An eigenvalue problem related to blowing-up solutions for a semilinear elliptic equation with the critical Sobolev exponent

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Abstract

We consider the eigenvalue problem

$$\begin{cases} -\Delta v = \lambda \left(c_0 p u_\varepsilon^{p-1} + \varepsilon \right) v & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \\ \|v\|_{L^\infty(\Omega)} = 1 \end{cases}$$

where $\Omega \subset \mathbb{R}^N (N \geq 5)$ is a smooth bounded domain, $c_0 = N(N-2)$, $p = (N+2)/(N-2)$ is the critical Sobolev exponent and $\varepsilon > 0$ is a small parameter. Here u_ε is a positive solution of

$$-\Delta u = c_0 u^p + \varepsilon u \text{ in } \Omega, \quad u|_{\partial\Omega} = 0$$

with the property that

$$\frac{\int_\Omega |\nabla u_\varepsilon|^2 dx}{\left(\int_\Omega |u_\varepsilon|^{p+1} dx \right)^{\frac{2}{p+1}}} \to S_N \quad \text{as } \varepsilon \to 0,$$

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where S_N is the best constant for the Sobolev inequality. In this paper, we show several asymptotic estimates for the eigenvalues $\lambda_{i,\varepsilon}$ and corresponding eigenfunctions $v_{i,\varepsilon}$ for $i = 1, 2, \cdots, N+1, N+2$.

1 Introduction

Consider the problem

$$(P_\varepsilon) \begin{cases} -\Delta u = c_0 u^p + \varepsilon u & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where $\Omega \subset \mathbb{R}^N (N \geq 4)$ is a smooth bounded domain, $c_0 = N(N-2)$, $p = (N+2)/(N-2)$ is the critical Sobolev exponent, and $\varepsilon > 0$ is a small parameter. In the following, u_ε will denote a positive solution of (P_ε) with the property

$$\frac{\int_\Omega |\nabla u_\varepsilon|^2 dx}{\left(\int_\Omega |u_\varepsilon|^{p+1} dx\right)^{\frac{2}{p+1}}} \to S_N \quad \text{as } \varepsilon \to 0, \tag{1.1}$$

where S_N is the best Sobolev constant in $\mathbb{R}^N$. By a result of Han [4] and Rey [5], solution sequence $\{u_\varepsilon\}$ satisfying (1.1) blows up at an interior point $x_0 \in \Omega$ in the sense that $\|u_\varepsilon\|_{L^\infty(\Omega)} \to \infty$ as $\varepsilon \to 0$ and the maximum point x_ε of u_ε accumulates to x_0. Moreover, x_0 has to be a critical point of the (positive) Robin function R defined as $R(x) = \lim_{z \to x} \left[\frac{1}{(N-2)\sigma_N} |x-z|^{2-N} - G(x,z)\right]$, where σ_N is the volume of the unit sphere in $\mathbb{R}^N$ and $G(x,z)$ is Green's function of $-\Delta$ with the Dirichlet boundary condition.

We are interested in some spectral properties of this blowing-up solution u_ε to (P_ε). For this purpose, let us consider the eigenvalue problem

$$\begin{cases} -\Delta v = \lambda \left(c_0 p u_\varepsilon^{p-1} + \varepsilon\right) v & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \\ \|v\|_{L^\infty(\Omega)} = 1. \end{cases} \tag{1.2}$$

In the following, the symbol $\|\cdot\|$ will denote $\|\cdot\|_{L^\infty(\Omega)}$. By a general theory, we know that there exists a countable sequence of eigenvalues $\lambda_{1,\varepsilon} \leq \lambda_{2,\varepsilon} \leq \cdots \leq \lambda_{i,\varepsilon} \leq \cdots \to +\infty$ and corresponding eignefunctions $v_{1,\varepsilon}, v_{2,\varepsilon}, \cdots, v_{i,\varepsilon}, \cdots$ with the orthogonal relation

$$\int_\Omega \left(c_0 p u_\varepsilon^{p-1} + \varepsilon\right) v_{i,\varepsilon} v_{j,\varepsilon} dx = 0, \quad (i \neq j). \tag{1.3}$$

To state the results, we introduce the scaled eigenfunctions

$$\tilde{v}_{i,\varepsilon}(y) = v_{i,\varepsilon}\left(\frac{y}{\|u_\varepsilon\|^{\frac{p-1}{2}}} + x_\varepsilon\right), \quad y \in \Omega_\varepsilon = \|u_\varepsilon\|^{(p-1)/2}(\Omega - x_\varepsilon). \tag{1.4}$$

Theorem 1.1 *Assume $N \geq 5$. As $\varepsilon \to 0$, we have*

$$\begin{aligned}
&\lambda_{1,\varepsilon} \to 1/p, \\
&\tilde{v}_{1,\varepsilon}(y) \to U(y) = \left(\frac{1}{1+|y|^2}\right)^{\frac{N-2}{2}} \quad in\ C^2_{loc}(\mathbb{R}^N), \\
&\|u_\varepsilon\|^2 v_{1,\varepsilon} \to (N-2)\sigma_N G(\cdot, x_0)\ in\ C^1_{loc}(\overline{\Omega} \setminus \{x_0\}).
\end{aligned}$$

Also, $\lambda_{1,\varepsilon}$ is simple for $\varepsilon > 0$ sufficiently small.

Theorem 1.2 *Assume $N \geq 6$. Then for $i = 2, 3, \cdots, N+1$, we have*

$$\tilde{v}_{i,\varepsilon}(y) \to \sum_{j=1}^N a_{i,j} \frac{y_j}{(1+|y|^2)^{\frac{N}{2}}} \quad in\ C^1_{loc}(\mathbb{R}^N), \tag{1.5}$$

$$\|u_\varepsilon\|^{2+\frac{2}{N-2}} v_{i,\varepsilon}(x) \to \sigma_N \sum_{j=1}^N a_{i,j} \left(\frac{\partial G}{\partial z_j}\right)(x,z)|_{z=x_0} \quad in\ C^1_{loc}(\overline{\Omega} \setminus \{x_0\}) \tag{1.6}$$

for some $\vec{a}_i = (a_{i,1}, a_{i,2}, \cdots, a_{i,N}) \neq \vec{0}$ as $\varepsilon \to 0$. In addition,

$$\|u_\varepsilon\|^{\frac{2N}{N-2}} (\lambda_{i,\varepsilon} - 1) \to M\mu_{i-1}, \quad \varepsilon \to 0, \tag{1.7}$$

where $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_N$ are eigenvalues of $Hess R(x_0)$ and

$$M = \frac{(N-2)\sigma_N^2}{2p \int_{\mathbb{R}^N} U^{p-1} |\nabla U|^2 dy} = \frac{\sigma_N \Gamma(N+2)}{(N+2)\Gamma(N/2+1)^2} > 0.$$

Furthermore, $\vec{a}_i$ is an eigenvector of $Hess R(x_0)$ corresponding to μ_{i-1} and $\vec{a}_i$ is perpendicular to $\vec{a}_j$ in $\mathbb{R}^N$ if $i \neq j$.

Theorem 1.3 *Assume $N \geq 6$. As $\varepsilon \to 0$, we have*

$$\tilde{v}_{N+2,\varepsilon}(y) \to b_{N+2} \frac{1-|y|^2}{(1+|y|^2)^{\frac{N}{2}}} \quad \text{in } C^1_{loc}(\mathbb{R}^N) \tag{1.8}$$

for some $b_{N+2} \neq 0$, and

$$\|u_\varepsilon\|^2 (\lambda_{N+2,\varepsilon} - 1) \to \Gamma, \tag{1.9}$$

where

$$\Gamma = \frac{(N-2)^2(N-4)\sigma_N^2 R(x_0)}{c_0 p(\frac{N-2}{2}) \int_{\mathbb{R}^N} \frac{(1-|y|^2)^2}{(1+|y|^2)^{N+2}} dy} = (N-2)(N-4)MR(x_0) > 0.$$

In [3], Grossi and Pacella considered the eigenvalue problem

$$\begin{cases} -\Delta v = \lambda \left(c_0(p-\varepsilon)u_\varepsilon^{p-\varepsilon-1}\right) v & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \\ \|v\|_{L^\infty(\Omega)} = 1 \end{cases}$$

on a smooth bounded domain $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$, where u_ε is a solution of the slightly subcritical problem

$$\begin{cases} -\Delta u = c_0 u^{p-\varepsilon} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with the property $\lim_{\varepsilon \to 0} \frac{\int_\Omega |\nabla u_\varepsilon|^2 dx}{\left(\int_\Omega |u_\varepsilon|^{p-\varepsilon+1} dx\right)^{\frac{2}{p-\varepsilon+1}}} = S_N$.

In addition to the qualitative properties of eigenfunctions, they obtained analogous results about the asymptotic behavior of eigenvalues and eigenfunctions as $\varepsilon \to 0$. We will prove above theorems along the line in [3]. However, we have to control additional linear term $\varepsilon u_\varepsilon$ in (P_ε), which causes some difficulties.

As for the qualitative properties of eigenfunctions, we have the same theorem in [3]. We omit the proof of the next theorem since the proof in [3] works well also in our case.

Theorem 1.4 *Assume $N \geq 6$. Define $N_{i,\varepsilon} = \{x \in \Omega \mid v_{i,\varepsilon}(x) = 0\}$ for $i \in \mathbb{N}$. Then for $\varepsilon > 0$ sufficiently small, we have the followings.*

(1) *The eigenfunctions* $v_{i,\varepsilon}$ *has only two nodal regions for* $i = 2, \cdots, N+1$.

(2) $\overline{N_{i,\varepsilon}} \cap \partial\Omega \neq \phi$ *if* Ω *is convex and* $i = 2, \cdots, N+1$.

(3) $\lambda_{N+2,\varepsilon}$ *is simple and* $v_{N+2,\varepsilon}$ *has only two nodal regions. Moreover* $\overline{N_{N+2,\varepsilon}} \cap \partial\Omega = \phi$.

2 Preliminaries

In this section, we collect lemmas which are needed in the proof.

Lemma 2.1 *The following identities hold true. For any* $i \in \mathbb{N}$ *and for any* $y \in \mathbb{R}^N$,

$$\int_{\partial\Omega} (x-y)\cdot\nu(\frac{\partial u_\varepsilon}{\partial\nu})(\frac{\partial v_{i,\varepsilon}}{\partial\nu})ds_x = (1-\lambda_{i,\varepsilon})\int_\Omega \left(c_0 p u_\varepsilon^{p-1} + \varepsilon\right) w_\varepsilon v_{i,\varepsilon} dx + 2\varepsilon\int_\Omega u_\varepsilon v_{i,\varepsilon}dx, \tag{2.1}$$

where $w_\varepsilon(x) = (x-y)\cdot\nabla u_\varepsilon + \frac{2}{p-1}u_\varepsilon$, *and*

$$\int_{\partial\Omega}(\frac{\partial u_\varepsilon}{\partial x_j})(\frac{\partial v_{i,\varepsilon}}{\partial\nu})ds_x = (1-\lambda_{i,\varepsilon})\int_\Omega\left(c_0 p u_\varepsilon^{p-1}+\varepsilon\right)(\frac{\partial u_\varepsilon}{\partial x_j})v_{i,\varepsilon}dx \tag{2.2}$$

where $\nu = \nu(x)$ *is the unit outer normal at* $x \in \partial\Omega$.

Proof. By an easy calculation, w_ε satisfies

$$-\Delta w_\varepsilon = (c_0 p u_\varepsilon^{p-1}+\varepsilon)w_\varepsilon + 2\varepsilon u_\varepsilon \quad \text{in } \Omega. \tag{2.3}$$

Then follow the proof of Lemma 4.3 and Lemma 5.1 in [3] with (2.3).

Denote

$$\tilde{u}_\varepsilon(y) = \frac{1}{\|u_\varepsilon\|}u_\varepsilon\left(\frac{y}{\|u_\varepsilon\|^{\frac{p-1}{2}}}+x_\varepsilon\right), \quad y\in\Omega_\varepsilon. \tag{2.4}$$

By a result in [4], we see

$$\tilde{u}_\varepsilon \to U(y) = \left(\frac{1}{1+|y|^2}\right)^{\frac{N-2}{2}} \text{ in } C^2_{loc}(\mathbb{R}^N)\cap H^1(\mathbb{R}^N). \tag{2.5}$$

Furthermore, we have

Theorem 2.2 *(Han [4] and Rey [5]) Assume $N \geq 4$ and let $x_\varepsilon \in \Omega$ be a point such that $u_\varepsilon(x_\varepsilon) = \|u_\varepsilon\|$. Then after passing to a subsequence, we have the followings: There exists a constant $C > 0$ independent of ε such that*

$$u_\varepsilon(x) \leq C \frac{\|u_\varepsilon\|}{(1 + \|u_\varepsilon\|^{p-1}|x - x_\varepsilon|^2)^{\frac{N-2}{2}}}, \quad (\forall x \in \Omega), \tag{2.6}$$

$$\|u_\varepsilon\| u_\varepsilon \to (N-2)\sigma_N G(\cdot, x_0) \text{ in } C^1_{loc}(\overline{\Omega} \setminus \{x_0\}), \tag{2.7}$$

as $\varepsilon \to 0$, and

$$\begin{aligned} &\lim_{\varepsilon \to 0} \varepsilon \|u_\varepsilon\|^{\frac{2(N-4)}{N-2}} = \frac{(N-2)^3}{2a_N} \sigma_N R(x_0) && (N \geq 5), \\ &\lim_{\varepsilon \to 0} \varepsilon \log \|u_\varepsilon\| = 4\sigma_4 R(x_0) && (N = 4), \end{aligned} \tag{2.8}$$

where $\sigma_N a_N = \int_{\mathbb{R}^N} U^2 dy$.

Theorem 2.3 *(Bianchi and Egnell [1]) The eigenvalue problem*

$$\begin{cases} -\Delta V_i = \lambda_i c_0 p U^{p-1} V_i & \text{in } \mathbb{R}^N, \\ V_i \in D^{1,2}(\mathbb{R}^N) \end{cases}$$

where $D^{1,2}(\mathbb{R}^N) = \{V \in L^{2N/(N-2)}(\mathbb{R}^N) : \int_{\mathbb{R}^N} |\nabla V|^2 dy < +\infty\}$, has eigenvalues

$$\lambda_1 = 1/p < \lambda_2 = \lambda_3 = \cdots = \lambda_{N+1} = \lambda_{N+2} = 1 < \lambda_{N+3} \leq \cdots$$

with eigenfunctions

$$\begin{aligned} &V_1 = U = \left(\frac{1}{1 + |y|^2} \right)^{\frac{N-2}{2}}, \quad V_i = \frac{\partial U}{\partial y_{i-1}}, (i = 2, \cdots, N+1), \\ &V_{N+2} = \frac{d}{d\lambda}\Big|_{\lambda=1} \lambda^{(N-2)/2} U(\lambda y) = y \cdot \nabla U + \frac{N-2}{2} U. \end{aligned}$$

Note that the pointwise estimate (2.6) is equivalent to

$$\tilde{u}_\varepsilon(y) \leq CU(y), \quad \forall y \in \Omega_\varepsilon. \tag{2.9}$$

Also, we need the following pointwise estimate for eigenfunctions. For the proof, see [2]. In the sequel, we assume always $N \geq 5$.

Lemma 2.4 *For any $i \in \mathbb{N}$, there exists a constant $C > 0$ independent of ε such that*

$$|\tilde{v}_{i,\varepsilon}(y)| \leq CU(y) \tag{2.10}$$

holds true for all $y \in \Omega_\varepsilon$.

By elliptic estimates, (2.9) and (2.10), there exists some V_i such that

$$\tilde{v}_{i,\varepsilon} \to V_i \quad \text{in } C^1_{loc}(\mathbb{R}^N) \quad (i \in \mathbb{N}).$$

Also we can check that $\int_{\Omega_\varepsilon} |\nabla \tilde{v}_{i,\varepsilon}|^2 dy \leq C$ (see [2]), so $V_i \in D^{1,2}(\mathbb{R}^N)$. Put $\lambda_i = \lim_{\varepsilon \to 0} \lambda_{i,\varepsilon}$. Then by (2.5) and the equation satisfied by $\tilde{v}_{i,\varepsilon}$, V_i satisfies

$$\begin{cases} -\Delta V_i = \lambda_i \left(c_0 p U^{p-1}\right) V_i \quad \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |\nabla V_i|^2 dy < \infty. \end{cases}$$

We see that $V_i \not\equiv 0$ by the estimate (2.10). Thus by Theorem 2.3, we have the following.

Lemma 2.5 *Suppose $\lambda_i = \lim_{\varepsilon \to 0} \lambda_{i,\varepsilon} = 1$. Then*

$$\tilde{v}_{i,\varepsilon} \to V_i = \sum_{j=1}^{N} a_{i,j} \frac{y_j}{(1+|y|^2)^{N/2}} + b_i \frac{1-|y|^2}{(1+|y|^2)^{N/2}} \quad \text{in } C^1_{loc}(\mathbb{R}^N) \tag{2.11}$$

as $\varepsilon \to 0$ for some $(a_{i,1}, a_{i,2}, \cdots, a_{i,N}, b_i) \neq (0, 0, \cdots, 0)$.

From Lemma 2.5, we can obtain the following convergence result. See [3].

Lemma 2.6 *Suppose $\lambda_i = \lim_{\varepsilon \to 0} \lambda_{i,\varepsilon} = 1$ and $b_i \neq 0$ in (2.11). Then we have*

$$\|u_\varepsilon\|^2 v_{i,\varepsilon} \to -(N-2) b_i \sigma_N G(\cdot, x_0) \text{ in } C^1_{loc}(\overline{\Omega} \setminus \{x_0\}) \quad \text{as } \varepsilon \to 0. \tag{2.12}$$

Now, since the blow-up point x_0 is an interior point of Ω, we may assume that there exists $\rho > 0$ such that $B(x_\varepsilon, 2\rho) \subset \Omega$ for any $\varepsilon > 0$ sufficiently small. We employ a cut-off function $\phi = \phi(x)$ such that $\phi \in C_0^\infty(B(x_\varepsilon, 2\rho))$, $0 \leq \phi \leq 1$ and $\phi \equiv 1$ on $B(x_\varepsilon, \rho)$. Denote

$$\psi_{j,\varepsilon}(x) = \phi(x) \left(\frac{\partial u_\varepsilon}{\partial x_j} \right), \quad j = 1, \cdots, N, \tag{2.13}$$

$$\psi_{N+1,\varepsilon}(x) = \phi(x) \left((x - x_\varepsilon) \cdot \nabla u_\varepsilon + \frac{2}{p-1} u_\varepsilon \right). \tag{2.14}$$

Then, as Lemma 3.1 in [3], we have the following lemma.

Lemma 2.7 *$u_\varepsilon, \{\psi_{j,\varepsilon}\}_{j=1,\cdots,N}, \psi_{N+1,\varepsilon}$ are linearly independent in $H_0^1(\Omega)$.*

Proof. Assume the contrary that there exist $\alpha_{0,\varepsilon}, \alpha_{1,\varepsilon}, \cdots, \alpha_{N,\varepsilon}, \alpha_{N+1,\varepsilon}$ such that $\sum_{j=0}^{N+1} \alpha_{j,\varepsilon}^2 \neq 0$ and

$$\alpha_{0,\varepsilon} u_\varepsilon + \sum_{j=1}^{N} \alpha_{j,\varepsilon}\psi_{j,\varepsilon} + \alpha_{N+1,\varepsilon}\psi_{N+1,\varepsilon} \equiv 0$$

in Ω. Without loss of generality, we may assume that $\sum_{j=0}^{N+1} \alpha_{j,\varepsilon}^2 = 1$.

First we claim that $\alpha_{0,\varepsilon} = 0$. Indeed, if $\alpha_{0,\varepsilon} \neq 0$, then we have $u_\varepsilon = \sum_{j=1}^{N+1} \beta_{j,\varepsilon}\psi_{j,\varepsilon}$ where $\beta_{j,\varepsilon} = -\alpha_{j,\varepsilon}/\alpha_{0,\varepsilon}$. Putting $x = x_\varepsilon$ to the both sides and noting $\nabla u_\varepsilon(x_\varepsilon) = 0$, we have $\|u_\varepsilon\| = \beta_{N+1,\varepsilon}\frac{2}{p-1}\|u_\varepsilon\|$, thus $\beta_{N+1,\varepsilon} = \frac{p-1}{2}$ if $\alpha_{0,\varepsilon} \neq 0$. On the other hand, by differentiating the equation of (P_ε) and noting $\phi \equiv 1$ on $B(x_\varepsilon, \rho)$, we see

$$-\Delta\psi_{j,\varepsilon} = \left(c_0 p u_\varepsilon^{p-1} + \varepsilon\right)\psi_{j,\varepsilon} \quad \text{on } B(x_\varepsilon, \rho), \quad (j = 1, \cdots, N). \tag{2.15}$$

Recall $w_\varepsilon(x) = (x - x_\varepsilon)\cdot\nabla u_\varepsilon(x) + \frac{2}{p-1}u_\varepsilon$ satisfies (2.3), thus

$$-\Delta\psi_{N+1,\varepsilon} = \left(c_0 p u_\varepsilon^{p-1} + \varepsilon\right)\psi_{N+1,\varepsilon} + 2\varepsilon u_\varepsilon \quad \text{on } B(x_\varepsilon, \rho). \tag{2.16}$$

Multiplying $\beta_{j,\varepsilon}$ to (2.15) and $\beta_{N+1,\varepsilon}$ to (2.16), and summing up, we have

$$-\Delta\left(\sum_{j=1}^{N+1}\beta_{j,\varepsilon}\psi_{j,\varepsilon}\right) = \left(c_0 p u_\varepsilon^{p-1} + \varepsilon\right)\left(\sum_{j=1}^{N+1}\beta_{j,\varepsilon}\psi_{j,\varepsilon}\right) + 2\varepsilon\beta_{N+1,\varepsilon}u_\varepsilon$$

on $B(x_\varepsilon, \rho)$. Moreover, since $u_\varepsilon = \sum_{j=1}^{N+1}\beta_{j,\varepsilon}\psi_{j,\varepsilon}$ is a solution to (P_ε), we have

$$-\Delta\left(\sum_{j=1}^{N+1}\beta_{j,\varepsilon}\psi_{j,\varepsilon}\right) = \left(c_0 u_\varepsilon^{p-1} + \varepsilon\right)\left(\sum_{j=1}^{N+1}\beta_{j,\varepsilon}\psi_{j,\varepsilon}\right).$$

Comparing both RHS's, we have $c_0(1-p)u_\varepsilon^{p-1} \equiv 2\varepsilon\beta_{N+1,\varepsilon}$ on $B(x_\varepsilon, \rho)$, which is impossible for $\beta_{N+1,\varepsilon} = \frac{p-1}{2} > 0$. Therefore we conclude that $\alpha_{0,\varepsilon} = 0$.

Next, we claim that $\alpha_{N+1,\varepsilon} = 0$. Indeed, putting $x = x_\varepsilon$ into $\sum_{j=1}^{N}\alpha_{j,\varepsilon}\psi_{j,\varepsilon} + \alpha_{N+1,\varepsilon}\psi_{N+1,\varepsilon} \equiv 0$ and noting $\phi(x_\varepsilon) = 1$ and $\nabla u_\varepsilon(x_\varepsilon) = 0$, we see $\alpha_{N+1,\varepsilon}(\frac{2}{p-1})u_\varepsilon(x_\varepsilon) = 0$. Thus we obtain $\alpha_{N+1,\varepsilon} = 0$.

Now, we obtain $\sum_{j=1}^N \alpha_{j,\varepsilon}\psi_{j,\varepsilon} \equiv 0$ on Ω. By scaling, this leads to

$$\sum_{j=1}^N \alpha_{j,\varepsilon}\phi_\varepsilon(y)\frac{\partial \tilde{u}_\varepsilon}{\partial y_j}(y) \equiv 0$$

for $y \in \Omega_\varepsilon$, where $\phi_\varepsilon(y) = \phi(\frac{y}{\|u_\varepsilon\|^{\frac{p-1}{2}}} + x_\varepsilon)$. Using $\tilde{u}_\varepsilon \to U$ in $C^2_{loc}(\mathbb{R}^N)$ as $\varepsilon \to 0$, we get that $\sum_{j=1}^N \alpha_j \frac{\partial U}{\partial y_j} \equiv 0$ on $\mathbb{R}^N$, where $\alpha_j = \lim_{\varepsilon\to 0}\alpha_{j,\varepsilon}$. Since $\frac{\partial U}{\partial y_j}$ are linearly independent, we have that $\alpha_j = 0$ for all $j = 1, 2, \cdots, N$. But this is impossible since $\sum_{j=1}^N \alpha_j^2 = \lim_{\varepsilon\to 0}(\sum_{j=1}^N \alpha_{j,\varepsilon}^2) = 1$. Thus we have proved Lemma 2.7.

3 Proof of Theorem 1.1

In this section, we prove Theorem 1.1. By the variational characterization of $\lambda_{1,\varepsilon}$, we have

$$\lambda_{1,\varepsilon} = \inf_{v\in H_0^1(\Omega)} \frac{\int_\Omega |\nabla v|^2 dx}{\int_\Omega \left(c_0 p u_\varepsilon^{p-1} + \varepsilon\right) v^2 dx}.$$

Inserting $v = u_\varepsilon$, we see

$$\lambda_{1,\varepsilon} \le \frac{\int_\Omega |\nabla u_\varepsilon|^2 dx}{\int_\Omega \left(c_0 p u_\varepsilon^{p-1} + \varepsilon\right) u_\varepsilon^2 dx} = \frac{\int_\Omega \left(c_0 u_\varepsilon^{p-1} + \varepsilon\right) u_\varepsilon^2 dx}{\int_\Omega \left(c_0 p u_\varepsilon^{p-1} + \varepsilon\right) u_\varepsilon^2 dx}.$$

By scaling, the right hand side can be estimated as

$$\begin{aligned}\lambda_{1,\varepsilon} &\le \frac{c_0 \int_{\Omega_\varepsilon} \tilde{u}_\varepsilon^{p+1} dy + \varepsilon\|u_\varepsilon\|^{-4/(N-2)}\int_{\Omega_\varepsilon}\tilde{u}_\varepsilon^2 dy}{c_0 p\int_{\Omega_\varepsilon} \tilde{u}_\varepsilon^{p+1} dy + \varepsilon\|u_\varepsilon\|^{-4/(N-2)}\int_{\Omega_\varepsilon}\tilde{u}_\varepsilon^2 dy} \\ &= \frac{c_0\int_{\mathbb{R}^N} U^{p+1}dy + o(1)}{c_0 p\int_{\mathbb{R}^N} U^{p+1}dy + o(1)}\end{aligned}$$

as $\varepsilon \to 0$, which implies $\limsup_{\varepsilon\to 0}\lambda_{1,\varepsilon} \le 1/p$. Hence by choosing a subsequence, we may assume that $\lambda_{1,\varepsilon} \to \lambda \in [0, 1/p]$. Now, $\tilde{v}_{1,\varepsilon}$ satisfies

$$\begin{cases} -\Delta\tilde{v}_{1,\varepsilon} = \lambda_{1,\varepsilon}\left(c_0 p\tilde{u}_\varepsilon^{p-1} + \frac{\varepsilon}{\|u_\varepsilon\|^{p-1}}\right)\tilde{v}_{1,\varepsilon} & \text{in } \Omega_\varepsilon, \\ \tilde{v}_{1,\varepsilon} = 0 & \text{on } \partial\Omega_\varepsilon. \end{cases}$$

As in the proof of Lemma 2.5, we see that $\tilde{v}_{1,\varepsilon}$ is bounded in $D^{1,2}(\mathbb{R}^N)$ and $\tilde{v}_{1,\varepsilon} \to V_1$ for some $0 \not\equiv V_1 \in D^{1,2}(\mathbb{R}^N)$. Letting $\varepsilon \to 0$, we see V_1 satisfies

$$\begin{cases} -\Delta V_1 = \lambda \left(c_0 p U^{p-1}\right) V_1 \quad \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |\nabla V_1|^2 dy < \infty, \quad \|V_1\|_{L^\infty(\mathbb{R}^N)} = 1. \end{cases}$$

Since there exists no eigenvalue λ less than $1/p$ by Theorem 2.3, we must have $\lambda = 1/p$ and $V_1 = U$.

Now, let us prove that $\lambda_{1,\varepsilon}$ is simple for small ε. Indeed, assume there exist two eigenfunctions $v_{1,\varepsilon}$ and $w_{1,\varepsilon}$ corresponding to $\lambda_{1,\varepsilon}$. Define $\tilde{v}_{1,\varepsilon}, \tilde{w}_{1,\varepsilon}$ as in (1.4). By the orthogonal property (1.3), we have

$$\int_{\Omega_\varepsilon} c_0 p \tilde{u}_\varepsilon^{p-1} \tilde{v}_{1,\varepsilon} \tilde{w}_{1,\varepsilon} dy + \varepsilon \|u_\varepsilon\|^{2-(p-1)N/2} \int_{\Omega_\varepsilon} \tilde{v}_{1,\varepsilon} \tilde{w}_{1,\varepsilon} dy = 0.$$

Since $\tilde{v}_{1,\varepsilon}, \tilde{w}_{1,\varepsilon} \to U$, the dominated convergence theorem implies $\int_{\mathbb{R}^N} U^{p+1} dy = 0$, which is a contradiction. The last claim will be proved just as in Proposition 1 in Han [4]. This finish the proof of Theorem 1.1.

4 Proof of Theorem 1.2

In this section, we prove Theorem 1.2 along the line of [3].

Proposition 4.1 *For $i = 2, \cdots, N+1$, we have*

$$\lambda_{i,\varepsilon} \leq 1 + \frac{C_1}{\|u_\varepsilon\|^{\frac{2N}{N-2}}} \tag{4.1}$$

for some $C_1 > 0$ and

$$\lim_{\varepsilon \to 0} \lambda_{i,\varepsilon} = 1. \tag{4.2}$$

Proof. By the variational characterization, $\lambda_{i,\varepsilon}$ can be expressed as

$$\lambda_{i,\varepsilon} = \inf_{W \subset H_0^1(\Omega), \dim(W) = i} \max_{v \in W} \frac{\int_\Omega |\nabla v|^2 dx}{\int_\Omega \left(c_0 p u_\varepsilon^{p-1} + \varepsilon\right) v^2 dx}.$$

We take

$$W = W_i = \operatorname{span}\{u_\varepsilon, \psi_{1,\varepsilon}, \cdots, \psi_{i-1,\varepsilon}\},$$

where $\psi_{j,\varepsilon}$ are defined in (2.13). For $a_0, a_1, \cdots, a_{i-1} \in \mathbb{R}$, we put

$$v = a_0 u_\varepsilon + \sum_{j=1}^{i-1} a_j \psi_{j,\varepsilon} = a_0 u_\varepsilon + \phi z_\varepsilon \in W_i,$$

where $z_\varepsilon = \sum_{j=1}^{i-1} a_j (\frac{\partial u_\varepsilon}{\partial x_j})$.

Calculating as in [3], we have

$$\max_{v \in W_i} \frac{\int_\Omega |\nabla v|^2 dx}{\int_\Omega (c_0 p u_\varepsilon^{p-1} + \varepsilon) v^2 dx} = \max_{a_0, a_1, \cdots, a_{i-1}} \left\{ 1 + \frac{N_\varepsilon}{D_\varepsilon} \right\}$$

where $N_\varepsilon = N_\varepsilon^1 + N_\varepsilon^2 + N_\varepsilon^3$,

$$N_\varepsilon^1 = a_0^2 c_0 (1-p) \int_\Omega u_\varepsilon^{p+1} dx,$$
$$N_\varepsilon^2 = 2 a_0 c_0 (1-p) \sum_{j=1}^{i-1} a_j \int_\Omega u_\varepsilon^p \phi (\frac{\partial u_\varepsilon}{\partial x_j}) dx,$$
$$N_\varepsilon^3 = \sum_{j,l=1}^{i-1} a_j a_l \int_\Omega |\nabla \phi|^2 (\frac{\partial u_\varepsilon}{\partial x_j})(\frac{\partial u_\varepsilon}{\partial x_l}) dx,$$

and $D_\varepsilon = D_\varepsilon^1 + D_\varepsilon^2 + D_\varepsilon^3$,

$$D_\varepsilon^1 = a_0^2 \int_\Omega (c_0 p u_\varepsilon^{p-1} + \varepsilon) u_\varepsilon^2 dx,$$
$$D_\varepsilon^2 = 2 a_0 \sum_{j=1}^{i-1} a_j \int_\Omega (c_0 p u_\varepsilon^{p-1} + \varepsilon) u_\varepsilon \phi (\frac{\partial u_\varepsilon}{\partial x_j}) dx,$$
$$D_\varepsilon^3 = \sum_{j,l=1}^{i-1} a_j a_l \int_\Omega (c_0 p u_\varepsilon^{p-1} + \varepsilon) \phi^2 (\frac{\partial u_\varepsilon}{\partial x_j})(\frac{\partial u_\varepsilon}{\partial x_l}) dx.$$

N_ε^2 and N_ε^3 can be estimated as the same way (3.24) and (3.25) in [3]:

$$\int_\Omega u_\varepsilon^p \phi (\frac{\partial u_\varepsilon}{\partial x_j}) dx = O(\frac{1}{\|u_\varepsilon\|^{p+1}}), \quad \int_\Omega |\nabla \phi|^2 (\frac{\partial u_\varepsilon}{\partial x_j})(\frac{\partial u_\varepsilon}{\partial x_l}) dx = O(\frac{1}{\|u_\varepsilon\|^2}). \quad (4.3)$$

Hence

$$N_\varepsilon^2 = O(\frac{1}{\|u_\varepsilon\|^{p+1}}), \quad N_\varepsilon^3 = O(\frac{1}{\|u_\varepsilon\|^2}).$$

As for D_ε^2, we write

$$\int_\Omega (c_0 p u_\varepsilon^{p-1} + \varepsilon) u_\varepsilon \phi(\frac{\partial u_\varepsilon}{\partial x_j}) dx = \int_\Omega \frac{c_0 p}{p+1} \phi(\frac{\partial u_\varepsilon^{p+1}}{\partial x_j}) dx + \frac{\varepsilon}{2} \int_\Omega \phi(\frac{\partial u_\varepsilon^2}{\partial x_j}) dx.$$

By integration by parts and (2.7), we have

$$D_\varepsilon^2 = O(\frac{1}{\|u_\varepsilon\|^{p+1}}) + O(\frac{\varepsilon}{\|u_\varepsilon\|^2}). \tag{4.4}$$

As for D_ε^3, by change of variables

$$x = \frac{y}{\|u_\varepsilon\|^{\frac{p-1}{2}}} + x_\varepsilon, \quad \frac{\partial u_\varepsilon}{\partial x_j}(x) = \|u_\varepsilon\|^{\frac{p-1}{2}+1} \frac{\partial \tilde{u}_\varepsilon}{\partial y_j}(y),$$

we see just as (3.26) in [3],

$$\begin{aligned}
&\int_\Omega u_\varepsilon^{p-1} \phi^2 (\frac{\partial u_\varepsilon}{\partial x_j})(\frac{\partial u_\varepsilon}{\partial x_l}) dx \\
&= \|u_\varepsilon\|^{p-1+2(\frac{p-1}{2}+1)-\frac{p-1}{2}N} \int_{\Omega_\varepsilon} \tilde{u}_\varepsilon^{p-1} \phi_\varepsilon^2(y) (\frac{\partial \tilde{u}_\varepsilon}{\partial y_j})(\frac{\partial \tilde{u}_\varepsilon}{\partial y_l}) dy \\
&= \|u_\varepsilon\|^{p-1} \left(\int_{\mathbb{R}^N} U^{p-1} \left(\frac{\partial U}{\partial y_j}\right) \left(\frac{\partial U}{\partial y_l}\right) dy + o(1) \right) \\
&= \|u_\varepsilon\|^{4/(N-2)} \left(\frac{\delta_{jl}}{N} \int_{\mathbb{R}^N} U^{p-1} |\nabla U|^2 dy + o(1) \right),
\end{aligned} \tag{4.5}$$

and

$$\begin{aligned}
&\int_\Omega \phi^2 (\frac{\partial u_\varepsilon}{\partial x_j})(\frac{\partial u_\varepsilon}{\partial x_l}) dx = \|u_\varepsilon\|^{2(\frac{p-1}{2}+1)-\frac{p-1}{2}N} \int_{\Omega_\varepsilon} \phi_\varepsilon^2(y) (\frac{\partial \tilde{u}_\varepsilon}{\partial y_j})(\frac{\partial \tilde{u}_\varepsilon}{\partial y_l}) dy \\
&= \int_{\mathbb{R}^N} \left(\frac{\partial U}{\partial y_j}\right) \left(\frac{\partial U}{\partial y_l}\right) dy + o(1),
\end{aligned} \tag{4.6}$$

where $\phi_\varepsilon(y)$ is defined as before. Here, we have used the fact $\nabla \tilde{u}_\varepsilon \to \nabla U$ in $L^2(\mathbb{R}^N)$ by (2.5). Thus by (4.5) and (4.6),

$$\begin{aligned}
D_\varepsilon^3 &= c_0 p \sum_{j=1}^{i-1} a_j^2 \|u_\varepsilon\|^{p-1} \left(\frac{1}{N} \int_{\mathbb{R}^N} U^{p-1} |\nabla U|^2 dy + o(1) \right) \\
&\quad + \varepsilon \sum_{j=1}^{i-1} a_j^2 \left(\frac{1}{N} \int_{\mathbb{R}^N} |\nabla U|^2 dy + o(1) \right).
\end{aligned} \tag{4.7}$$

Now, by testing $(a_0, a_1, \cdots, a_{i-1}) = (0, 1, \cdots, 1)$, we have

$$\begin{aligned}
&\max_{v\in W_i} \frac{\int_\Omega |\nabla v|^2 dx}{\int_\Omega \left(c_0 p u_\varepsilon^{p-1} + \varepsilon\right) v^2 dx} = \max_{(a_0,a_1,\cdots,a_{i-1})\in\mathbb{R}^i} \left\{1 + \frac{N_\varepsilon}{D_\varepsilon}\right\} \\
&\geq 1 + \frac{\sum_{j,l=1}^{i-1} \int_\Omega |\nabla\phi|^2 (\frac{\partial u_\varepsilon}{\partial x_j})(\frac{\partial u_\varepsilon}{\partial x_l}) dx}{\sum_{j,l=1}^{i-1} \int_\Omega (c_0 p u_\varepsilon^{p-1} + \varepsilon)\phi^2 (\frac{\partial u_\varepsilon}{\partial x_j})(\frac{\partial u_\varepsilon}{\partial x_l}) dx} \\
&= 1 + \frac{O(\frac{1}{\|u_\varepsilon\|^2})}{\|u_\varepsilon\|^{p-1} + O(\varepsilon)}.
\end{aligned}$$

Thus we have some $C_0 > 0$ such that

$$\max_{v\in W_i} \frac{\int_\Omega |\nabla v|^2 dx}{\int_\Omega \left(c_0 p u_\varepsilon^{p-1} + \varepsilon\right) v^2 dx} \geq 1 + \frac{C_0}{\|u_\varepsilon\|^{p+1}}, \tag{4.8}$$

just as (3.27) in [3].

Let $(a_{0,\varepsilon}, a_{1,\varepsilon}, \cdots, a_{i-1,\varepsilon}) \in \mathbb{R}^i$ be a maximizer of $\max_{(a_0,a_1,\cdots,a_{i-1})\in\mathbb{R}^i} \left\{1 + \frac{N_\varepsilon}{D_\varepsilon}\right\}$. We may assume that $\sum_{j=1}^{i-1} a_{j,\varepsilon}^2 = 1$. From the above estimates and (4.8), we check that $\|u_\varepsilon\|^2 a_{0,\varepsilon}^2$ is uniformly bounded in ε as (3.30) in [3], thus we have

$$\begin{aligned}
&\max_{v\in W_i} \frac{\int_\Omega |\nabla v|^2 dx}{\int_\Omega \left(c_0 p u_\varepsilon^{p-1} + \varepsilon\right) v^2 dx} \\
&= \left\{1 + \frac{N_\varepsilon}{D_\varepsilon}\right\}\Big|_{(a_0,a_1,\cdots,a_{i-1})=(a_{0,\varepsilon},a_{1,\varepsilon},\cdots,a_{i-1,\varepsilon})} \\
&= 1 + \frac{a_{0,\varepsilon}^2 c_0(1-p)\|u_\varepsilon\|^2 \int_\Omega u_\varepsilon^{p+1} dx + a_{0,\varepsilon} O(\frac{1}{\|u_\varepsilon\|^{p-1}}) + O(1)}{a_{0,\varepsilon}^2 \|u_\varepsilon\|^2 \int_\Omega (c_0 p u_\varepsilon^{p-1} + \varepsilon) u_\varepsilon^2 dx + O(\frac{1}{\|u_\varepsilon\|^{p-1}}) + O(\varepsilon) + O(\|u_\varepsilon\|^{p+1}) + O(\varepsilon\|u_\varepsilon\|^2)} \\
&\leq 1 + \frac{O(\frac{1}{\|u_\varepsilon\|^{p}}) + O(1)}{O(\frac{1}{\|u_\varepsilon\|^{p-1}}) + O(\varepsilon) + O(\|u_\varepsilon\|^{p+1}) + O(\varepsilon\|u_\varepsilon\|^2)} \\
&\leq 1 + \frac{C_1}{\|u_\varepsilon\|^{p+1}}
\end{aligned}$$

for some $C_1 > 0$. This proves (4.1).

By using (4.1), we obtain (4.2) just as in [3]. Thus the proof of Proposition 4.1 is finished.

Lemma 4.2 *Let $i \in \mathbb{N}$ be such that $\lim_{\varepsilon\to 0}\lambda_{i,\varepsilon} = 1$. If b_i in (2.11) of Lemma 2.5 is not 0, then we have*

$$\lambda_{i,\varepsilon} - 1 = \frac{1}{\|u_\varepsilon\|^2}(C_2 + o(1)) \quad \text{as } \varepsilon \to 0 \tag{4.9}$$

for some $C_2 > 0$ independent of ε.

Proof. Assume $b_i \neq 0$. We use the integral identity (2.1) in Lemma 2.1 with $y = x_\varepsilon$. The LHS of (2.1) can be written as

$$\begin{aligned}
&\frac{1}{\|u_\varepsilon\|^3}\int_{\partial\Omega}(x - x_\varepsilon)\cdot\nu(\frac{\partial\|u_\varepsilon\|u_\varepsilon}{\partial\nu})(\frac{\partial\|u_\varepsilon\|^2 v_{i,\varepsilon}}{\partial\nu})ds_x \\
&= \frac{1}{\|u_\varepsilon\|^3}\left[-(N-2)^2\sigma_N^2 b_i\int_{\partial\Omega}(x - x_0)\cdot\nu\left(\frac{\partial G}{\partial\nu}(x, x_0)\right)^2 ds_x + o(1)\right] \\
&= \frac{1}{\|u_\varepsilon\|^3}\left[-(N-2)^3\sigma_N^2 R(x_0) b_i + o(1)\right].
\end{aligned} \tag{4.10}$$

Here we have used (2.7), (2.12) and the fact $\int_{\partial\Omega}((x-x_0)\cdot\nu)\left(\frac{\partial G}{\partial\nu}(x,x_0)\right)^2 ds_x = (N-2)R(x_0)$.

On the other hand, the RHS of (2.1) $= I_1 + I_2 + I_3$, where

$$\begin{aligned}
I_1 &= (1 - \lambda_{i,\varepsilon})c_0 p\int_\Omega u_\varepsilon^{p-1} w_\varepsilon v_{i,\varepsilon}dx, \\
I_2 &= (1 - \lambda_{i,\varepsilon})\varepsilon\int_\Omega w_\varepsilon v_{i,\varepsilon}dx, \quad I_3 = 2\varepsilon\int_\Omega u_\varepsilon v_{i,\varepsilon}dx
\end{aligned}$$

and, as before, $w_\varepsilon(x) = (x - x_\varepsilon)\cdot\nabla u_\varepsilon + \frac{2}{p-1}u_\varepsilon$. Denote

$$\tilde{w}_\varepsilon(y) = \frac{1}{\|u_\varepsilon\|}w_\varepsilon(\frac{y}{\|u_\varepsilon\|^{\frac{p-1}{2}}} + x_\varepsilon) = y\cdot\nabla_y\tilde{u}_\varepsilon(y) + \frac{2}{p-1}\tilde{u}_\varepsilon(y) \tag{4.11}$$

for $y \in \Omega_\varepsilon$. By (2.5), we see

$$\tilde{w}_\varepsilon \to y\cdot\nabla U + \frac{N-2}{2}U = \left(\frac{N-2}{2}\right)\frac{1 - |y|^2}{(1+|y|^2)^{N/2}}, \quad \text{in } C^1_{loc}(\mathbb{R}^N).$$

Thus,

$$
\begin{aligned}
I_1 &= (1-\lambda_{i,\varepsilon})c_0 p\|u_\varepsilon\|^{p-(p-1)N/2}\int_{\Omega_\varepsilon}\tilde{u}_\varepsilon^{p-1}\tilde{w}_\varepsilon\tilde{v}_{i,\varepsilon}(y)dy \\
&= (1-\lambda_{i,\varepsilon})c_0 p\|u_\varepsilon\|^{-1}\times \\
&\times\left[\int_{\mathbb{R}^N}U^{p-1}\left(y\cdot\nabla U+\frac{2}{p-1}U\right)\left(\sum_{j=1}^N a_{i,j}\frac{y_j}{(1+|y|^2)^{N/2}}+b_i\frac{1-|y|^2}{(1+|y|^2)^{N/2}}\right)dy+o(1)\right] \\
&= (1-\lambda_{i,\varepsilon})\|u_\varepsilon\|^{-1}b_i c_0 p\left(\frac{N-2}{2}\right)\left[\int_{\mathbb{R}^N}U^{p-1}\frac{(1-|y|^2)^2}{(1+|y|^2)^N}dy+o(1)\right].
\end{aligned}
$$

Analogously,

$$
I_2 = (1-\lambda_{i,\varepsilon})\varepsilon\left(\frac{N-2}{2}\right)\|u_\varepsilon\|^{-(N+2)/(N-2)}\left[b_i\int_{\mathbb{R}^N}\frac{(1-|y|^2)^2}{(1+|y|^2)^N}dy+o(1)\right],
$$

$$
I_3 = 2\varepsilon\|u_\varepsilon\|^{-(N+2)/(N-2)}b_i\left[\int_{\mathbb{R}^N}U(y)\frac{1-|y|^2}{(1+|y|^2)^{N/2}}dy+o(1)\right].
$$

Dividing both sides of (4.10) $= I_1+I_2+I_3$ by $b_i\neq 0$, and calculating with (2.8) when $N\geq 5$, we obtain the result for

$$
C_2 = \frac{(N-2)^2(N-4)\sigma_N^2 R(x_0)}{c_0 p(\frac{N-2}{2})\int_{\mathbb{R}^N}\frac{(1-|y|^2)^2}{(1+|y|^2)^{N+2}}dy}. \tag{4.12}
$$

Now, by Proposition 4.1 and Lemma 4.2, a contradiction is obvious if b_i in (2.11) is not 0. Thus we have (1.5) in Theorem 1.2.

(1.6) is a direct consequence of Lemma 3.3 in [6] below. Note that now $\|v_{i,\varepsilon}\|=1$ while $\|v_{i,\varepsilon}\|=\|u_\varepsilon\|$ in [6].

Lemma 4.3 *Assume $N\geq 6$. For $i=2,\cdots,N+1$, let $b_i=0$ and $\vec{a}_i=(a_{i,1},\cdots,a_{i,N})\neq 0$ in (2.11). Then we have*

$$
\|u_\varepsilon\|^{2+2/(N-2)}v_{i,\varepsilon}\to\sigma_N\sum_{j=1}^N a_{i,j}\left(\frac{\partial G}{\partial z_j}(x,z)\right)\Big|_{z=x_0}
$$

in $C^1_{loc}(\overline{\Omega}\setminus\{x_0\})$.

Now, we prove (1.7). We return to (2.2). By (2.7) and Lemma 4.3, we see

$$
\begin{aligned}
&\text{LHS of (2.2)} = \frac{1}{\|u_\varepsilon\|^{3+2/(N-2)}}\int_{\partial\Omega}(\frac{\partial\|u_\varepsilon\|u_\varepsilon}{\partial x_j})(\frac{\partial\|u_\varepsilon\|^{2+2/(N-2)}v_{i,\varepsilon}}{\partial\nu})ds_x \\
&= \frac{1}{\|u_\varepsilon\|^{3+2/(N-2)}}\left[(N-2)\sigma_N^2\sum_{k=1}^N a_{i,k}\int_{\partial\Omega}\left(\frac{\partial G}{\partial x_i}\right)\frac{\partial}{\partial\nu_x}\left(\frac{\partial G}{\partial z_k}\right)(x,x_0)ds_x + o(1)\right] \\
&= \frac{1}{\|u_\varepsilon\|^{3+2/(N-2)}}\left[\frac{N-2}{2}\sigma_N^2\sum_{k=1}^N a_{i,k}\frac{\partial^2 R}{\partial z_i\partial z_k}(z)\Big|_{z=x_0} + o(1)\right],
\end{aligned}
$$

where we have used the fact $\int_{\partial\Omega}\left(\frac{\partial G}{\partial x_i}\right)\frac{\partial}{\partial\nu_x}\left(\frac{\partial G}{\partial z_j}\right)(x,x_0)ds_x = \frac{1}{2}\frac{\partial^2 R}{\partial z_i\partial z_j}(z)\Big|_{z=x_0}$.

On the other hand, RHS of (2.2) $= I + II$ where

$$
I = (1-\lambda_{i,\varepsilon})c_0p\int_\Omega u_\varepsilon^{p-1}(\frac{\partial u_\varepsilon}{\partial x_j})v_{i,\varepsilon}dx, \quad II = (1-\lambda_{i,\varepsilon})\varepsilon\int_\Omega(\frac{\partial u_\varepsilon}{\partial x_j})v_{i,\varepsilon}dx.
$$

As before, we have

$$
I = \frac{(\lambda_{i,\varepsilon}-1)}{\|u_\varepsilon\|^{(N-4)/(N-2)}}\frac{c_0p}{N(N-2)}a_{i,j}\left[\int_{\mathbb{R}^N}U^{p-1}|\nabla U|^2dy + o(1)\right],
$$

and

$$
II = \frac{(\lambda_{i,\varepsilon}-1)}{\|u_\varepsilon\|^{N/(N-2)}}\varepsilon\frac{1}{N(N-2)}a_{i,j}\left[\int_{\mathbb{R}^N}|\nabla U|^2dy + o(1)\right].
$$

Multiplying $\|u_\varepsilon\|^{3+2/(N-2)}$ to the both sides of (2.2) and recalling (2.8), we see that

$$
\begin{aligned}
&\frac{N-2}{2}\sigma_N^2\sum_{k=1}^N a_{i,k}\frac{\partial^2 R}{\partial z_k\partial z_j}(z)\Big|_{z=x_0} \\
&= (\lambda_{i,\varepsilon}-1)a_{i,j}\left\{\|u_\varepsilon\|^{2N/(N-2)}p\int_{\mathbb{R}^N}U^{p-1}|\nabla U|^2dy + O(\|u_\varepsilon\|^{4/(N-2)})\right\}
\end{aligned}
$$

holds for any $j = 1, \cdots, N$. Hence

$$
(\lambda_{i,\varepsilon}-1)\|u_\varepsilon\|^{2N/(N-2)} \to M\eta_i, \quad \text{as } \varepsilon \to 0,
$$

where

$$M = \frac{(\frac{N-2}{2})\sigma_N^2}{p\int_{\mathbb{R}^N} U^{p-1}|\nabla U|^2 dy}, \quad \eta_i = \frac{\sum_{k=1}^N a_{i,k}\frac{\partial^2 R}{\partial z_k \partial z_j}(x_0)}{a_{i,j}}.$$

By the definition of η_i, we have $\sum_{k=1}^N \frac{\partial^2 R}{\partial z_k \partial z_j}(x_0) a_{i,k} = \eta_i a_{i,j}$, thus η_i is an eigenvalue of the Hessian matrix of R at x_0 and $\vec{a}_i$ is a corresponding eigenvector. If $i \neq j$, we see that $\vec{a}_i$ and $\vec{a}_j$ is perpendicular to each other in $\mathbb{R}^N$, because of (1.3).

Thus, all η_i is one of N eigenvalues of Hess$R(x_0)$ and we have $\eta_i = \mu_{i-1}$ for $i = 2, \cdots, N+1$. This ends the proof of Theorem 1.2.

5 Proof of Theorem 1.3

In this section, we prove Theorem 1.3. First, we prove

Lemma 5.1

$$\lambda_{N+2,\varepsilon} \to 1 \quad \textit{as } \varepsilon \to 0. \tag{5.1}$$

Proof. Since we know $\liminf_{\varepsilon\to 0} \lambda_{N+2,\varepsilon} \geq 1$ by Proposition 4.1, we have to check that $\limsup_{\varepsilon\to 0} \lambda_{N+2,\varepsilon} \leq 1$. For this purpose, we use a variational characterization of $\lambda_{N+2,\varepsilon}$ to obtain

$$\lambda_{N+2,\varepsilon} \leq \max_{v\in W} \frac{\int_\Omega |\nabla v|^2 dx}{\int_\Omega \left(c_0 p u_\varepsilon^{p-1} + \varepsilon\right) v^2 dx}, \tag{5.2}$$

where $W = \text{span}\{u_\varepsilon, \phi(\frac{\partial u_\varepsilon}{\partial x_1}), \cdots, \phi(\frac{\partial u_\varepsilon}{\partial x_N}), \phi w_\varepsilon\}$, ϕ is a cut-off function as in Lemma 2.7, and, as before, $w_\varepsilon(x) = (x - x_\varepsilon)\cdot\nabla u_\varepsilon + \frac{2}{p-1}u_\varepsilon$. For $a_0, a_1, \cdots, a_N, d \in \mathbb{R}$, we set $\hat{z}_\varepsilon(x) = \sum_{j=1}^N a_j(\frac{\partial u_\varepsilon}{\partial x_j}) + d w_\varepsilon(x)$. Direct calculation shows that $\hat{z}_\varepsilon$ satisfies the equation

$$-\Delta \hat{z}_\varepsilon = \left(c_0 p u_\varepsilon^{p-1} + \varepsilon\right)\hat{z}_\varepsilon + 2\varepsilon d u_\varepsilon.$$

We test (5.2) by $v = a_0 u_\varepsilon + \phi \hat{z}_\varepsilon \in W$.

As in the proof of Proposition 4.1, we have

$$\max_{v\in W} \frac{\int_\Omega |\nabla v|^2 dx}{\int_\Omega (c_0 p u_\varepsilon^{p-1} + \varepsilon) v^2 dx} = \max_{a_0, a_1, \cdots, a_N, d} \left\{1 + \frac{\hat{N}_\varepsilon}{\hat{D}_\varepsilon}\right\},$$

where $\hat{N}_\varepsilon = \hat{N}_\varepsilon^1 + \hat{N}_\varepsilon^2 + \hat{N}_\varepsilon^3 + \hat{N}_\varepsilon^4$,

$$\hat{N}_\varepsilon^1 = a_0^2 c_0 (1-p) \int_\Omega u_\varepsilon^{p+1} dx,$$

$$\hat{N}_\varepsilon^2 = 2a_0 c_0 (1-p) \int_\Omega u_\varepsilon^p \phi \hat{z}_\varepsilon dx$$

$$= 2a_0 c_0 (1-p) \left\{ \sum_{j=1}^N a_j \int_\Omega u_\varepsilon^p (\frac{\partial u_\varepsilon}{\partial x_j}) \phi dx + d \int_\Omega u_\varepsilon^p \phi w_\varepsilon(x) dx \right\},$$

$$\hat{N}_\varepsilon^3 = \int_\Omega |\nabla \phi|^2 \hat{z}_\varepsilon^2 dx = \sum_{j,l=1}^N a_j a_l \int_\Omega |\nabla \phi|^2 (\frac{\partial u_\varepsilon}{\partial x_j})(\frac{\partial u_\varepsilon}{\partial x_l}) dx$$

$$+ 2d \sum_{j=1}^N \int_\Omega |\nabla \phi|^2 (\frac{\partial u_\varepsilon}{\partial x_j}) w_\varepsilon dx + d^2 \int_\Omega |\nabla \phi|^2 w_\varepsilon^2 dx,$$

$$\hat{N}_\varepsilon^4 = 2d\varepsilon \int_\Omega \phi^2 \hat{z}_\varepsilon u_\varepsilon dx = 2d\varepsilon \sum_{j=1}^N a_j \int_\Omega \phi^2 u_\varepsilon (\frac{\partial u_\varepsilon}{\partial x_j}) dx + 2d^2 \varepsilon \int_\Omega \phi^2 u_\varepsilon w_\varepsilon dx,$$

and $\hat{D}_\varepsilon = \hat{D}_\varepsilon^1 + \hat{D}_\varepsilon^2 + \hat{D}_\varepsilon^3$,

$$\hat{D}_\varepsilon^1 = a_0^2 \int_\Omega (c_0 p u_\varepsilon^{p-1} + \varepsilon) u_\varepsilon^2 dx,$$

$$\begin{aligned}
\hat{D}_\varepsilon^2 &= 2a_0 \int_\Omega (c_0 p u_\varepsilon^{p-1} + \varepsilon) u_\varepsilon \phi \hat{z}_\varepsilon dx \\
&= 2a_0 c_0 p \sum_{j=1}^N a_j \int_\Omega u_\varepsilon^p \phi (\frac{\partial u_\varepsilon}{\partial x_j}) dx + 2a_0 \sum_{j=1}^N a_j \varepsilon \int_\Omega u_\varepsilon \phi (\frac{\partial u_\varepsilon}{\partial x_j}) dx \\
&+ 2a_0 c_0 p d \int_\Omega u_\varepsilon^p \phi w_\varepsilon dx + 2a_0 d \varepsilon \int_\Omega u_\varepsilon \phi w_\varepsilon dx,
\end{aligned}$$

$$\begin{aligned}
\hat{D}_\varepsilon^3 &= \int_\Omega (c_0 p u_\varepsilon^{p-1} + \varepsilon) \phi^2 \hat{z}_\varepsilon^2 dx \\
&= \sum_{j,l=1}^N \int_\Omega (c_0 p u_\varepsilon^{p-1} + \varepsilon) \phi^2 (a_j \frac{\partial u_\varepsilon}{\partial x_j} + d w_\varepsilon)(a_l \frac{\partial u_\varepsilon}{\partial x_l} + d w_\varepsilon) dx \\
&= c_0 p \sum_{j,l=1}^N a_j a_l \int_\Omega u_\varepsilon^{p-1} \phi^2 (\frac{\partial u_\varepsilon}{\partial x_j})(\frac{\partial u_\varepsilon}{\partial x_l}) dx + \varepsilon \sum_{j,l=1}^N a_j a_l \int_\Omega \phi^2 (\frac{\partial u_\varepsilon}{\partial x_j})(\frac{\partial u_\varepsilon}{\partial x_l}) dx \\
&+ 2c_0 p d \sum_{j=1}^N a_j \int_\Omega u_\varepsilon^{p-1} \phi^2 w_\varepsilon (\frac{\partial u_\varepsilon}{\partial x_j}) dx + 2\varepsilon d \sum_{j=1}^N a_j \int_\Omega \phi^2 w_\varepsilon (\frac{\partial u_\varepsilon}{\partial x_j}) dx \\
&+ c_0 p d^2 \int_\Omega u_\varepsilon^{p-1} \phi^2 w_\varepsilon^2 dx + \varepsilon d^2 \int_\Omega \phi^2 w_\varepsilon^2 dx.
\end{aligned}$$

Let $(a_0, a_1, \cdots, a_N, d)$ denote a maximizer of $\max_{a_0, a_1, \cdots, a_N, d} \left\{ 1 + \frac{\hat{N}_\varepsilon}{\hat{D}_\varepsilon} \right\}$ which is normalized as $a_0^2 + \sum_{j=1}^N a_j^2 + d^2 = 1$. Since the case $a_0 = 1$ is obvious, we consider only the case $\sum_{j=1}^N a_j^2 + d^2 \neq 0$.

We calculate, as the derivation of (7.8), (7.9), (7.10) in [3],

$$\begin{aligned}
\int_\Omega u_\varepsilon^p \phi w_\varepsilon dx &= \int_\Omega u_\varepsilon^p \phi \left((x - x_\varepsilon) \cdot \nabla u_\varepsilon + \frac{2}{p-1} u_\varepsilon \right) dx \\
&= \int_\Omega \frac{\phi}{p+1} \sum_{j=1}^N \frac{\partial}{\partial x_j} \left\{ (x_j - (x_\varepsilon)_j) u_\varepsilon^{p+1} \right\} - \left(\frac{N}{p+1} - \frac{2}{p-1} \right) u_\varepsilon^{p+1} \phi dx \\
&= -\frac{1}{p+1} \int_\Omega \sum_{j=1}^N \frac{\partial \phi}{\partial x_j} (x_j - (x_\varepsilon)_j) u_\varepsilon^{p+1} dx = O(\frac{1}{\|u_\varepsilon\|^{p+1}}),
\end{aligned} \tag{5.3}$$

and

$$\int_\Omega |\nabla\phi|^2(\frac{\partial u_\varepsilon}{\partial x_j})w_\varepsilon dx = O(\frac{1}{\|u_\varepsilon\|^2}), \quad \int_\Omega |\nabla\phi|^2 w_\varepsilon^2 dx = O(\frac{1}{\|u_\varepsilon\|^2}) \tag{5.4}$$

since (2.7) and $\nabla\phi \equiv 0$ near x_0. Thus by (4.3), (5.3), (5.4), we have

$$\hat{N}_\varepsilon^2 = O(\frac{1}{\|u_\varepsilon\|^{p+1}}), \quad \hat{N}_\varepsilon^3 = O(\frac{1}{\|u_\varepsilon\|^2}).$$

Also, as (7.11), (7.12) in [3], we have

$$\int_\Omega u_\varepsilon^{p-1}\phi^2(\frac{\partial u_\varepsilon}{\partial x_j})w_\varepsilon dx = \|u_\varepsilon\|^{2/(N-2)}o(1), \tag{5.5}$$

and

$$\int_\Omega u_\varepsilon^{p-1}\phi^2 w_\varepsilon^2 dx = \left(\frac{N-2}{2}\right)^2 \int_{\mathbb{R}^N} U^{p-1}(y)\left(\frac{(1-|y|^2)}{(1+|y|^2)^{N/2}}\right)^2 dy + o(1). \tag{5.6}$$

Since

$$\int_\Omega \phi^2 u_\varepsilon(\frac{\partial u_\varepsilon}{\partial x_j})dx = \frac{1}{2}\int_\Omega \phi^2\frac{\partial}{\partial x_j}u_\varepsilon^2 dx = -\frac{1}{2}\int_\Omega \frac{\partial\phi^2}{\partial x_j}u_\varepsilon^2 dx = O(\frac{1}{\|u_\varepsilon\|^2}), \tag{5.7}$$

and

$$\begin{aligned}
&\int_\Omega \phi^2 u_\varepsilon w_\varepsilon dx = \int_\Omega \phi^2 u_\varepsilon\left((x-x_\varepsilon)\cdot\nabla u_\varepsilon + \frac{2}{p-1}u_\varepsilon\right)dx \\
&= \int_\Omega \phi^2\frac{1}{2}\sum_{j=1}^N \frac{\partial}{\partial x_j}\left((x_j-(x_\varepsilon)_j)u_\varepsilon^2\right)dx + \left(\frac{2}{p-1}-\frac{N}{2}\right)\int_\Omega \phi^2 u_\varepsilon^2 dx \\
&= -\int_\Omega \frac{1}{2}u_\varepsilon^2\sum_{j=1}^N \frac{\partial\phi^2(x)}{\partial x_j}(x_j-(x_\varepsilon)_j)dx - \int_\Omega \phi^2 u_\varepsilon^2 dx \\
&= O(\frac{1}{\|u_\varepsilon\|^2}) - \frac{1}{\|u_\varepsilon\|^{4/(N-2)}}\left(\int_{\mathbb{R}^N} U^2 dy + o(1)\right) = O(\frac{1}{\|u_\varepsilon\|^{4/(N-2)}}),
\end{aligned} \tag{5.8}$$

$\hat{N}_\varepsilon^4$ can be estimated as

$$\hat{N}_\varepsilon^4 = O(\frac{\varepsilon}{\|u_\varepsilon\|^2}) + O(\frac{\varepsilon}{\|u_\varepsilon\|^{4/(N-2)}}) = O(\frac{1}{\|u_\varepsilon\|^2})$$

by (5.7), (5.8) and (2.8). Therefore, we have

$$\begin{aligned}\hat{N}_\varepsilon &= \hat{N}_\varepsilon^1 + \hat{N}_\varepsilon^2 + \hat{N}_\varepsilon^3 + \hat{N}_\varepsilon^4 \\ &= a_0^2 c_0(1-p)\int_\Omega u_\varepsilon^{p+1}dx + O(\frac{1}{\|u_\varepsilon\|^{p+1}}) + O(\frac{1}{\|u_\varepsilon\|^2}) \le O(\frac{1}{\|u_\varepsilon\|^2}).\end{aligned}$$

Furthermore, by change of variables, we see

$$\int_\Omega \phi^2(\frac{\partial u_\varepsilon}{\partial x_j})w_\varepsilon dx = O(\frac{1}{\|u_\varepsilon\|^{2/(N-2)}}), \quad \int_\Omega \phi^2 w_\varepsilon^2 dx = O(\frac{1}{\|u_\varepsilon\|^{4/(N-2)}}) \tag{5.9}$$

Thus we have

$$\hat{D}_\varepsilon^2 = O(\frac{1}{\|u_\varepsilon\|^{p+1}}) + O(\frac{\varepsilon}{\|u_\varepsilon\|^2}) + O(\frac{\varepsilon}{\|u_\varepsilon\|^{4/(N-2)}}) = O(\frac{1}{\|u_\varepsilon\|^2})$$

by (4.4), (5.3), (5.8) and (2.8), and

$$\begin{aligned}\hat{D}_\varepsilon^3 &= c_0 p\left(\sum_{j=1}^N a_j^2\right)\|u_\varepsilon\|^{4/(N-2)}\left(\frac{1}{N}\int_{\mathbb{R}^N} U^{p-1}|\nabla U|^2 dy + o(1)\right) \\ &+ O(\varepsilon) + d\left(\sum_{j=1}^N a_j\right)o(\|u_\varepsilon\|^{2/(N-2)}) + O(\frac{\varepsilon}{\|u_\varepsilon\|^{2/(N-2)}}) \\ &+ d^2\left(c_0 p\left(\frac{N-2}{2}\right)^2\int_{\mathbb{R}^N} U^{p-1}(y)\left(\frac{(1-|y|^2)}{(1+|y|^2)^{N/2}}\right)^2 dy + o(1)\right) \\ &+ O(\frac{1}{\|u_\varepsilon\|^{4/(N-2)}})\end{aligned}$$

by (4.7), (5.5), (5.6) and (5.9).

From these, we can estimate $\hat{D}_\varepsilon$ from below, just as (7.14) in [3]:

$$\begin{aligned}\hat{D}_\varepsilon &\ge \hat{D}_\varepsilon^2 + \hat{D}_\varepsilon^3 \\ &\ge \gamma_1\|u_\varepsilon\|^{4/(N-2)}\left(\sum_{j=1}^N a_j^2\right) + d\left(\sum_{j=1}^N a_j\right)o(\|u_\varepsilon\|^{2/(N-2)}) + \gamma_2 d^2 \\ &\ge (\gamma_1/2)\|u_\varepsilon\|^{4/(N-2)}\left(\sum_{j=1}^N a_j^2\right) + (\gamma_2/2)d^2 \ge \delta\end{aligned}$$

for some $\gamma_1, \gamma_2 > 0$ and $\delta > 0$, because $\sum_{j=1}^N a_j^2$ and d^2 can not vanish at the same time. Therefore, we have

$$\limsup_{\varepsilon\to 0} \lambda_{N+2,\varepsilon} \le \limsup_{\varepsilon\to 0} \left\{ 1 + \frac{\hat{N}_\varepsilon}{\hat{D}_\varepsilon} \right\} \le 1 + \lim_{\varepsilon\to 0} \frac{O(\frac{1}{\|u_\varepsilon\|^2})}{\delta} = 1.$$

Since we have checked (5.1), we know by Lemma 2.5 that

$$\tilde{v}_{N+2,\varepsilon} \to \sum_{k=1}^N a_{N+2,k} \frac{y_k}{(1+|y|^2)^{N/2}} + b_{N+2} \frac{1-|y|^2}{(1+|y|^2)^{N/2}}$$

in $C^1_{loc}(\mathbb{R}^N)$. Now, for fixed ε, $v_{N+2,\varepsilon}$ and $v_{i,\varepsilon}$ is orthogonal in the sense of (1.3) for $i = 2, \cdots, N+1$. From this, we have $\vec{a}_{N+2} \cdot \vec{a}_i = 0$ for any $i = 2, \cdots N+1$. Since $\vec{a}_i$ are linearly independent in $\mathbb{R}^N$, we have that $\vec{a}_{N+2} = \vec{0}$. Thus we obtain (1.8).

Since $b_{N+2} \ne 0$, Lemma 2.6 assures that

$$\|u_\varepsilon\|^2 v_{N+2,\varepsilon} \to -(N-2)\sigma_N b_{N+2} G(\cdot, x_0), \text{ in } C^1_{loc}(\overline{\Omega} \setminus \{x_0\}) \quad \text{as } \varepsilon \to 0.$$

Then, we can repseat the same proof of Lemma 4.2 (with $i = N+2$) to obtain

$$\|u_\varepsilon\|^2 (\lambda_{N+2,\varepsilon} - 1) \to \Gamma,$$

where $\Gamma = C_2$ in (4.12). Calculation shows $C_2 = (N-2)(N-4)MR(x_0)$. This proves Theorem 1.3.

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