BAND SURGERY ON KNOTS AND LINKS

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ABSTRACT

We give some relationships of the Jones and Q polynomials between two links which are related by a band surgery. Then we consider two applications: The first one is to an evaluation of the ribbon-fusion number, the least fusion number of a ribbon knot. The second one is to DNA knot theory, helping us to understand the action of the Xer site-specific recombination at *psi* site.

Keywords: Knot; Link; Band surgery; Jones polynomial; Q polynomial; ribbon-fusion number; DNA knot; Xer recombination.

Mathematics Subject Classification 2000: 57M25, 57M27

1. Introduction

Let L be an oriented link, and $b : I \times I \to S^3$ an embedding such that $b(I \times I) \cap L = b(I \times \partial I)$, where I is a closed interval. Let $L' = (L - b(I \times \partial I)) \cup b(\partial I \times I)$, which is another link. If L' has the orientation compatible with the orientation of $L - b(I \times I) \cap L$ and $b(\partial I \times I)$, then L' is called the link obtained from L by the *band surgery* along the band b. Then there is a relation between the signatures of L and L' due to Murasugi; see Eq. (2.2). In this paper, we give further relationships in terms of the Jones polynomial (Theorem 2.2) and the Q polynomial (Theorem 3.1). Then we apply these relations in two ways: The first application is to estimate the ribbon-fusion number of a ribbon knot. A knot is a *ribbon knot* if it is a knot obtained from a trivial $(m + 1)$-component link by doing band surgery along m bands for some m. We call the least number of such m the *ribbon-fusion number.* There is an estimation for this number due to Sakuma, which is given in terms of the Nakanishi index (Proposition 4.2). Using the above-mentioned relationships we deduce Theorems 4.3 and 4.4, which can give a sharper estimation (Examples 4.6, 4.7).

The second application is to consider a problem whether a given knot with $(2n + 1)$ crossings is related to a $(2, 2n)$ torus link or not by a band surgery, which was brought from the study of a DNA site-specific recombination. More precisely, Bath, Sherratt, and Colloms [1] have shown that the action of the Xer site-specific recombination at *psi* site is the change from a $(2, 2n)$ torus link to a $(2n+1)$-crossing knot by a band surgery. So characterizing such change is an important problem. Applying Theorems 2.2 and 3.1, we will show the 7 crossing knots 7_3, 7_6 cannot be obtained from a $(2, 6)$ torus link (Proposition 5.4), and the 9 crossing knots 9_{15}, 9_{17}, 9_{31} cannot be obtained from a $(2, 8)$ torus link (Propositions 5.6 and 5.7).

Notation. For knots with up to 10 crossings we use Rolfsen notation [23, Appendix C].

2. The Jones Polynomial

In this section, we give a relationship of the Jones polynomials of two links that are related by a band surgery. Before that we review a classical result for the signature of these links due to Murasugi. Let L_+, L_-, L_0 be three links that are identical except near one point where they are as in Fig. 1; we call (L_+, L_-, L_0) a *skein triple*. Then Murasugi [19, Lemma 7.1] has shown:

$$|\sigma(L_\pm) - \sigma(L_0)| \leq 1. \tag{2.1}$$

Since we may consider the link L_+ or L_- is obtained from L_0 by a band surgery, and vice versa, two links L and L' which are related by a band surgery satisfy:

$$|\sigma(L) - \sigma(L')| \leq 1. \tag{2.2}$$

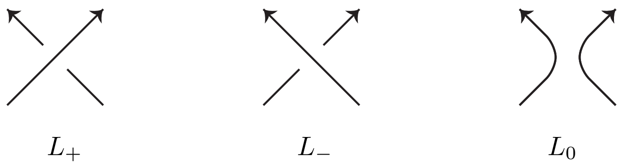


Fig. 1. A skein triple.

The *Jones polynomial* $V(L;t) \in \boldsymbol{Z}[t^{\pm 1/2}]$ [8], is an invariant of the isotopy type of an oriented link L, which is defined by the following formulas:

$$V(U;t) = 1; \tag{2.3}$$

$$t^{-1}V(L_+;t) - tV(L_-;t) = \left(t^{1/2} - t^{-1/2}\right) V(L_0;t), \tag{2.4}$$

where U is the unknot and (L_+, L_-, L_0) is a skein triple.

We put $\omega = e^{i\pi/3}$. For a knot K, Lickorish and Millett [14, Theorem 3] have shown:

$$V(L;\omega) = \pm i^{c(L)-1}(i\sqrt{3})^d, \tag{2.5}$$

where $c(L)$ is the number of the components of L, $d = \dim H_1(\Sigma(L); \boldsymbol{Z}_3)$ with $\Sigma(L)$ the double cover of S^3 branched over L; cf. [15]. Note that $V(L;\omega)$ means the value of $V(L;t)$ at $t^{1/2} = e^{i\pi/6}$, whence $t^{1/2} - t^{-1/2} = i$.

The following lemma is due to Miyazawa [17].

Lemma 2.1.

$$\frac{V(L_+;\omega)}{V(L_-;\omega)} \in \left\{\pm 1,\ i\sqrt{3}^{\pm 1}\right\} \tag{2.6}$$

Proof. For the skein triple (L_+, L_-, L_0), we consider another oriented link L_∞ which is one of the diagram of Fig. 2, the choice being (i) if $c(L_+) < c(L_0)$ and (ii) if $c(L_+) > c(L_0)$.

Fig. 2. Two choices of the oriented link L_∞.

Then by [2, Theorem 2] for the case (i) we have

$$V(L_+;t) - tV(L_-;t) + t^{3\lambda}(t-1)V(L_\infty;t) = 0, \tag{2.7}$$

where λ is the linking number of the right-hand component of L_0 in Fig. 1 with the remainder of L_0, and for the case (ii) we have

$$V(L_+;t) - tV(L_-;t) + t^{3(\mu-\frac{1}{2})}(t-1)V(L_\infty;t) = 0, \tag{2.8}$$

where μ is the linking number of the botttom-right and top-left component L_+ in in Fig. 1 with the remainder of L_+.

We consider the case (i). Putting $t = \omega$ in (2.7), we have

$$x - \omega + (-1)^\lambda(\omega - 1)y = 0, \tag{2.9}$$

where $x = V(L_+;\omega)/V(L_-;\omega)$ and $y = V(L_\infty;\omega)/V(L_-;\omega)$. Then by Eq. (2.5) there are four cases:

(a) $(x, y) = (\alpha, \beta)$;
(b) $(x, y) = (\alpha, \beta i)$;
(c) $(x, y) = (\alpha i, \beta)$;
(d) $(x, y) = (\alpha i, \beta i)$,

where α, β are real numbers. For the case (a), we have $\alpha = 1$, $\beta = (-1)^\lambda$; for the case (b), we have $\alpha = -1$, $\beta = (-1)^{\lambda+1}\sqrt{3}$; for the case (c), we have $\alpha = \sqrt{3}$, $\beta = (-1)^{\lambda+1}$; for the case (d), we have $\alpha = \sqrt{3}^{-1}$, $\beta = (-1)^{\lambda+1}\sqrt{3}^{-1}$, obtaining the result.

For the case (ii) we can prove similarly. □

Theorem 2.2. *Let L and L' be two links related with a band surgery such that $c(L) < c(L')$. Then*

$$\frac{V(L;\omega)}{V(L';\omega)} \in \left\{\pm i,\ -\sqrt{3}^{\pm 1}\right\} \tag{2.10}$$

Proof. From the condition there is a skein triple (L_+, L_-, L_0) such that L_+ and L_0 are isotopic to L and L', respectively. Put $x = V(L_+;\omega)/V(L_-;\omega)$ and $z = V(L_0;\omega)/V(L_-;\omega)$. Then by Eq. (2.4), we have

$$\omega^{-1}x - \omega = iz, \tag{2.11}$$

and so

$$\frac{V(L;\omega)}{V(L';\omega)} = \frac{x}{z} = \frac{ix}{\omega^{-1}x - \omega}. \tag{2.12}$$

By Lemma 2.1 we obtain (2.2). □

By using Eq. (2.5), Theorem 2.2 immediately implies the following.

Corollary 2.3. *Suppose that a knot K is obtained from a 2-component link L by a band surgery. Then*

$$V(K;\omega) \in \begin{cases} \{\pm 1,\ -i\sqrt{3}\epsilon\} & \text{if } V(L;\omega) = i\epsilon; \\ \{-\epsilon,\ \pm i\sqrt{3},\ -3\epsilon\} & \text{if } V(L;\omega) = \sqrt{3}\epsilon, \end{cases} \tag{2.13}$$

where $\epsilon = \pm 1$.

3. The Q Polynomial

In this section, we give a relationship of the Q polynomials of two links that are related by a band surgery. The *Q polynomial* $Q(L;z) \in \boldsymbol{Z}[z^{\pm 1}]$ [4,6] is an invariant of the isotopy type of an unoriented link L, which is defined by the following formulas:

$$Q(U;z) = 1; \tag{3.1}$$

$$Q(L_+;z) + Q(L_-;z) = z\left(Q(L_0;z) + Q(L_\infty;z)\right), \tag{3.2}$$

where U is the unknot and L_+, L_-, L_0, L_∞ are four unoriented links that are identical except near one point where they are as in Fig. 3. We call $(L_+, L_-, L_0, L_\infty)$ an *unoriented skein quadruple*.

We put $\rho(K) = Q\left(K; (\sqrt{5}-1)/2\right)$. For a knot K, Jones [9] has shown:

$$\rho(K) = \pm\sqrt{5}^{r}, \tag{3.3}$$

where $r = \dim H_1(\Sigma(K); \boldsymbol{Z}_5)$ with $\Sigma(K)$ the double cover of S^3 branched over K. Furthermore, Rong [24] deduced some information on the values $\rho(L_-)/\rho(L_\infty)$,

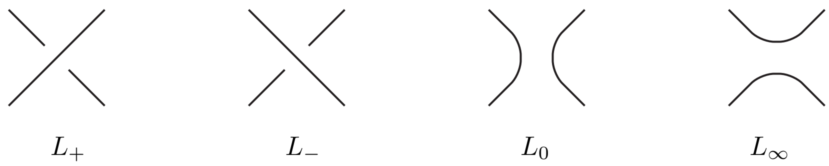


Fig. 3. An unoriented skein quadruple.

$\rho(L_0)/\rho(L_\infty)$, $\rho(L_+)/\rho(L_\infty)$, where $(L_+, L_-, L_0, L_\infty)$ is an unoriented skein quadruple. Using these values, we have the following, which is analogous to a criterion on the unknotting number of a knot due to Stoimenow [25, Theorem 4.1]; cf. [10].

Theorem 3.1. *If two links L and L' are related by a band surgery, then*

$$\rho(L)/\rho(L') \in \left\{ \pm 1, \sqrt{5}^{\pm 1} \right\}. \tag{3.4}$$

Proof. From the condition there is an unoriented skein quadruple $(L_+, L_-, L_0, L_\infty)$ such that L_0 and L_∞ are isotopic to L and L', respectively. Then from the proof of Theorem 2 in [24], we have $\rho(L_0)/\rho(L_\infty) \in \{\pm 1, \sqrt{5}^{\pm 1}\}$, obtaining the result. □

By using (3.3), Theorem 3.1 immediately implies the following.

Corollary 3.2. *Suppose that a knot K is obtained from a link L by a band surgery. Then*

$$\rho(K) \in \begin{cases} \{\pm 1, \sqrt{5}\epsilon\} & \text{if } \rho(L) = \epsilon; \\ \{1, \pm\sqrt{5}, 5\} & \text{if } \rho(L) = \sqrt{5}, \end{cases} \tag{3.5}$$

where $\epsilon = \pm 1$.

4. The Ribbon-Fusion Number of a Ribbon Knot

In this section, we apply the theorems given in the previous sections to an evaluation of the ribbon-fusion number of a ribbon knot. A knot is said to be a *ribbon knot of m-fusions* if it is a knot obtained from a trivial $(m+1)$-component link by doing band surgery along m bands. More precisely, it has the form

$$S_0^1 \cup S_1^1 \cup \cdots \cup S_m^1 \cup \bigcup_{i=1}^{m} f_i(\partial I \times I) - \operatorname{int}\left(\bigcup_{i=1}^{m} f_i(I \times \partial I)\right), \tag{4.1}$$

where $S_0^1 \cup S_1^1 \cup \cdots \cup S_m^1$ is a trivial link of m components and $f_i : I \times I \to S^3$ $(i = 1, 2, \ldots, m)$ are disjoint embeddings such that

$$f_i(I \times \partial I) \cup S_j = \begin{cases} f_i(I, 0) & \text{if } j = 0; \\ f_i(I, 1) & \text{if } j = i; \\ \emptyset & \text{if otherwise.} \end{cases} \tag{4.2}$$

By a *ribbon knot* we mean a ribbon knot of m-fusions for some m; see [16,27]. The least number of such m is the *ribbon-fusion number* of K, which we denote by $\mathrm{rf}(K)$.

Remark 4.1. In [3,22,26] the ribbon-fusion number is called the ribbon number.

If K and K' are ribbon knots, then it is easy to see

$$\mathrm{rf}(K \# K') \leq \mathrm{rf}(K) + \mathrm{rf}(K'). \tag{4.3}$$

Also, for any n-bridge knot K, the connected sum of K and its mirror image $K!$, $K \# K!$ is a ribbon knot (cf. [23, 8E30]), which satisfies

$$\mathrm{rf}(K \# K!) \leq n - 1. \tag{4.4}$$

Bleiler and Eudave-Muñoz [3] have shown a composite knot with ribbon-fusion number one has a summand that is two-bridge. Then Tanaka [26] proved that there exist composite ribbon-fusion number one knots with arbitrarily large bridge numbers.

The *Nakanishi index* of a knot K, denoted by $m(K)$, is the minimum size among all square Alexander matrix of K, provided that $m(K) = 0$ if and only if an Alexander matrix of K is equivalent to the 1×1 matrix with entry 1 as presentation matrices; see [11, p. 72]. Then Makoto Sakuma has given a lower bound of the ribbon-fusion number using the Nakanishi index of a knot [22, Proposition 2].

Proposition 4.2. *For a ribbon knot K,*

$$\mathrm{rf}(K) \geq m(K)/2. \tag{4.5}$$

As applications of Theorems 2.2 and 3.1 we give other lower bounds for the ribbon-fusion number.

Theorem 4.3. *If $\mathrm{rf}(K) = n$, then*

$$V(K; \omega) \in \left\{ 1, \pm(i\sqrt{3})^k, 3^n \;\middle|\; k = 1, 2, \ldots, 2n - 1 \right\}. \tag{4.6}$$

In particular, if K is a ribbon knot with $V(K; \omega) = -3^n$, then $\mathrm{rf}(K) > n$.

Proof. We use induction on n. If $\mathrm{rf}(K) = 1$, then K is obtained from the trivial 2-component link U^2 by a band surgery. Thus since $V(U^2; \omega) = -\sqrt{3}$, by Corollary 2.3 we obtain Eq. (4.6) with $n = 1$.

Suppose that Eq. (4.6) holds for $n = j$. If $\mathrm{rf}(K) = j + 1$, then K is obtained from the split union of a knot K' with $\mathrm{rf}(K') = j$ and the trivial knot, $K' \sqcup U$, by a band surgery. Then since $V(K' \sqcup U; \omega) = -\sqrt{3}V(K'; \omega)$, by Theorem 2.2 we have $V(K; \omega)/V(K'; \omega) \in \{\,1,\ \pm i\sqrt{3},\ 3\,\}$. Hence we obtain Eq. (4.6) with $n = j + 1$. □

Theorem 4.4. *If* $\mathrm{rf}(K) = n$*, then*

$$\rho(K) \in \left\{\, 1, \pm\sqrt{5}^k, 5^n \;\middle|\; k = 1, 2, \ldots, 2n-1 \,\right\}. \tag{4.7}$$

In particular, if K *is a ribbon knot with* $\rho(K) = -5^n$*, then* $\mathrm{rf}(K) > n$.

Proof. We use induction on n. If $\mathrm{rf}(K) = 1$, then K is obtained from the trivial 2-component link U^2 by a band surgery. Thus since $\rho(U^2) = \sqrt{5}$, by Corollary 3.2 we obtain Eq. (4.7) with $n = 1$.

Suppose that Eq. (4.7) holds for $n = j$. If $\mathrm{rf}(K) = j + 1$, then K is obtained from the split union of a knot K' with $\mathrm{rf}(K') = j$ and the trivial knot, $K' \sqcup U$, by a band surgery. Then by Theorem 3.1 we have $\rho(K)/\rho(K' \sqcup U) \in \{\pm 1, \sqrt{5}^{\pm 1}\}$. Since $Q(U^2; z) = 2z^{-1} - 1$ and $\rho(U^2) = \sqrt{5}$, $\rho(K' \sqcup U) = \rho(K')\rho(U^2) = \sqrt{5}\rho(K')$, and so we have $\rho(K)/\rho(K') \in \{1, \pm\sqrt{5}, 5\}$. Hence we obtain Eq. (4.7) with $n = j + 1$. □

Theorems 4.3 and 4.4 immediately imply:

Corollary 4.5. *If a knot* K *satisfies either* $V(K; \omega) = -1$ *or* $\rho(K) = -1$*, then* K *is not a ribbon knot.*

We denote the connected sum of n copies of a knot K by $\overset{n}{\#} K$.

Example 4.6. Let $J_{r,s}$ be the connected sum of r copies of the knot 6_1 and s copies of its mirror image $6_1!$. Suppose that $r \geq s$. Then putting $J_{r,s} = \left(\overset{r-s}{\#}\, 6_1\right) \# \left(\overset{s}{\#}(6_1 \# 6_1!)\right)$, we have $\mathrm{rf}(J_{r,s}) \leq r$. In fact, the knot 6_1 is a ribbon knot of 1-fusion (see [11, Appendix F.5]), and also the connected sum $6_1 \# 6_1!$ is a ribbon knot of 1-fusion since 6_1 is a 2-bridge knot. On the other hand, by Proposition 4.2, $\mathrm{rf}(J_{r,s}) \geq (r + s)/2$. Let us consider the case $s = r - 2$. Since $V(6_1; \omega) = i\sqrt{3}$, $V(6_1; \omega) = -i\sqrt{3}$ (cf. [12, Table 3.1]), we have $V(J_{r,r-2}; \omega) = -3^{r-1}$. Thus by Theorem 4.3 $\mathrm{rf}(J_{r,r-2}) \geq r$, and so $\mathrm{rf}(J_{r,r-2}) = r$, which cannot be deduced from Proposition 4.2.

Example 4.7. Let K_n be the connected sum of n copies of the knot 8_8, $(n - 1)$ copies of the knot $8_8!$, and the knot 8_9;

$$K_n = 8_9 \# 8_8 \# \left(\overset{n-1}{\#}\, (8_8 \# 8_8!)\right) \tag{4.8}$$

Then we have $\mathrm{rf}(K_n) = n + 1$. In fact, the knots 8_8 and 8_9 are ribbon knots of 1-fusion (see [11, Appendix F.5]), and the connected sum $8_8 \# 8_8!$ is also a ribbon knot of 1-fusion since 8_8 is a 2-bridge knot. Thus $\mathrm{rf}(K_n) \leq n+1$. On the other hand,

$\rho(8_8) = \sqrt{5}$, $\rho(8_9) = -\sqrt{5}$ (cf. [4, Table]) and so $\rho(K_n) = -5^n$. Thus by Theorem 4.4 $\mathrm{rf}(K_n) > n$. Note that using Proposition 4.2, we only have $\mathrm{rf}(K_n) \geq n$.

5. Band Surgery from a $(2, 2n)$ Torus Link to a $(2n + 1)$-Crossing Knot

The motivation of this section is the study of Bath, Sherratt, and Colloms [1] of a DNA site-specific recombination; they showed that the action of the Xer site-specific recombination at *psi* site is the change from a $(2, 2n)$ torus link to a $(2n+1)$-crossing knot by a band surgery. So characterizing such change is an important problem. In this section, we consider a problem whether a given knot with $(2n + 1)$ crossings is related to a $(2, 2n)$ torus link or not by a band surgery. Also, DNA knots or links are mainly of 2 bridge, so we consider this problem for 2-bridge knots with 7 or 9 crossings. Applying Corollary 2.3 or Corollary 3.2 we can conclude that some knot cannot be related with a $(2, 2n)$ torus link by a band surgery.

5.1. *Torus links*

First, we calculate some values for torus links needed to apply Eq. (2.2) and Corollaries 2.3 and 3.2. For a positive integer m, we denote by T_m the oriented torus knot or link of type $(2, m)$ with m crossings as shown in Fig. 4.

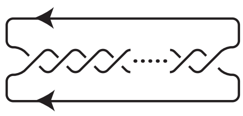
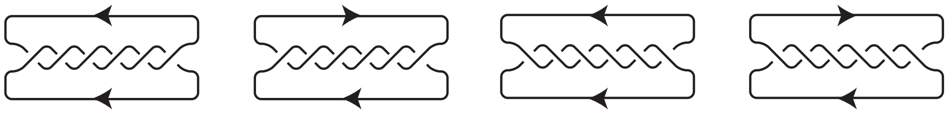


Fig. 4. The oriented torus knot or link of type $(2, m)$, T_m.

If m is even, then T_m is a 2-component link. We denote by T'_{2n} the oriented torus link obtained from T_{2n} by reversing the orientation of one component, and $T_{2n}!$, $T'_{2n}!$ the mirror images of T_{2n}, T'_{2n}, respectively. Fig. 5 shows torus links T_6, T'_6, $T_6!$, $T'_6!$.



T_6 T'_6 $T_6!$ $T'_6!$

Fig. 5. Oriented torus links of type $(2, 6)$.

Lemma 5.1. *The torus links T_{2n}, T'_{2n}, $T_{2n}!$, $T'_{2n}!$ have the linking numbers, the signatures, and the values of the Jones polynomials at $t = \omega$ as in Table 1.*

Table 1. The linking number, the signature, and the Jones polynomial at $t = \omega$ of the torus links of type $(2, 2n)$.

L	$\mathrm{lk}(L)$	$\sigma(L)$	$V(L;\omega)$ (mod 6)					
			$n \equiv 0$	$n \equiv 1$	$n \equiv 2$	$n \equiv 3$	$n \equiv 4$	$n \equiv 5$
T_{2n}	$-n$	$2n-1$	$-\sqrt{3}$	i	i	$\sqrt{3}$	$-i$	$-i$
T'_{2n}	n	-1	$-\sqrt{3}$	$-i$	i	$-\sqrt{3}$	$-i$	i
$T_{2n}!$	n	$-2n+1$	$-\sqrt{3}$	$-i$	$-i$	$\sqrt{3}$	i	i
$T'_{2n}!$	$-n$	1	$-\sqrt{3}$	i	$-i$	$-\sqrt{3}$	i	$-i$

Proof. The signatures of T_{2n} and $T_{2n}!$ are given in [5, Theorem 5.2]; cf. [21, Theorem 7.5.1]. We obtain the signatures of T'_{2n} and $T'_{2n}!$ by the following formula due to Murasugi [20, Theorem 1]:

$$\sigma(L') = \sigma(L) + 2\mathrm{lk}(L), \tag{5.1}$$

where L is an oriented 2-component link with linking number $\mathrm{lk}(L)$ and L' a link obtained from L by reversing the orientation of one component

Now we consider the Jones polynomial of T'_n. Since (T'_{2n}, T'_{2n-2}, U) is a skein triple, from Eq. (2.4) we have

$$t^{-1}V(T'_{2n};t) - tV(T'_{2n-2};t) = t^{1/2} - t^{-1/2}. \tag{5.2}$$

Then

$$\begin{aligned} V(T'_{2n};t) - \mu^{-1} &= t^2\left(V(T'_{2n-2};t) - \mu^{-1}\right) \\ &= t^{2n}\left(V(T'_0;t) - \mu^{-1}\right) \\ &= t^{2n}\left(\mu - \mu^{-1}\right), \end{aligned} \tag{5.3}$$

where $\mu = V(U^2;t) = -t^{1/2} - t^{-1/2}$. Then

$$\begin{aligned} V(T'_{2n};\omega) &= -\sqrt{3}^{-1} + \omega^{2n}\left(-\sqrt{3} + \sqrt{3}^{-1}\right) \\ &= \begin{cases} -\sqrt{3} & \text{if } n \equiv 0 \pmod 3; \\ -i & \text{if } n \equiv 1 \pmod 3; \\ i & \text{if } n \equiv 2 \pmod 3. \end{cases} \end{aligned} \tag{5.4}$$

Since $V(T'_{2n}!;t) = V(T'_{2n};t^{-1})$ [7, Theorem 3] and $\omega^{-1} = \overline{\omega}$, the complex conjugate of ω, we have $V(T'_{2n}!;\omega) = \overline{V(T'_{2n};\omega)}$. Since $V(T_{2n};t) = t^{-3n}V(T'_{2n};t)$ [13,18], we have $V(T_{2n};\omega) = (-1)^n V(T'_{2n};\omega)$. Similarly, we have $V(T_{2n}!;\omega) = (-1)^n V(T'_{2n}!;\omega)$. Then we obtain Table 1. □

Let $\rho_m = \rho(T_m)$. Then we have the following.

Lemma 5.2.

$$\rho_m = \begin{cases} \sqrt{5} & \textit{if } m \equiv 0 \pmod 5; \\ 1 & \textit{if } m \equiv 1, 4 \pmod 5; \\ -1 & \textit{if } m \equiv 2, 3 \pmod 5. \end{cases} \tag{5.5}$$

Proof. From an unoriented skein quadruple $(T_{m+1}, T_{m-1}, T_m, U^2)$, where U^2 is the trivial 2-component link, by Eq. (3.2) we have

$$Q(T_{m+1}; z) + Q(T_{m-1}; z) = z\left(Q(T_m; z) + Q(U^2; z)\right). \tag{5.6}$$

Since $\rho(U^2) = \sqrt{5}$, we have

$$\rho_{m+1} + \rho_{m-1} = \frac{\sqrt{5} - 1}{2}\left(\rho_m + \sqrt{5}\right). \tag{5.7}$$

Using $\rho_0 = \sqrt{5}$, $\rho_1 = 1$, we obtain Eq. (5.5). □

Combining Theorem 3.1 and Lemma 5.2, we obtain immediately the following.

Corollary 5.3. *Suppose that a knot K is obtained from a torus link of type $(2, 2n)$ by a band surgery. Then*

$$\rho(K) \in \begin{cases} \{\ 1, \pm\sqrt{5}, 5\ \} & \textit{if } n \equiv 0 \pmod 5; \\ \{\ \pm 1, -\sqrt{5}\ \} & \textit{if } n \equiv 1, 4 \pmod 5; \\ \{\ \pm 1, \sqrt{5}\ \} & \textit{if } n \equiv 2, 3 \pmod 5. \end{cases} \tag{5.8}$$

5.2. *7-crossing 2-bridge knots*

We consider the problem whether a 7-crossing 2-bridge knot is related to a $(2, 6)$ torus link or not by a band surgery. According to Shimokawa, the knots 7_1, 7_2, 7_4 are obtained from a $(2, 6)$ torus link.

First, we consider applying Corollary 2.3. The γ-values of $(2, 6)$ torus links are $\pm i\sqrt{3}$ from Table 1. Then we can apply Corollary 2.3 for a knot K with $V(K; \omega) = \pm 1$. Note that the determinant of such a knot is $\not\equiv 0 \pmod 3$; see Eq. (2.5). Since the determinants of the knots 7_3, 7_5, 7_6, 7_7 are 13, 17, 19, 21, respectively, we should test this method except for the knot 7_7. Then for the knot 7_3 and 7_6 we can obtain the result.

Proposition 5.4. *The knots 7_3 and 7_6 cannot be obtained from a $(2, 6)$ torus link by a band surgery.*

Proof. Suppose that the knot 7_6 is related with a $(2,6)$ torus link by a band surgery. Since $\sigma(7_6) = 2$ (cf. [12, Table 8.1]), by Eq. (2.2) $T_6'!$ should be such a torus link. From Table 1 $V(T_6'!;\omega) = -\sqrt{3}$, and so by Corollary 2.3 $V(7_6;\omega) \in \{1, \pm i\sqrt{3}, 3\}$, which is a contradiction since $V(7_6;\omega) = -1$ (cf. [12, Table 3.1]).

For the knot 7_3, the proof is similar. Since $\sigma(7_3) = -4$, we have to consider the link $T_6!$ Suppose that 7_3 is related with $T_6!$ by a band surgery. Since $V(T_6!;\omega) = \sqrt{3}$, $V(7_3;\omega) \in \{-1, \pm i\sqrt{3}, -3\}$, which is a contradiction since $V(7_3;\omega) = 1$. □

Remark 5.5. Kawauchi has proved that 7_3 and 7_7 cannot be obtained from a $(2,6)$ torus link by a band surgery using the Alexander invariants. Also, Darcy, Ishihara, Shimokawa have given a characterization of band surgery for the knots 7_2 and 7_4. So the question whether the knot 7_5, whose signature is -4, is related by a band surgery to a $(2,6)$ torus link $T_6!$ or not remains open.

For a 7 crossing knot, we cannot apply Corollary 5.3. In fact, in order to apply Corollary 5.3 the knot should satisfy $\rho(K) = -\sqrt{5}$. Then the determinant of such a knot is $\equiv 0 \pmod 5$; see Eq. (3.3).

5.3. 9-*crossing* 2-*bridge knots*

We consider the problem whether a 9-crossing 2-bridge knot is related to a $(2,8)$ torus link or not by a band surgery. Since $(2,8)$ torus links have signatures ± 1 or ± 7 (Table 1), a knot with signature ± 4 is never related to $(2,8)$ torus links by a band surgery by Eq. (2.2). The following knots have signature ± 4: 9_4, 9_7, 9_{10}, 9_{11}, 9_{13}, 9_{18}, 9_{20}, 9_{23}; see [11, Appendix F.3]. Also, it is easy to see that the knots 9_1, 9_2 are related to a $(2,8)$ torus link by a band surgery.

First, we consider applying Corollary 2.3. The γ-values of $(2,8)$ torus links are ± 1 (Table 1), and so we can apply Corollary 2.3 for a knot K with $V(K;\omega) = \pm i\sqrt{3}$. Note that the determinant of such a knot is $\equiv 0 \pmod 3$; see Eq. (2.5). Thus we apply this method for the knots 9_6, 9_{15}, 9_{17}, whose determinants are 27, 39, 39, respectively.

Proposition 5.6. *The knots 9_{15} and 9_{17} cannot be obtained from a $(2,8)$ torus link by a band surgery.*

Proof. The proof is similar to that of Proposition 5.4. We list the necessary data:

$$\sigma(9_{15}) = -2, \quad V(9_{15};\omega) = -i\sqrt{3}; \tag{5.9}$$

$$\sigma(9_{17}) = 2, \quad V(9_{17};\omega) = i\sqrt{3}. \tag{5.10}$$

□

Next, we consider applying Corollary 5.3. We can apply Corollary 2.3 for a knot K with $\rho(K) = -\sqrt{5}$. Note that the determinant of such a knot is $\equiv 0 \pmod 5$. Thus we apply this method for the knots 9_6, 9_{15}, 9_{17}.

Proposition 5.7. *The knot 9_{31} cannot be obtained from a $(2,8)$ torus link by a band surgery.*

Proof. Suppose that the knot 9_{31} is related with a $(2,8)$ torus link by a band surgery. By Lemma 5.2 $\rho(T_8) = -1$, and so by Corollary 3.2 $\rho(9_{31}) \in \{\pm 1, -\sqrt{5}\}$, which is a contradiction since $\rho(9_{31}) = \sqrt{5}$. (Note that $Q(9_{31}) = -7 + 12z + 36z^2 - 22z^3 - 58z^4 - 4z^5 + 28z^7 + 14z^8 + 2z^9$, which is obtained from the Kauffman F polynomial listed in [11, Appendix F.6].) □

For the following 9 crossing 2-bridge knots we cannot decide whether they are related to a $(2,8)$ torus link or not by a band surgery using our method:

$$9_k, \quad k = 3, 5, 6, 8, 9, 12, 14, 19, 21, 26, 27. \tag{5.11}$$

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