positive minimal solutions with respect to $\kappa > 0$, we will use (3.3). First we establish the following result.

Lemma 3.5. *For every $w \in H^2(\mathbb{R}^N)$ and $\kappa \in \mathbb{R}$, it follows*

$$(w + U_{j_0})^p_+ - (U_{j_0-1})^p_+ \in H^{-2}(\mathbb{R}^N).$$

Proof. First we fix $\kappa > 0$. For $w \in H^2(\mathbb{R}^N)$, it is sufficient to show that

$$|\int_{\mathbb{R}^N} ((w + U_{j_0})^p_+ - (U_{j_0-1})^p_+)\psi dx| \leq c\|\psi\| \text{ for all } \psi \in H^2(\mathbb{R}^N) \tag{3.5}$$

for some $c > 0$. Since $V_{j_0} = U_{j_0} - U_{j_0-1}$, we have

$$(w + U_{j_0})^p_+ - (U_{j_0-1})^p_+ = (w + V_{j_0} + U_{j_0-1})^p_+ - (U_{j_0-1})^p_+.$$

Next we can easily show that there exists $c > 0$ such that

$$|(t + s)^p_+ - s^p_+| \leq c(|t|^p + s^{p-1}|t|) \text{ for all } s \geq 0 \text{ and } t \in \mathbb{R}.$$

Then we have

$$l.h.s\ of\ (3.5) \leq c \int_{\mathbb{R}^N} |w + V_{j_0}|^p |\psi| dx + c \int_{\mathbb{R}^N} U^{p-1}_{j_0-1} |w + V_{j_0}||\psi| dx$$

$$\leq c \int_{\mathbb{R}^N} |w|^p |\psi| dx + c \int_{\mathbb{R}^N} |V_{j_0}|^p |\psi| dx$$

$$+ c \int_{\mathbb{R}^N} |U_{j_0-1}|^{p-1} |w||\psi| dx + c \int_{\mathbb{R}^N} |U_{j_0-1}|^{p-1} |V_{j_0}||\psi| dx.$$

By Lemma 3.2 and 3.3, we know that $U_{j_0-1} \in L^p(\mathbb{R}^N)$ and $V_{j_0} \in H^2(\mathbb{R}^N)$. Moreover since $p < \frac{N}{N-4}$, it follows $\|\psi\|_{L^{2p}(\mathbb{R}^N)} \leq c\|\psi\|$ by the Sobolev inequality. Thus we obtain

$$\int_{\mathbb{R}^N} |V_{j_0}|^p |\psi| dx \leq \|V_{j_0}\|^p_{L^{2p}} \|\psi\|_{L^2} \leq c\|V_{j_0}\|^p_{L^{2p}} \|\psi\|,$$

$$\int_{\mathbb{R}^N} |U_{j_0-1}|^p |V_{j_0}||\psi| dx \le \|U_{j_0-1}\|_{L^p}^{p-1} \|V_{j_0}\|_{L^{2p}} \|\psi\|_{L^{2p}}.$$

Arguing similarly, (3.5) follows for $\kappa > 0$.

Finally for $\kappa \le 0$, it follows $(w + U_{j_0})_+^p - (U_{j_0-1})_+^p = (w + U_{j_0})_+^p$ because $U_{j_0-1} < 0$. Since $(u+v)_+ \le u_+ + v_+$, we have

$$(w + U_{j_0})_+^p \le (w_+ + (U_{j_0})_+)^p \le w_+^p \le |w|^p \in L^2(\mathbb{R}^N) \subset H^{-2}(\mathbb{R}^N).$$

Thus we obtain $(w + U_{j_0})_+^p - (U_{j_0-1})_+^p \in H^{-2}(\mathbb{R}^N)$ for $\kappa \le 0$. □

Next we prepare three lemmas to prove the equivalence in Proposition 3.4.

Lemma 3.6. *For $\kappa > 0$, if $w \in L^p(\mathbb{R}^N)$ satisfies (3.3), then $w \in C(\mathbb{R}^N)$ and $w(x) \to 0$ as $|x| \to \infty$.*

Proof. By Lemma 3.2 and 3.3, we know that $U_{j_0} \in L^p(\mathbb{R}^N)$, $V_{j_0} \in L^q(\mathbb{R}^N)$ for all $q \in [1, \infty]$. Then we can see that $(w + U_{j_0})^p - (U_{j_0-1})^p \in L^1(\mathbb{R}^N)$. From (3.3) and by the Hausdorff-Young inequality, we obtain $w \in L^r(\mathbb{R}^N)$ for all $r \in [1, \frac{N}{N-4})$. Since $q_0 < \frac{N}{N-4}$, it follows $w \in L^{q_0}(\mathbb{R}^N)$. Arguing similarly as in Lemma 3.3, we have

$$(w + U_{j_0})^p - (U_{j_0-1})^p \in L^{r_{j_0}}(\mathbb{R}^N).$$

Then by Remark 2.5 (i), we obtain $w \in C(\mathbb{R}^N)$ and $w(x) \to 0$ as $|x| \to \infty$. □

Lemma 3.7. *For $\kappa > 0$, let $u \in L^p_{loc}(\mathbb{R}^N)$ be a solution of (1.1). Then*

$$u = G*[u^p] + \kappa F \text{ a.e. in } \mathbb{R}^N$$

and $u \ge U_j$ a.e. in $\mathbb{R}^N$ for all $j \in \mathbb{N} \cup \{0\}$.

Proof. We observe that if u satisfies (1.1), then $u = G*[u^p] + \kappa F = G*[u^p] + U_0$ a.e. in $\mathbb{R}^N$. Since u is positive, we have $u > U_0$. Next we suppose that $u - U_{j-1} > 0$ for some $j \in \mathbb{N}$. Then we have $u - U_j = G*[u^p - (U_{j-1})^p] > 0$. Thus we obtain $u > U_j$ for all $j \in \mathbb{N}$. □

Lemma 3.8. *If $u \in L^p_{loc}(\mathbb{R}^N)$ satisfies (1.1), then $u(x) \approx G(x)$ as $|x| \to \infty$.*

Proof. For $\varepsilon \in (0, c_2)$, we put

$$t_{1,\varepsilon} = \frac{c_1 - \sqrt{c_1^2 - 4(c_2 - \varepsilon)}}{2}, t_{2,\varepsilon} = \frac{c_1 + \sqrt{c_1^2 - 4(c_2 - \varepsilon)}}{2},$$

$L_{i,\varepsilon} = -\Delta + t_{i,\varepsilon} I, i = 1, 2$. Taking ε smaller if necessary, we may assume that $p\sqrt{t_{1,\varepsilon}} > \sqrt{t_1}$. We also denote by $G_{i,\varepsilon}$ the fundamental solutions of $L_{i,\varepsilon}$ respectively. By the definition, it follows $L - \varepsilon I = L_{1,\varepsilon} L_{2,\varepsilon}$. Thus the fundamental solution of $L - \varepsilon I$ is given by $G_{1,\varepsilon} * G_{2,\varepsilon}$.

Now by Proposition 3.1, there exists $R_\varepsilon > 0$ such that $u \le \varepsilon^{\frac{1}{p-1}}$ in $\mathbb{R}^N \setminus B_{R_\varepsilon}$. Then we have

$$Lu - \varepsilon u = u^p - \varepsilon u + \kappa f = (u^{p-1} - \varepsilon) u \chi_{B_{R_\varepsilon}} + (u^{p-1} - \varepsilon) u (1 - \chi_{B_{R_\varepsilon}}) + \kappa f$$

$$\le u^p \chi_{B_{R_\varepsilon}} + \kappa f \text{ in } \mathcal{S}'(\mathbb{R}^N),$$

where χ is the characteristic function. Thus we obtain

$$u(x) \le G_{1,\varepsilon}*G_{2,\varepsilon}*[u^p \chi_{B_{R_\varepsilon}}](x) + \kappa \sum_{i=1}^{m} \alpha_i G_{1,\varepsilon}*G_{2,\varepsilon}(x - a_i).$$

Since $t_{2,\varepsilon} \ge t_{1,\varepsilon}$, it follows $u(x) = O(G_{1,\varepsilon}(x))$ as $|x| \to \infty$ and hence $u(x)^p = o(G(x))$ because $p\sqrt{t_{1,\varepsilon}} > \sqrt{t_1}$. Then by Remark 2.5 (ii), we obtain

$$u(x) = G*[u^p](x) + \kappa F(x) \approx G(x) \text{ as } |x| \to \infty.$$

□

Now we are ready to prove Proposition 3.4.

Proof of Proposition 3.4. $(ii) \Rightarrow (i)$: We can readily see the claim follows.
$(i) \Rightarrow (iii)$: By the definition of U_{j_0}, we can see that w satisfies (3.4). Next by Lemma 3.8, we have $u \in L^1(\mathbb{R}^N)$ and hence $w = u - U_{j_0} \in L^1(\mathbb{R}^N)$. Since $w \in C(\mathbb{R}^N)$ and $w(x) \to 0$ as $|x| \to \infty$, it follows $w \in L^r(\mathbb{R}^N)$ for all $r \in [1, \infty]$. Especially $w \in L^{q_0}(\mathbb{R}^N)$. Since w, $V_{j_0} \in L^\infty(\mathbb{R}^N)$, we have

$$(w + U_{j_0})^p - (U_{j_0-1})^p \le p(w + U_{j_0})^{p-1}(w + V_{j_0}) \in L^{\frac{q_0}{p-1}}(\mathbb{R}^N).$$

Moreover from $V_{j_0} = U_{j_0} - U_{j_0-1} \le U_{j_0}$ and w, $U_{j_0} \in L^p(\mathbb{R}^N)$, we also have $(w + U_{j_0})^p - (U_{j_0-1})^p \in L^1(\mathbb{R}^N)$. Since $\frac{q_0}{p-1} > \frac{p}{p-1} > \frac{N}{4} > \frac{2N}{N+4}$, we can apply the Hardy-Littlewood-Sobolev inequality to obtain

$$\|L_1 w\|_{L^2} \le c\|(w + U_{j_0})^p - (U_{j_0-1})^p\|_{L^{\frac{2N}{N+4}}}.$$

This implies $w \in H^2(\mathbb{R}^N)$. Finally by Lemma 3.7, it follows that $u > 0$ implies $w = u - U_{j_0} > 0$. Thus w is a weak positive solution of (3.4).

$(iii) \Rightarrow (ii)$. Since $p < \frac{N}{N-4}$, it follows that $w \in H^2(\mathbb{R}^N)$ implies $w \in L^p(\mathbb{R}^N)$. Then by Lemma 3.6, we obtain $w \in C(\mathbb{R}^N)$ and $w(x) \to 0$ as $|x| \to \infty$. □

We finish this section by collecting basic properties of solutions of (1.1). This result follows from Proposition 3.1, Lemma 3.2 (ii) and 3.8.

Proposition 3.9. *For $\kappa > 0$, let $u \in L^p_{loc}(\mathbb{R}^N)$ be a solution of (1.1). Then $u \in C^4(\mathbb{R}^N \setminus \bigcup_{i=1}^m \{a_i\})$, $u(x) \approx G(x)$ as $|x| \to \infty$ and $\frac{u(x)}{G(x-a_i)} \to \kappa\alpha_i$ as $x \to \alpha_i$, $1 \le i \le m$.*

4 Existence and non-existence of positive solutions

In this section, we study the existence and non-existence of positive solutions of (1.1). First we prove the existence of a positive solution of (1.1) for small κ via the Implicit Function Theorem. Next we will prove the existence of a positive minimal solution and its non-degeneracy if κ belongs to some range. We will also obtain the non-existence result for large κ.

Now we put

$$\kappa^* := \sup\{\kappa > 0; (1.1) \text{ has at least one positive solution.}\}$$

First we show the set of which we take the supremum is nonempty.

Proposition 4.1. *There exists $\kappa_0 > 0$ such that for all $\kappa \in (0, \kappa_0]$, problem $(1.1)_\kappa$ has a positive solution.*

Proof. By Lemma 3.5, we can define $\Phi : \mathbb{R} \times H^2(\mathbb{R}^N) \mapsto H^{-2}(\mathbb{R}^N)$ by

$$\Phi(\kappa, w) := Lw - (w + U^\kappa_{j_0})^p_+ + (U^\kappa_{j_0-1})^p_+.$$

Then we can see that $\Phi \in C^2(\mathbb{R} \times H^2(\mathbb{R}^N))$ and

$$\Phi_w(\kappa, w)\phi = L\phi - p(w + U^\kappa_{j_0})^{p-1}_+ \phi \text{ for all } \phi \in H^2(\mathbb{R}^N).$$

Since $U^\kappa_0 = \kappa F$, it follows that $U^0_0 = 0$. Thus we have $U^0_j = 0$ for all $j \in \mathbb{N}$. This implies $\Phi_w(0,0)\phi = L\phi$ and $\Phi(0,0) = 0$. Since $\Phi_w(0,0)$ is invertible, we can apply the Implicit Function Theorem. Then there exists $\kappa_0 > 0$ such that for all $\kappa \in [-\kappa_0, \kappa_0]$, there exists $w_\kappa \in H^2(\mathbb{R}^N)$ so that $\Phi(\kappa, w_\kappa) = 0$. This implies w_κ satisfies

$$Lw_\kappa = (w_\kappa + U^\kappa_{j_0})^p_+ - (U^\kappa_{j_0-1})^p_+ \text{ in } \mathbb{R}^N.$$

We put $u := w_\kappa + U^\kappa_{j_0}$. We show that u is positive for $\kappa > 0$. By the definition of U_j, we have

$$w_\kappa = G*[(w_\kappa + U^\kappa_{j_0})^p_+] - G*[(U^\kappa_{j_0-1})^p_+] = G*[(w_\kappa + U^\kappa_{j_0})^p_+] + \kappa F - U^\kappa_{j_0}.$$

Since $G > 0$ and $F > 0$, we obtain $u > 0$. By Proposition 3.4, it follows that u is a positive solution of (1.1). □

Next we prove the existence of a positive minimal solution. We observe that if w is a positive minimal solution of (3.3), then $u := w + U_{j_0}$ is a positive minimal solution of (1.1).

Lemma 4.2. *For given $\kappa > 0$, suppose that there exists $\tilde{w} \in C(\mathbb{R}^N)$ such that $\tilde{w}(x) \geq 0$, $\tilde{w}(x) \to 0$ as $|x| \to \infty$ and*

$$\tilde{w} \geq G*[(\tilde{w} + U^\kappa_{j_0})^p - (U^\kappa_{j_0-1})^p].$$

Then $(3.3)_\kappa$ has a positive minimal solution w_κ and it satisfies $w_\kappa(x) \leq \tilde{w}(x)$. Especially if problem $(3.3)_\kappa$ has a positive solution $w(x)$, then $(3.3)_\kappa$ has a positive minimal solution $w_\kappa(x)$.

Proof. We define a sequence $\{w_n\}$ as follows: $w_0 \equiv 0$ and

$$w_n := U_{n+j_0} - U_{j_0} = \sum_{j=j_0+1}^{n+j_0} V_j \text{ for } n \in \mathbb{N}.$$

By Lemma 3.3, it follows $w_n \in C(\mathbb{R}^N)$ and $w_n(x) \to 0$ as $|x| \to \infty$. Since $V_j \geq 0$, we also have $0 < w_n < w_{n+1}$ for all $n \in \mathbb{N}$.

Now we observe that

$$\tilde{w} \geq G*[(\tilde{w} + U_{j_0})^p - (U_{j_0-1})^p] > G*[(U_{j_0})^p - (U_{j_0-1})^p] = V_{j_0+1} = w_1.$$

Next we suppose by induction that $w_n < \tilde{w}$ for some $n \in \mathbb{N}$. Then we have

$$\tilde{w} \geq G*[(\tilde{w} + U_{j_0})^p - (U_{j_0-1})^p] > G*[(w_n + U_{j_0})^p - (U_{j_0-1})^p]$$

$$= G*[(U_{n+j_0})^p - (U_{j_0-1})^p] = U_{n+1+j_0} - U_{j_0} = w_{n+1}.$$

Thus we obtain

$$0 = w_0 < w_1 < \cdots < w_n < w_{n+1} < \ \cdots \ < \tilde{w} \text{ in } \mathbb{R}^N \text{ for all } n \in \mathbb{N}.$$

We put $w_\kappa(x) := \lim_{n\to\infty} w_n(x)$. Then it follows $0 < w_\kappa \le \tilde{w}$ in $\mathbb{R}^N$.

Now by the definition, we have for $x \in \mathbb{R}^N$

$$w_n(x) = G*[(w_{n-1} + U_{j_0})^p - (U_{j_0-1})^p](x)$$

$$= \int_{\mathbb{R}^N} G(x-y)[(w_{n-1} + U_{j_0})^p - (U_{j_0-1})^p](y)dy$$

$$< \int_{\mathbb{R}^N} G(x-y)[(\tilde{w} + U_{j_0})^p - (U_{j_0-1})^p](y)dy \le \tilde{w}(x).$$

By the assumption, it follows $y \mapsto G(x-y)[(\tilde{w}+U_{j_0})^p - (U_{j_0-1})^p](y)$ is integrable for every $x \in \mathbb{R}^N$. Then by the Lebesgue's convergence theorem, we obtain

$$w_\kappa(x) = G*[(w_\kappa + U_{j_0})^p - (U_{j_0-1})^p](x)$$

for every $x \in \mathbb{R}^N$. This implies $w_\kappa \in C(\mathbb{R}^N)$, $w_\kappa(x) \to 0$ as $|x| \to \infty$ and w_κ satisfies $(3.3)_\kappa$.

Next we show that w_κ is minimal. If w is a positive solution of $(3.3)_\kappa$, then

$$0 < w = G*[(w + U_{j_0})^p - (U_{j_0-1})^p] \text{ in } \mathbb{R}^N.$$

This implies w satisfies the assumptions of this lemma. Then we have $w_n \le w$ and hence $w_\kappa \le w$ in $\mathbb{R}^N$. Thus w_κ is a positive minimal solution of $(3.3)_\kappa$.

□

Next we prove the continuation and the ordering property of positive minimal solutions. To this aim, we need the following lemma.

Lemma 4.3. *If* $0 < \kappa_1 < \kappa_2$, *then*

$$(w + U_j^{\kappa_2})^p - (U_{j-1}^{\kappa_2})^p > (w + U_j^{\kappa_1})^p - (U_{j-1}^{\kappa_1})^p \text{ a.e. in } \mathbb{R}^N$$

for all $j \in \mathbb{N}$ *and* $w \in H^2(\mathbb{R}^N)$ *with* $w(x) \ge 0$.

Proof. First we prove $U_j^{\kappa_2} > U_j^{\kappa_1}, j \in \mathbb{N} \cup \{0\}$ and $V_j^{\kappa_2} > V_j^{\kappa_1}$ for $j \in \mathbb{N}$. Since $U_0^\kappa = \kappa F$, it follows $U_0^{\kappa_2} > U_0^{\kappa_1}$. By the definitions, we have

$$U_1^{\kappa_2} = G*[U_0^{\kappa_2}] + \kappa_2 F > G*[U_0^{\kappa_1}] + \kappa_1 F = U_1^{\kappa_1}.$$

Then by induction, we obtain $U_j^{\kappa_2} > U_j^{\kappa_1}$.

Next we observe that $V_1^\kappa = G*[(U_0^\kappa)^p]$. By the monotonicity of U_j^κ with respect to κ, we have $V_1^{\kappa_2} > V_1^{\kappa_1}$. We recall that $V_2^{\kappa_2}$ was defined as follows:

$$V_2^{\kappa_2} = G*[(V_1^{\kappa_2} + U_0^{\kappa_2})^p - (U_0^{\kappa_2})^p].$$

Since the map: $s \mapsto (t+s)^p - s^p$ is increasing for every fixed $t \geq 0$, we have

$$V_2^{\kappa_2} > G*[(V_1^{\kappa_2} + U_0^{\kappa_1})^p - (U_0^{\kappa_1})^p].$$

Moreover by the monotonicity of the map: $t \mapsto (t+s)^p - s^p$, we obtain $V_2^{\kappa_2} > V_2^{\kappa_1}$. By induction, it follows $V_j^{\kappa_2} > V_j^{\kappa_1}$ for all $j \in \mathbb{N}$.

Finally we observe that

$$(w + U_j^{\kappa_2})^p - (U_{j-1}^{\kappa_2})^p = (w + V_j^{\kappa_2} + U_{j-1}^{\kappa_2})^p - (U_{j-1}^{\kappa_2})^p.$$

Arguing as above, we obtain the claim. □

Using Lemma 4.3, we can obtain the continuation and the ordering property of positive minimal solutions with respect to κ. Actually the ordering property is one of the key to prove the non-degeneracy of positive minimal solutions.

Lemma 4.4. *For given $\tilde{\kappa} > 0$, suppose that $(3.3)_{\tilde{\kappa}}$ has a positive minimal solution $w_{\tilde{\kappa}}$. Then for any $\kappa \in (0, \tilde{\kappa})$, $(3.3)_\kappa$ has a positive minimal solution w_κ and it satisfies $w_\kappa < w_{\tilde{\kappa}}$ in $\mathbb{R}^N$.*

Proof. By Lemma 4.3, we have

$$w_{\tilde{\kappa}} = G*[(w_{\tilde{\kappa}} + U_{j_0}^{\tilde{\kappa}})^p - (U_{j_0-1}^{\tilde{\kappa}})^p] > G*[(w_{\tilde{\kappa}} + U_{j_0}^{\kappa})^p - (U_{j_0-1}^{\kappa})^p].$$

for $0 < \kappa < \tilde{\kappa}$. Thus $w_{\tilde{\kappa}}$ satisfies the assumptions of Lemma 4.2. Then there exists a positive minimal solution w_κ of (3.3) for all $\kappa \in (0, \tilde{\kappa})$. By Lemma 4.2, it follows $w_\kappa \leq w_{\tilde{\kappa}}$. Using Lemma 4.3 again, we obtain

$$w_{\tilde{\kappa}} > G*[(w_\kappa + U_{j_0}^{\kappa})^p - (U_{j_0-1}^{\kappa})^p] = w_\kappa \text{ in } \mathbb{R}^N.$$

□

Next we prove the non-degeneracy of positive minimal solutions. To do this, we have to establish the existence of the first eigenvalue.

Now let $a(x)$ be a function such that $a \in L^q \cap L^{\frac{N}{4}}(\mathbb{R}^N)$ for some $q > \frac{N}{4}$ and $a(x) > 0$ for all $x \in \mathbb{R}^N$. We consider the following weighted eigenvalue problem:

$$L\phi(x) = \lambda a(x)\phi(x), \ x \in \mathbb{R}^N, \ \phi \in H^2(\mathbb{R}^N). \tag{4.1}$$

We put $A(\phi) := \int_{\mathbb{R}^N} a(x)\phi^2 dx$ for $\phi \in H^2(\mathbb{R}^N)$.

Lemma 4.5. *Problem (4.1) has a first eigenvalue $\lambda_1 > 0$ and the corresponding eigenfunction $\phi_1(x)$ has the constant sign. Moreover λ_1 is characterized as follows:*

$$\lambda_1 = \inf_{\phi \in H^2(\mathbb{R}^N)\setminus\{0\}} \frac{\|\phi\|^2}{A(\phi)}.$$

Especially for any $v \in H^2(\mathbb{R}^N)$, it follows $\|v\|^2 \geq \lambda_1 A(v)$.

Proof. Now we consider the following minimizing problem:

$$\mu := \inf\{\|\phi\|^2; \phi \in H^2(\mathbb{R}^N), A(\phi) = 1\}. \tag{4.2}$$

Since $a \in L^q \cap L^{\frac{N}{4}}(\mathbb{R}^N)$ for some $q > \frac{N}{4}$, A is bounded and compact on $H^2(\mathbb{R}^N)$. Then by the compactness of A, there exists a minimizer $\phi_1 \in H^2(\mathbb{R}^N)$ such that $\mu = \|\phi_1\|^2 > 0$, $A(\phi_1) = 1$. By Weinberger's result ([44] Chapter 3), μ corresponds to the first eigenvalue λ_1 of (4.1) and the minimizer ϕ_1 corresponds to the first eigenfunction. Since $A(\phi_1) = 1$, it follows $\phi_1(x) \not\equiv 0$. Thus it remains to show that ϕ_1 has the constant sign.

Now we consider the function:

$$\psi_1 := L_2^{\frac{1}{2}}\phi_1 = (-\Delta + t_2 I)^{\frac{1}{2}}\phi_1 = \mathcal{F}^{-1}[(t_2 + |\cdot|^2)^{\frac{1}{2}}\mathcal{F}\phi_1(\cdot)]$$

where $\mathcal{F}$ denotes the Fourier transform. Since $\phi_1 \in H^2(\mathbb{R}^N)$, we have $\psi_1 \in H^1(\mathbb{R}^N)$ and hence $|\psi_1| \in H^1(\mathbb{R}^N)$. We also define the function E_2 by $L_2^{\frac{1}{2}}E_2 = \delta_0$. Then it follows $E_2 > 0$, $\phi_1 = E_2 * \psi_1$ and $E_2 * |\psi_1| \in H^2(\mathbb{R}^N)$. We observe that if $(\psi_1)_+ \equiv 0$, then $\psi_1(x) \leq 0, \not\equiv 0$. Since $\phi_1(x)$ is the eigenfunction of (4.1), we have

$$\phi_1 = \lambda_1 G * (a\phi_1) = \lambda_1 G * [aE_2 * \psi_1] < 0.$$

Similarly if $(\psi_1)_- \equiv 0$, then $\phi_1(x) > 0$. Thus we may assume that $(\psi_1)_+ \not\equiv 0$ and $(\psi_1)_- \not\equiv 0$.

Now we suppose the following inequality holds.

$$\frac{\|E_2 * (\psi_1)_+\|^2}{A(E_2 * (\psi_1)_+)} \leq \frac{\|E_2 * (\psi_1)_-\|^2}{A(E_2 * (\psi_1)_-)}. \tag{4.3}$$

Next we rewrite $(L\phi_1, \phi_1)_{L^2}$ as follows:

$$(L\phi_1, \phi_1)_{L^2} = (L_1 L_2 \phi_1, \phi_1)_{L^2} = \|L_1^{\frac{1}{2}} L_2^{\frac{1}{2}}\phi_1\|_{L^2}^2 = \|\psi_1\|_{H^1}^2$$

$$= \|(\psi_1)_+\|_{H^1}^2 + \|(\psi_1)_-\|_{H^1}^2 = \|L_1^{\frac{1}{2}}(\psi_1)_+\|_{L^2}^2 + \|L_1^{\frac{1}{2}}(\psi_1)_-\|_{L^2}^2$$

$$= \|E_2 * (\psi_1)_+\|^2 + \|E_2 * (\psi_1)_-\|^2,$$

where $\|u\|_{H^1}^2 = \int_{\mathbb{R}^N}(|\nabla u|^2 + t_1 u^2)dx = \|L_1^{\frac{1}{2}}u\|_{L^2}^2$. Then by the characterization of λ_1 and from (4.3), we obtain

$$\lambda_1 = \frac{(L\phi_1, \phi_1)_{L^2}}{A(\phi_1)} = \frac{\|E_2 * (\psi_1)_+\|^2 + \|E_2 * (\psi_1)_-\|^2}{A(E_2 * (\psi_1)_+) + A(E_2 * (\psi_1)_-)} \geq \frac{\|E_2 * (\psi_1)_+\|^2}{A(E_2 * (\psi_1)_+)}.$$

This implies $E_2 * (\psi_1)_+$ also attains the infimum of (4.2) because $E_2 * (\psi)_+ \in H^2(\mathbb{R}^N)$. Thus it follows

$$L(E_2 * (\psi_1)_+) = \lambda_1 a E_2 * (\psi_1)_+, \ E_2 * (\psi_1)_+ = \lambda_1 G * [aE_2 * (\psi_1)_+].$$

From this equation and the fact $G = G_1 * G_2$, we have

$$(\psi_1)_+ = \lambda_1 G_1 * E_2 * [aE_2 * (\psi_1)_+] > 0.$$

This contradicts to $(\psi_1)_- \not\equiv 0$. If the reverse inequality holds in (4.3), we obtain $(\psi_1)_+ \equiv 0$. Thus ϕ_1 has the constant sign. □

Remark 4.6. *In the proof of Lemma 4.5, we proved* $(-\Delta + t_2 I)^{\frac{1}{2}}\phi_1$ *is positive to show* $\phi_1 > 0$. *If we could know that* $\phi_1 \geq 0, \not\equiv 0$, *then we can show* $\phi_1 > 0$ *in a simpler way. In fact we put* $\psi_1 = L_2\phi_1$. *Then* $\psi_1(x)$ *satisfies*

$$L_1\psi_1 = \lambda_1 a\phi_1 \geq 0, \not\equiv 0.$$

By the maximum principle, it follows $\psi_1 > 0$. *Using the maximum principle again, we obtain* $\phi_1 > 0$.

In the second order case, we can easily see that $\phi_1 \geq 0$ *because* $|\phi_1| \in H^1(\mathbb{R}^N)$. *However in the fourth order case, we can not know* $\phi_1 \geq 0$ *readily because* $|\phi_1| \notin H^2(\mathbb{R}^N)$ *in general. This is one of difficult parts when we consider higher order elliptic problems.*

Here we introduce an another approach due to [13] when the weight function is regular. In [13], they established the positivity of the first eigenfunction of the following weighted eigenvalue problem:

$$\Delta^2 u = \lambda g(x)u \text{ in } \mathbb{R}^N, \ u \in D^{2,2}(\mathbb{R}^N) \setminus \{0\}, \tag{4.4}$$

where $g \in L^{\frac{N}{4}} \cap L^\infty(\mathbb{R}^N) \setminus \{0\}$, $g \geq 0$ *and* g *is locally Hölder continuous. Their strategy is the followings. First they considered the weighted eigenvalue problem on the ball* $B_n(0) := \{x \in \mathbb{R}^N; |x| \leq n\}$ *with Navier boundary condition:*

$$\Delta^2 u = \lambda g(x)u \text{ in } B_n(0), u = \Delta u = 0 \text{ on } \partial B_n(0). \tag{4.5}$$

Let $\phi_{1,n}$ *be a first eigenfunction of (4.5). Then we can see that* $\phi_{1,n}(x) \to \phi_1(x)$ *a.e.* $x \in \mathbb{R}^N$ *as* $n \to \infty$ *and* ϕ_1 *is a first eigenfunction of (4.4). This leads* $\phi_1 \geq 0, \not\equiv 0$ *if we could show* $\phi_{1,n} \geq 0, \not\equiv 0$ *in* $B_n(0)$.

To prove the non-negativity of $\phi_{1,n}$, *they used the fixed point theorem on the cone:*

$$C := \{u \in C^1(\overline{B_n(0)}); u = 0 \text{ on } \partial B_n(0), u(x) \geq 0 \text{ for all } x \in B_n(0)\}.$$

Their argument works even if we replace Δ^2 *by* L. *However in our problem, the weight function* a *is singular. In this case, eigenfunctions of (4.1) are continuous but can never be* C^1*-smooth at each singular points* a_i *in general.*

There is an another approach to prove $\phi_{1,n} \geq 0, \not\equiv 0$ *when the problem is biharmonic (see [6]). For simplicity, we consider the following eigenvalue problem in a bounded domain* Ω *with the Navier boundary condition:*

$$\Delta^2\phi = \lambda\phi \text{ in } \Omega, \ \phi = \Delta\phi = 0 \text{ on } \partial\Omega.$$

Then the first eigenvalue is characterized as an infimum of the Rayleigh quotient: $\inf_{u \in H^2 \cap H_0^1(\Omega)\setminus\{0\}} \frac{\|\Delta u\|_{L^2}^2}{\|u\|_{L^2}^2}$. *By the compactness of the embedding* $H^2 \cap H_0^1(\Omega) \hookrightarrow L^2(\Omega)$, *we can easily obtain the existence of a minimizer* ϕ_1. *To prove* ϕ_1 *has the constant sign, we argue in the following way. For the minimizer* ϕ_1, *there exists a unique solution to the problem:*

$$-\Delta w = |\Delta\phi_1| \text{ in } \Omega, \ w = 0 \text{ on } \partial\Omega.$$

By the maximum principle, it follows $w > 0$ *in* Ω. *Then we notice that*

$$-\Delta(w \pm \phi_1) = -\Delta w \mp \Delta\phi_1 = |\Delta\phi_1| \mp \Delta\phi_1 \geq 0, w \pm \phi_1 = 0 \text{ on } \partial\Omega.$$

This leads $\phi_1 \equiv \mp w$ in Ω or $w > |\phi_1|$ in Ω. If $w > |\phi_1|$, then we obtain $\frac{\|\Delta\phi_1\|_{L^2}^2}{\|\phi_1\|_{L^2}^2} > \frac{\|\Delta w\|_{L^2}^2}{\|w\|_{L^2}^2}$. This contradicts to the fact ϕ_1 is the minimizer. Thus ϕ_1 has the constant sign.

However in our problem, we can not argue in this way because this argument does not give a control of $\|\nabla\phi_1\|_{L^2}^2$.

Now for given $\tilde{\kappa} > 0$, suppose $(3.3)_{\tilde{\kappa}}$ has a positive minimal solution $w_{\tilde{\kappa}}$. Then for $0 < \kappa < \tilde{\kappa}$, there exists a positive minimal solution w_κ of $(3.3)_\kappa$ by Lemma 4.4. We put $u_\kappa = w_\kappa + U_{j_0}^\kappa$. Now we are ready to prove the non-degeneracy of minimal solutions.

Proposition 4.7. *Let λ_1^κ be the first eigenvalue of the linearized eigenvalue problem:*

$$L\phi = \lambda a_\kappa \phi, \text{ in } \mathbb{R}^N, \phi \in H^2(\mathbb{R}^N),\ a_\kappa(x) := pu_\kappa(x)^{p-1}. \tag{4.6}$$

Then it follows $\lambda_1^\kappa > 1$.

Proof. First we establish the existence of the first eigenvalue. Now by Proposition 3.9, it follows $u_\kappa \in L^q(\mathbb{R}^N)$ for all $q \in [1, \frac{N}{N-4})$. Since $1 < p < \frac{N}{N-4}$, we can take $\delta > 0$ so that $\delta < \frac{N}{4(p-1)}(\frac{N}{N-4} - p)$. Then we have $(\frac{N}{4} + \delta)(p-1) < \frac{N}{N-4}$. This implies $a_\kappa \in L^{\frac{N}{4}+\delta} \cap L^{\frac{N}{4}}(\mathbb{R}^N)$. Thus we can apply Lemma 4.5. We denote by ϕ_1 the first eigenfunction. Then $\phi_1 > 0$.

By Lemma 4.5, we already know that $\lambda_1^\kappa > 0$. We prove $\lambda_1^\kappa > 1$. We put $z := w_{\tilde{\kappa}} - w_\kappa$. By Lemma 4.4, it follows $z > 0$ in $\mathbb{R}^N$. Moreover by Lemma 4.3, we have

$$Lw_{\tilde{\kappa}} = (w_{\tilde{\kappa}} + U_{j_0}^{\tilde{\kappa}})^p - (U_{j_0-1}^{\tilde{\kappa}})^p > (w_{\tilde{\kappa}} + U_{j_0}^{\kappa})^p - (U_{j_0-1}^{\kappa})^p.$$

Thus we obtain

$$Lz = Lw_{\tilde{\kappa}} - Lw_\kappa$$

$$= (w_{\tilde{\kappa}} + U_{j_0}^{\tilde{\kappa}})^p - (U_{j_0-1}^{\tilde{\kappa}})^p - (w_\kappa + U_{j_0}^{\kappa})^p + (U_{j_0-1}^{\kappa})^p$$

$$> (w_{\tilde{\kappa}} + U_{j_0}^{\kappa})^p - (w_\kappa + U_{j_0}^{\kappa})^p > p(w_\kappa + U_{j_0}^{\kappa})^{p-1} z = a_\kappa z. \tag{4.7}$$

Multiplying $\phi_1(x)$ in (4.7) and integrating over $\mathbb{R}^N$, we get

$$(Lz, \phi_1)_{L^2} > \int_{\mathbb{R}^N} a_\kappa z \phi_1 dx.$$

On the other hand from (4.1), we have

$$(Lz, \phi_1)_{L^2} = (L\phi_1, z)_{L^2} = \lambda_1^\kappa \int_{\mathbb{R}^N} a_\kappa \phi_1 z dx.$$

Since $a_\kappa(x)$, $z(x)$ and $\phi_1(x)$ are positive functions, it follows $\lambda_1^\kappa > 1$. □

Now let $0 < \kappa < \kappa^*$. By Lemma 4.4, there exists a positive minimal solution w_κ of (3.3). We put $u_\kappa = w_\kappa + U_{j_0}^\kappa$. Then by Proposition 4.7, we have $\lambda_1^\kappa > 1$. Thus we obtain the following result.

Proposition 4.8. *For any $\kappa \in (0, \kappa^*)$, problem $(1.1)_\kappa$ has a positive minimal solution u_κ. Moreover let λ_1^κ be the first eigenvalue of the linearized eigenvalue problem:*

$$L\phi = \lambda p u_\kappa^{p-1}\phi, \;\; \phi \in H^2(\mathbb{R}^N).$$

Then it follows $\lambda_1^\kappa > 1$ and

$$\|\phi\|^2 \geq \lambda_1^\kappa \int_{\mathbb{R}^N} p u_\kappa^{p-1}\phi^2 dx \text{ for all } \phi \in H^2(\mathbb{R}^N).$$

Finally in this section, we prove $\kappa^* < \infty$. We put

$$\overline{\kappa} := (\inf_{\phi \in H^2 \setminus \{0\}} \frac{\|\phi\|^2}{\overline{A}(\phi)})^{\frac{1}{p-1}}, \;\; \overline{A}(\phi) := p \int_{\mathbb{R}^N} F^{p-1}\phi^2 dx$$

where $F(x) = \sum_{i=1}^m \alpha_i G(x - a_i)$. Since $F \in L^q(\mathbb{R}^N)$ for all $q \in [1, \frac{N}{N-4})$ and $1 < p < \frac{N}{N-4}$, the number $\overline{\kappa}$ is well-defined and positive.

Proposition 4.9. *Problem (1.1) has no positive solution for any $\kappa > \overline{\kappa}$. This implies $\kappa^* \leq \overline{\kappa} < \infty$.*

Proof. We suppose by contradiction that (1.1) has a positive solution for some $\kappa > \overline{\kappa}$. Then by Lemma 4.2 and 4.4, there exists a positive minimal solution $u_{\overline{\kappa}}$ of $(1.1)_{\overline{\kappa}}$. Moreover by Proposition 4.7, we have $\lambda_1^\kappa > 1$.

On the other hand by Lemma 3.7, it follows $u_{\overline{\kappa}} > \overline{\kappa} F$. Thus we obtain

$$1 < \lambda_1^{\overline{\kappa}} = \inf_{\phi \in H^2 \setminus \{0\}} \frac{\|\phi\|^2}{\int_{\mathbb{R}^N} a_{\overline{\kappa}}\phi^2 dx} < \inf_{\phi \in H^2 \setminus \{0\}} \frac{\|\phi\|^2}{\overline{A}(\phi)} \times \frac{1}{\overline{\kappa}^{p-1}} = 1.$$

This is a contradiction. Thus (1.1) has no positive solution for all $\kappa > \overline{\kappa}$. ☐

5 Existence of a second solution

For $0 < \kappa < \kappa^*$, let u_κ be a positive minimal solution of (1.1). To find a second positive solution, we will use the Mountain Pass Theorem.

We consider a new equation:

$$Lv = (v + u_\kappa)^p - u_\kappa^p \text{ in } \mathbb{R}^N, v \in H^2(\mathbb{R}^N). \tag{5.1}$$

Since $Lu_\kappa = u_\kappa^p + \kappa f$, we can see that if v is a solution of (5.1), then $v + u_\kappa$ is a solution of (1.1). We define the energy functional $I : H^2(\mathbb{R}^N) \mapsto \mathbb{R}$ by

$$I(v) = \frac{1}{2}\|v\|^2 - \int_{\mathbb{R}^N} H(u_\kappa, v)dx.$$

Here for $a > 0$ and $s \in \mathbb{R}$, we put

$$H(a, s) = \frac{1}{p+1}(a + s_+)^{p+1} - \frac{1}{p+1}a^{p+1} - a^p s_+, \;\; h(a, s) = (a + s_+)^p - a^p.$$

We recall that u_κ is singular at each a_i. Thus we have to treat nonlinear terms carefully. First we prepare basic properties of the nonlinear terms $H(a, s)$ and $h(a, s)$.

Lemma 5.1. *Let $a > 0$ be arbitrary given. Then these properties hold.*
(i) $h(a, s) \le C(s^p + a^{p-1}s)$ for all $s \ge 0$ and some $C > 0$.
(ii) $H(a, s + t) - H(a, s) - h(a, s)t \le C(a^{p-1}t^2 + s^{p-1}t^2 + t^{p+1})$ for all $s, t \ge 0$ and some $C > 0$.
(iii) $h(a, s)s - 2H(a, s) \ge 0$ for all $s \ge 0$.
(iv) For any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$H(a, s) - \frac{p}{2}a^{p-1}s^2 \le \varepsilon a^{p-1}s^2 + C_\varepsilon s^{p+1} \text{ for all } s \ge 0.$$

(v) Let $c_p := \min\{1, p - 1\}$. Then

$$h(a, s)s - (2 + c_p)H(a, s) + \frac{c_p}{2}pa^{p-1}s^2 \ge 0 \text{ for all } s \ge 0.$$

Proof. (i) We can write $h(a, s) = p\int_0^p (a + \theta)^{p-1}d\theta$. Then we can see that $h(a, s) \le C(a^{p-1}s + s^p)$. Since $H(a, s) = \int_0^s h(a, \theta)d\theta$, the second claim follows.
(ii) By definitions, we have

$$H(a, s + t) - H(a, s) - h(a, s)t = \int_0^t \int_0^\theta p(a + s + \tau)^{p-1}d\tau d\theta.$$

Then we can show that

$$H(a, s + t) - H(a, s) - h(a, s)t \le C(a^{p-1}t^2 + s^{p-1}t^2 + t^{p+1}).$$

(iii) We notice that

$$h(a, s)s - 2H(a, s) = \int_0^s p(p - 1)(a + \theta)^{p-2}(s - \theta)\theta d\theta.$$

From this expression, we obtain $h(a, s)s - 2H(a, s) \ge 0$.
(iv) We observe that

$$H(a, s) - \frac{p}{2}a^{p-1}s^2 = p(p - 1)\int_0^s (a + \theta)^{p-2}(s - \theta)^2 d\theta.$$

Then we have

$$p(p - 1)\int_0^s (a + \theta)^{p-2}(s - \theta)^2 d\theta \le Ca^{p-2}t^3 + Ct^{p+1} \le \varepsilon a^{p-1}t^2 + C_\varepsilon t^{p+1}.$$

Here we used the Young inequality.
(v) For $s \ge 0$, we define

$$J(s) := h(a, s)s - (2 + c_p)H(a, s) + \frac{c_p}{2}pa^{p-1}s^2.$$

Then it follows $J(0) = J'(0) = J''(0) = 0$ and

$$J'''(s) = p(p - 1)(1 - c_p)(a + s)^{p-2}a + p(p - 1)(p - 1 - c_p)(a + s)^{p-2}s \ge 0.$$

Integrating third times, we obtain $J(s) \ge 0$ for all $s \ge 0$. □

Next we show that the functional I is C^1. Although the proof is almost same as the second order case [32], we give the proof here for the sake of completeness.

Lemma 5.2. *The functional I is C^1 and*

$$I'(v)\psi = (v,\psi) - \int_{\mathbb{R}^N} h(u_\kappa, v)\psi dx, \;\; \psi \in H^2(\mathbb{R}^N).$$

Proof. First we observe that for $v \in H^2(\mathbb{R}^N)$, we have by Lemma 5.1 (i),

$$I(v) \le \frac{1}{2}\|v\|^2 + \int_{\mathbb{R}^N} |H(u_\kappa, v)| dx \le \frac{1}{2}\|v\|^2 + C\int_{\mathbb{R}^N} (u_\kappa^{p-1}v^2 + |v|^{p+1}) dx$$

$$\le \frac{1}{2}\|v\|^2 + C\|u_\kappa\|_{L^p}^{p-1}\|v\|_{L^{2p}}^2 + C\|v\|_{L^{p+1}}^{p+1}.$$

Here $v \in L^{2p}(\mathbb{R}^N)$ because $p < \frac{N}{N-4}$. Since $u_\kappa \in L^p(\mathbb{R}^N)$, I is well-defined in $H^2(\mathbb{R}^N)$.

Next we show that for $v, \psi \in H^2(\mathbb{R}^N)$,

$$\frac{1}{\|\psi\|}|I(v+\psi) - I(v) - ((v,\psi) - \int_{\mathbb{R}^N} h(u_\kappa, v)\psi dx)| \to 0 \text{ as } \|\psi\| \to 0. \tag{5.2}$$

In fact, we have by Lemma 5.1 (ii),

$$|I(v+\psi) - I(v) - ((v,\psi) - \int_{\mathbb{R}^N} h(u_\kappa, v)\psi dx)|$$

$$\le \|\psi\|^2 + \int_{\mathbb{R}^N} |H(u_\kappa, v+\psi) - H(u_\kappa, v) - h(u_\kappa, v)\psi| dx$$

$$\le \|\psi\|^2 + C\int_{\mathbb{R}^N} (u_\kappa^{p-1}\psi^2 + v^{p-1}\psi^2 + |\psi|^{p+1}) dx$$

$$\le C(1 + \|u\|_{L^p}^{p-1} + \|v\|_{L^p}^{p-1} + \|\psi\|^{p-1})\|\psi\|^2.$$

Thus we obtain (5.2) and it implies

$$I'(v)\psi = (v,\psi) - \int_{\mathbb{R}^N} h(u_\kappa, v)\psi dx.$$

Finally we prove that I' is continuous. We show that if $v_n \rightharpoonup v$ in $H^2(\mathbb{R}^N)$, then

$$\int_{\mathbb{R}^N} h(u_\kappa, v_n)\psi dx \to \int_{\mathbb{R}^N} h(u_\kappa, v)\psi dx$$

for all $\psi \in H^2(\mathbb{R}^N)$. This implies I' is continuous. First we observe that for $R > 0$,

$$\int_{B_R} h(u_\kappa, v_n)\psi dx \to \int_{B_R} h(u_\kappa, v)\psi dx \tag{5.3}$$

In fact since $v_n \rightharpoonup v$ in $H^2(\mathbb{R}^N)$, we may assume that $v_n \to v$ in $L^{2p}(B_R)$ and $v_n(x) \le V(x) \in L^{2p}(B_R)$ a.e. $x \in B_R(0)$ for some $V \in L^{2p}(B_R)$. Then we have $h(u_\kappa, v_n)\psi \to h(u_\kappa, v)\psi$ a.e. $x \in B_R$. Moreover by Lemma 5.1 (i), we have

$$h(u_\kappa, v_n)|\psi| \le C(v_n^p + u_\kappa^{p-1}v_n)|\psi| \le C(V^{p-1} + u_\kappa^{p-1}V)|\psi| \in L^1(B_R).$$

By Lebesgue's convergence theorem, we obtain (5.3). Next we show that for any $\varepsilon > 0$, there exists $R_\varepsilon > 0$ such that

$$\int_{|x|\ge R_\varepsilon} h(u_\kappa, v_n)\psi dx \le \varepsilon. \tag{5.4}$$

For $R > 0$, we have

$$\int_{|x|\ge R} h(u_\kappa, v_n)\psi dx \le C \int_{|x|\ge R} (v_n^p + u_\kappa^{p-1}v_n)|\psi| dx$$

$$\le C(\|v_n\|^p_{L^{2p}(\mathbb{R}^N)}\|\psi\|_{L^2(\mathbb{R}^N\setminus B_R(0))} + \|u_\kappa\|^{p-1}_{L^p(\mathbb{R}^N)}\|v_n\|_{L^{2p}(\mathbb{R}^N)}\|\psi\|_{L^{2p}(\mathbb{R}^N\setminus B_R(0))})$$

$$\le C(\|\psi\|_{L^2(\mathbb{R}^N\setminus B_R(0))} + \|\psi\|_{L^{2p}(\mathbb{R}^N\setminus B_R(0))}),$$

because $\|v_n\|_{L^{2p}(\mathbb{R}^N)}$ is bounded. Since $\psi \in H^2(\mathbb{R}^N)$, we can choose large $R_\varepsilon > 0$ so that

$$\|\psi\|_{L^2(\mathbb{R}^N\setminus B_R(0))} + \|\psi\|_{L^{2p}(\mathbb{R}^N\setminus B_R(0))} \le \varepsilon.$$

Thus we obtain (5.4). In a similar way, we have $\int_{|x|\ge R_\varepsilon} h(u_\kappa, v)\psi dx \le \varepsilon$. Since ε is arbitrary, we have if $v_n \rightharpoonup v$ in $H^2(\mathbb{R}^N)$, then

$$\int_{\mathbb{R}^N} h(u_\kappa, v_n)\psi dx \to \int_{\mathbb{R}^N} h(u_\kappa, v)\psi dx.$$

□

Lemma 5.3. *(i) Suppose $I'(v_n) \to 0$ and $v_n \rightharpoonup v_0$ in $H^2(\mathbb{R}^N)$ for some v_0, then $I'(v_0) = 0$.*
(ii) If $I'(v_0) = 0$, then $v_0 \ge 0$. Moreover if $v_0 \not\equiv 0$, then $v_0 > 0$ in $\mathbb{R}^N$.

Proof. (i) In the proof of Lemma 5.2, we proved if $v_n \rightharpoonup v_0$ in $H^2(\mathbb{R}^N)$, then

$$\int_{\mathbb{R}^N} h(u_\kappa, v_n)\psi dx \to \int_{\mathbb{R}^N} h(u_\kappa, v_0)\psi dx$$

for all $\psi \in H^2(\mathbb{R}^N)$. This implies $0 = \lim_{n\to\infty} I'(v_n)\psi = I'(v_0)\psi$.

(ii) We observe that if $I'(v_0) = 0$, then v_0 satisfies

$$v_0(x) = G*[((v_0)_+ + u_\kappa)^p - u_\kappa^p](x) \ge 0.$$

We put $w_0(x) := L_2 v_0(x)$. If $v_0 \not\equiv 0$, then $w_0(x)$ satisfies

$$L_1 w_0 = (v_0 + u_\kappa)^p - u_\kappa^p \ge 0, \not\equiv 0.$$

By the maximum principle, we have $w_0(x) > 0$. Using the maximum principle again, we obtain $v_0(x) > 0$. □

Now we consider the problem at infinity and the corresponding functional:

$$Lv(x) = v(x)_+^p, \; x \in \mathbb{R}^N, \tag{5.5}$$

$$I_\infty(v) := \frac{1}{2}\|v\|^2 - \frac{1}{p+1}\int_{\mathbb{R}^N} {v_+}^{p+1} dx,$$

$$c_\infty := \inf\{I_\infty(v); I'_\infty(v) = 0, \; v \in H^2 \setminus \{0\}\}.$$

Proposition 5.4. *There exists $v_\infty(x)$ such that $I_\infty(v_\infty) = c_\infty$, $I'_\infty(v_\infty) = 0$ and $v_\infty > 0$ in $\mathbb{R}^N$.*

Proof. First we show that the set:

$$M := \{v \in H^2(\mathbb{R}^N) \setminus \{0\}; I'_\infty(v) = 0\} \neq \emptyset.$$

In fact, let $\phi \in H^2_{rad}(\mathbb{R}^N)$ be a radial function such that $\phi(x) > 0$ for all $x \in \mathbb{R}^N$. Then we can see that $I_\infty(t_0\phi) < 0$ for some $t_0 > 0$. We define the Mountain Pass level m_∞ by

$$m_\infty := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_\infty(\gamma(t)),$$

$$\Gamma := \{\gamma \in C([0,1], H^2_{rad}(\mathbb{R}^N)); \gamma(0) = 0, \gamma(1) = t_0\phi\}.$$

Then it follows $m_\infty > 0$. Now the embedding $H^2_{rad}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ is compact for any $q \in (2, \frac{2N}{N-4})$ (see [45] section 1.5). Then by the Mountain Pass Theorem, there exists $\tilde{v} \in H^2_{rad}(\mathbb{R}^N) \setminus \{0\}$ such that $I'_\infty(\tilde{v}) = 0$ and $I_\infty(\tilde{v}) = m_\infty$. Especially it follows $M \neq \emptyset$.

Next we show there is $v_\infty(x)$ such that $I_\infty(v_\infty) = \inf_{v \in M} I_\infty(v)$. We observe that if $v \in M$, then

$$I_\infty(v) = \frac{1}{2}\|v\|^2 - \frac{1}{p+1}\int_{\mathbb{R}^N} v_+^{p+1} dx = (\frac{1}{2} - \frac{1}{p+1})\|v\|^2 \geq 0.$$

Thus $\inf_{v \in M} I_\infty(v)$ is well-defined. Let $v_n \in M$ be a sequence such that $I_\infty(v_n) \to \inf_{v \in M} I_\infty(v)$. Then v_n is bounded in $H^2(\mathbb{R}^N)$. Since $v_n \in M$, v_n is a bounded Palais-Smale sequence. By concentration compactness principle, there exist $k \in \mathbb{N} \cup \{0\}$, $v_0 \in H^2(\mathbb{R}^N)$, $w^i \in H^2(\mathbb{R}^N) \setminus \{0\}$, $\{y_n^i\} \subset \mathbb{R}^N$ $i = 1, \cdots, k$ such that

$$I_\infty(v_n) \to I_\infty(v_0) + \sum_{i=1}^k I_\infty(w^i), \; \|v_n - v_0 - \sum_{i=1}^k w^i(\cdot - y_n^i)\| \to 0,$$

$$I'_\infty(v_0) = I'_\infty(w^i) = 0, \; |y_n^i| \to \infty, \; |y_n^i - y_n^{i'}| \to \infty.$$

If $v_0 \not\equiv 0$, then $k = 0$. In this case $v_0(x)$ satisfies $I_\infty(v_0) = \inf_{v \in M} I_\infty(v)$. If $v_0 \equiv 0$, then it follows $k = 1$ and $w^1(x)$ attains $\inf_{v \in M} I_\infty(v)$. Thus there exists $v_\infty(x) \in M$ such that $I_\infty(v_\infty) = \inf_{v \in M} I_\infty(v)$.

Finally we show $v_\infty(x) > 0$. Since $v_\infty \in M$, it follows $v_\infty(x) \not\equiv 0$. Next we observe that v_∞ satisfies $v_\infty(x) = G * [(v_\infty)_+^p](x) \geq 0$ because v_∞ is a solution of (5.5). Putting $L_2 v_\infty = w_\infty$, then w_∞ satisfies the following system:

$$L_2 w_\infty = v_\infty^p, \; L_1 v_\infty = w_\infty.$$

By the maximum principle, it follows $w_\infty(x) > 0$ and hence $v_\infty(x) > 0$. □

Remark 5.5. *For the general second order scalar field equation:*

$$-\Delta u = h(u) \text{ in } \mathbb{R}^N,$$

it is well-known that the least energy solution is radially symmetric by the Schwarz symmetrization [4]. In the fourth order case, we don't know whether the least energy solution of (5.5) is radially symmetric or not.

Next we prove the functional I has the Mountain Pass geometry.

Lemma 5.6. *(i) There exists α, $\rho > 0$ such that $I(v) \geq \alpha$ for all $v \in H^2(\mathbb{R}^N)$ with $\|v\| = \rho$.*
(ii) $\lim_{t\to\infty} I(tv_\infty) = -\infty$ *and* $\sup_{t>0} I(tv_\infty) < c_\infty$.

Proof. (i) First we observe that

$$I(v) = \frac{1}{2}\|v\|^2 - \int_{\mathbb{R}^N} H(u_\kappa, v)dx$$

$$= \frac{1}{2}(\|v\|^2 - p\int_{\mathbb{R}^N} u_\kappa^{p-1}v^2dx) - \int_{\mathbb{R}^N} H(u_\kappa, v) - \frac{p}{2}u_\kappa^{p-1}v^2dx.$$

By Lemma 5.1 (iv), we have

$$\int_{\mathbb{R}^N} H(u_\kappa, v) - \frac{p}{2}u_\kappa^{p-1}v^2dx \leq \varepsilon\int_{\mathbb{R}^N} u_\kappa^{p-1}v^2dx + C_\varepsilon\|v\|_{L^{p+1}}^{p+1}.$$

By Proposition 4.8, we obtain

$$I(v) \geq \frac{1}{2}(1 - \frac{1}{\lambda_1^\kappa})\|v\|^2 - \frac{\varepsilon}{p\lambda_1^\kappa}\|v\|^2 - C_\varepsilon\|v\|^{p+1}.$$

We choose $\varepsilon < \frac{p}{2}(1 - \lambda_1^\kappa)$. Then we can take small $\rho > 0$ so that the claim follows.

(ii) For $t > 0$, we consider $I(tv_\infty)$. Then it follows

$$I(tv_\infty) = \frac{t^2}{2}\|v_\infty\|^2 - \int_{\mathbb{R}^N} H(u_\kappa, tv_\infty)dx$$

$$= \frac{t^2}{2}\|v_\infty\|^2 - \int_{\mathbb{R}^N}\int_0^{tv_\infty} (u_\kappa + s_+)^p - u_\kappa^p ds dx.$$

On the other hand, we have

$$I_\infty(tv_\infty) = \frac{t^2}{2}\|v_\infty\|^2 - \int_{\mathbb{R}^N}\int_0^{tv_\infty} s_+^p ds dx.$$

Thus we obtain

$$I(tv_\infty) = I_\infty(tv_\infty) - \int_{\mathbb{R}^N}\int_0^{tv_\infty} ((u_\kappa + s_+)^p - u_\kappa^p - s_+^p)ds dx.$$

By the convexity, it follows $I(tv_\infty) < I_\infty(tv_\infty)$ for all $t > 0$. Next we put $\alpha(t) := I_\infty(tv_\infty)$. We can easily show that $\alpha(t) \leq \alpha(1) = c_\infty$ for all $t \in [0, \infty)$ and $\alpha(t) \to -\infty$ as $t \to \infty$. Thus we obtain $\sup_{t>0} I(tv_\infty) < c_\infty$ and $\lim_{t\to\infty} I(tv_\infty) = -\infty$.

□

Lemma 5.7. *(i) Let $\{v_n\}$ be a sequence such that $I(v_n) \to c \in \mathbb{R}$ and $I'(v_n) \to 0$. Then v_n is bounded in $H^2(\mathbb{R}^N)$.*
(ii) Let v_n be a sequence such that $I(v_n) \to c \in \mathbb{R}$, $I'(v_n) \to 0$ and $\|v_n\|$ is bounded. Then there exist $k \in \mathbb{N} \cup \{0\}$, $v_0 \in H^2$, $w^i \in H^2 \setminus \{0\}$, $\{y_n^i\}$, $i = 1, \cdots, k$ such that

$$I(v_n) \to I(v_0) + \sum_{i=1}^{k} I_\infty(w^i), \|v_n - v_0 - \sum_{i=1}^{k} w^i(\cdot - y_n^i)\| \to 0,$$

$$I'_\infty(w^i) = 0,\ I'(v_0) = 0, |y_n^i| \to \infty,\ |y_n^i - y_n^{i'}| \to \infty.$$

(iii) If $I(v_n) \to c \in (0, c_\infty)$ and $I'(v_n) \to 0$, then $\{v_n\}$ has a convergent subsequence.

Proof. (i) By assumptions, we have

$$\frac{1}{2}\|v_n\|^2 - \int_{\mathbb{R}^N} H(u_\kappa, v_n)dx = I(v_n) = c + o(1),$$

$$\|v_n\|^2 - \int_{\mathbb{R}^N} h(u_\kappa, v_n)v_n dx = I'(v_n)v_n = o(1)\|v_n\|.$$

Thus we obtain

$$\int_{\mathbb{R}^N} h(u_\kappa, v_n)v_n - (2 + c_p)H(u_\kappa, v_n)dx + \frac{c_p}{2}\|v_n\|^2 = (2 + c_p)c + o(1)\|v_n\| + o(1).$$

On the other hand by Lemma 5.1 (v), we have

$$\int_{\mathbb{R}^N} h(u_\kappa, v_n)v_n - (2 + c_p)H(u_\kappa, v_n)dx \geq -\frac{c_p}{2}\int_{\mathbb{R}^N} pu_\kappa^{p-1}v_n^2 dx.$$

By Proposition 4.8, we obtain

$$\frac{c_p}{2}(1 - \frac{1}{\lambda_1^\kappa})\|v_n\|^2 \leq (2 + c_p)c + o(1)\|v_n\| + o(1).$$

Thus $\{v_n\}$ is bounded in $H^2(\mathbb{R}^N)$.

(ii) In the proof of Lemma 5.2, we proved if $v_n \rightharpoonup v_0$ in $H^2(\mathbb{R}^N)$, then it follows $\int_{\mathbb{R}^N} h(u_\kappa, v_n)\psi dx \to \int_{\mathbb{R}^N} h(u_\kappa, v_0)\psi dx$ for all $\psi \in H^2(\mathbb{R}^N)$. Then arguing as in [29] (Theorem 5.1), we obtain the claims.

(iii) Let $\{v_n\}$ be a sequence such that $I(v_n) \to c \in (0, c_\infty)$ and $I'(v_n) \to 0$. From (i), $\{v_n\}$ is bounded in $H^2(\mathbb{R}^N)$. Thus $\{v_n\}$ is a bounded Palais-Smale sequence. Then from (ii), there exists $v_0 \in H^2(\mathbb{R}^N)$ such that $I'(v_0) = 0$. By Lemma 5.1 (iii), it follows

$$I(v_0) = \frac{1}{2}\|v_0\|^2 - \int_{\mathbb{R}^N} H(u_\kappa, v_0)dx = \int_{\mathbb{R}^N} \frac{1}{2}h(u_\kappa, v_0)v_0 - H(u_\kappa, v_0)dx \geq 0.$$

Thus from (ii), we obtain $c = I(v_0) + \sum_{i=1}^{k} I_\infty(w^i) \geq kc_\infty$. On the other hand, we have $c < c_\infty$. Thus it follows $k = 0$ and hence $v_n \to v_0$ in $H^2(\mathbb{R}^N)$.

□

Proof of Theorem 1.1. (i) By Lemma 5.6 -5.7, we can apply the Mountain Pass Theorem. We denote by v_κ the Mountain Pass solution of (5.1). Then by Lemma 5.3 (ii), $v_\kappa(x) > 0$. We put $u^\kappa(x) := v_\kappa(x) + u_\kappa(x)$. Then $u^\kappa(x)$ is a positive solution of (1.1) and it satisfies $u_\kappa(x) < u^\kappa(x)$ for all $x \in \mathbb{R}^N$.

(ii) By Proposition 3.9, u_κ and $u^\kappa(x)$ satisfy the desired properties.

□

Acknowledgment

The authors would like to thank Professor Yuki Naito for discussions and helpful comments.

References

[1] S. Adachi and K. Tanaka. *Existence of positive solutions for a class of nonhomogeneous elliptic equations in* $\mathbb{R}^N$, Nonlinear Anal. **48** (2002), 685-705.

[2] H. Amann and P. Quittner. *Elliptic boundary value problems involving measures: existence, regularity and multiplicity*, Adv. Diff. Eqns. **3** (1998), 753-813.

[3] N. Aronszajn and K. T. Smith. *Theory of Bessel potentials I*, Ann. Inst. Fourier **11** (1961), 385-475.

[4] H. Berestycki and P. L. Lions. *Nonlinear Scalar field equations I. Existence of a ground state*, Arch. Rat. Mech. Anal. **82** (1983), 313-345.

[5] F. Bernis, J. Garcia-Azorero and I. Peral. *Existence and multiplicity of nontrivial solutions in semilinear critical problems of fourth order*, Adv. Diff. Eqns. **1** (1996), 219-240.

[6] E. Berchio, F. Gazzola and E. Mitidieri. *Positivity preserving property for a class of biharmonic elliptic problems*, J. Diff. Eqns. **229** (2006), 1-23.

[7] E. Berchio, F. Gazzola and T. Weth. *Critical growth biharmonic elliptic problems under Steklov-type boundary conditions*, Adv. Diff. Eqns. **12** (2007), 381-406.

[8] E. Berchio, F. Gazzola and T. Weth. *Radial symmetry of positive solutions to nonlinear polyharmonic Dirichlet problems*, J. Reine Angew. Math. **620** (2008), 165-183.

[9] M. F. Bidaut-Véron and L. Vivier. *An elliptic semilinear equation with source term involving boundary measures: the subcritical case*, Rev. Mat. Iberoamericana **16** (2000), 477-513.

[10] M. F. Bidaut-Véron and C. Yarur. *Semilinear elliptic equations and systems with measure data: existence and a priori estimates*, Adv. Diff. Eqns. **7** (2002), 257-296.

[11] H. Brezis and W. A. Strauss. *Semi-linear second order elliptic equations in* L^1, J. Math. Soc. Japan **25** (1973), 565-590.

[12] D. M. Cao and H. S. Zhou. *Multiple positive solutions of nonhomogeneous semilinear elliptic equations in* $\mathbb{R}^N$, Proc. Royal Soc. Edin. **126A** (1996), 443-463.

[13] J. Chabrowski and João Marcos do Ó. *On some fourth-order semilinear elliptic problems in* $\mathbb{R}^N$, Nonlinear Anal. **49** (2002), 861-884.

[14] C. C. Chen and C. S. Lin. *Existence of positive weak solutions with a prescribed singular set of semilinear elliptic equations*, J. Geom. Anal. **9** (1999), 221-246.

[15] K. J. Chen. *Exactly two entire positive solutions for a class of nonhomogeneous elliptic equations*, Diff. Int. Eqns. **17** (2004), 1-16.

[16] Y. Deng and Y. Li. *Existence of multiple positive solutions for a semilinear elliptic equation*, Adv. in Diff. Eqns. **2** (1997), 361-382.

[17] Y. Deng and Y. Li. *Existence and bifurcation of the positive solutions for a semilinear equation with critical exponent*, J. Diff. Eqns. **130** (1996), 179-200.

[18] Z. Djadli, A. Malchiodi and M. Ould Ahmedou. *Prescribing a fourth order conformal invariant on the standard sphere I: A perturbation result*, Commun. Contemp. Math. **4** (2002), 375-408.

[19] Z. Djadli, A. Malchiodi and M. Ould Ahmedou. *Prescribing a fourth order conformal invariant on the standard sphere II: Blow-up analysis and applications*, Ann. Sc. Norm. Super. Pisa Cl. Sci. **1** (2002), 387-434.

[20] J. Duoandikoetxea. *Fourier analysis*, Graduate Studies in Math. **29** (2004) AMS. Providence.

[21] D. E. Edmunds, D. Fortunato and E. Jannelli. *Critical exponents, critical dimensions and the biharmonic operator*, Arch. Rat. Mech. Anal. **112** (1990), 269-289.

[22] V. Felli, E. Hebey and R. Robert. *Fourth order equations of critical Sobolev growth. Energy function and solutions of bounded energy in the conformally flat case*, NoDEA. **12** (2005), 171-213.

[23] P. R. Garabedian. *Partial differential equations*, second edition (1986) Chelsea, New York.

[24] H. C. Grunau and G. Sweers. *Positivity for equations involving polyharmonic operators with Dirichlet boundary conditions*, Math. Ann. **307** (1997), 589-626.

[25] H. C. Grunau and G. Sweers. *Sign change for the Green function and for the first eigenfunction of equations of clamped-plate type*, Arch. Rat. Mech. Anal. **150** (1999), 179-190.

[26] H. C. Grunau and G. Sweers. *The maximum principle and positive principal eigenfunctions for polyharmonic equations*, Reaction diffusion systems (Trieste, 1995), 163-182 Lect. Notes in Pure and Appl. Math. **194** (1998), Dekker, New York.

[27] E. Hebey, F. Robert and Y. Wen. *Compactness and global estimates for a fourth order equation of critical Sobolev growth arising from conformal geometry*, Comm. Contemp. Math. **8** (2006), 9-65.

[28] N. Hirano. *Existence of entire positive solutions for nonhomogeneous elliptic equations*, Nonlinear Anal. **29** (1997), 889-901.

[29] L. Jeanjean and K. Tanaka. *A positive solution for a nonlinear Schrödinger equation on* $\mathbb{R}^N$, Indiana Univ. Math. J. **54** (2005), 443-464.

[30] R. A. Johnson, X. Pan and Y. Yi. *Singular solutions of the elliptic equation* $\Delta u - u + u^p = 0$, Ann. Mat. Pura Appl. **166** (1994), 203-225.

[31] P. L. Lions. *Isolated singularities in semilinear problems*, J. Diff. Eqns. **38** (1980), 441-450.

[32] Y. Naito and T. Sato. *Positive solutions for semilinear elliptic equations with singular forcing terms*, J. Diff. Eqns. **235** (2007), 435-483.

[33] S. Nazarov and G. Sweers. *A hinged plate equation and iterated Dirichlet Laplace operator on domains with concave corners*, J. Diff. Eqns. **233** (2007), 151-180.

[34] M. Ramos. *Uniform estimates for the biharmonic operator in* $\mathbb{R}^N$ *and applications*, Comm. Appl. Anal. **8** (2004), 435-457.

[35] T. Sato. *Positive solutions with weak isolated singularities to some semilinear elliptic equations*, Tohoku Math. J. **47** (1995), 55-80.

[36] T. Sato. *Positive solutions to some semilinear elliptic equations with nonnegative forcing terms*, preprint.

[37] H. S. Shapiro and M. Tegmark. *An elementary proof that the biharmonic Green function of an eccentric ellipse changes sign*, SIAM Rev. **36** (1994), 99-101.

[38] N. Stavrakakis. *Semilinear biharmonic problems on* $\mathbb{R}^N$, Reaction diffusion systems (Trieste, 1995), 365-376. Lect. Notes in Pure and Appl. Math. **194** (1998), Dekker, New York.

[39] N. Stavrakakis and G. Sweers. *Positivity for a noncooperative system of elliptic equations in* $\mathbb{R}^N$, Adv. Diff. Eqns. **4** (1999), 115-136.

[40] E. M. Stein. *Singular integrals and differentiability properties of functions*, (1970) Princeton Univ. Press, Princeton.

[41] G. Sweers. *No Gidas-Ni-Nirenberg type result for semilinear biharmonic problems*, Math. Nachr. **246/247** (2002), 202-206.

[42] L. Véron. *Elliptic equations involving measures*, in Handbook of differential equations I. stationary partial differential equations, (2004) North-Holland, Amsterdam.

[43] T. Watanabe. *Two positive solutions for an inhomogeneous scalar field equation*, preprint.

[44] H. F. Weinberger. *Variational methods for eigenvalue approximation*, Regional conference series in applied mathematics **15** (1994) SIAM. Pliladelphia.

[45] M. Willem. *Minimax theorems*, Prog. in nonlinear Diff. Eqns. and their applications, **24** (1996) Birkhäuser, Boston.

[46] X. P. Zhu. *A perturbation result on positive entire solutions of a semilinear elliptic equation*, J. Diff. Eqns. **92** (1991), 163-178.