PLURIHARMONIC MAPS AND SUBMANIFOLDS

J.-H. ESCHENBURG

Abstract. The following notes contain a slightly extended version of four lectures given in the Differential Geometry Seminar at Osaka City University on March 12 and 13, 2009. We reported on various approaches to pluriharmonic maps, twistor theory and the DPW method and we discussed recent applications to submanifold geometry

Contents

1. Introduction

A (simply connected) Kähler manifold M allows for a family of parallel rotations R_θ on its tangent bundle, namely multiplication by the complex scalars $e^{i\theta}$. Given a smooth map $f : M \to S$ into some symmetric space $S = G/K$, one may ask when $df \circ \mathsf{R}_\theta$ is the differential of another map f_θ. When S has semi-definite curvature operator (e.g. when S is compact or dual to a compact symmetric space), this holds if and only if f is pluriharmonic,[1] i.e. the $(1, 1)$ part of its hessian vanishes, $(\nabla df)^{(1,1)} = \nabla'' d' f = 0$. Then $(f_\theta)_{\theta\in[0,2\pi]}$ is called *associated family* of f. Equating $df \circ \mathsf{R}_\theta$ with df_θ needs an identification of the two tangent spaces $T_{f(x)}S$ and $T_{f_\theta(x)}S$ by an element $\Phi_\theta(x) \in G$. This defines a smooth map $\Phi : \mathbb{S}^1 \times M \to G$, the so called *extended solution*

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[1]Without the semi-definitness, the "if" statement is unknown

(introduced by Uhlenbeck [U]). Further, if a *frame F for f* is given, i.e. a smooth map $F : M \to G$ with $f = \pi \circ F$ for the projection $\pi : G \to G/K$, then $F_\theta = \Phi_\theta F$ is a frame for f_θ, and the family of maps (F_θ) is called an *extended framing* of f, [BP, DPW]. Extended solutions and extended framings are two different descriptions of pluriharmonic maps. We shall discuss the advantages of both notions in various applications.

Extended framings take values in the twisted loop group $\Lambda_\sigma G$. There is some freedom in the choice of F which is fixed by passing to the quotient space $\hat{Z} = \Lambda_\sigma G/K$ where σ is the involution of G corresponding to the symmetric space $S = G/K$. We may view $\hat{Z}$ as *universal twistor space*: Every pluriharmonic map f is the projection of a holomorphic and "superhorizontal" map $\hat{f} : M \to \hat{Z}$. The complexified loop group $\Lambda_\sigma G^c$ acts on $\hat{Z}$ preserving the holomorphic structure as well as the superhorizontal distribution. Applied to $\hat{f}$, these transformations give new pluriharmonic maps which are called *dressing transformations* of f.

An important special case happens when the associated family is constant, $f_\theta = f$ (*isotropic* case). Then $\hat{f}$ takes values in a finite dimensional complex homogeneous subspace of $\Lambda_\sigma G/K$, a *twistor space*. We will discuss this notion in detail. There are two special cases where all isotropic pluriharmonic maps can be written down explicitly: when S is a complex Grassmannian or a quaternionic symmetric space (Wolf space). For all other compact symmetric spaces S, superhorizontality is a complicated nonholonomic condition, and the general solution is unknown.

The general (non-isotropic) case can be solved by the DPW method (Dorfmeister-Pedit-Wu [DPW]). However, its application is substantially more difficult than in the surface case ($\dim_{\mathbb{C}} M = 1$) since in higher dimensions it requires solving the *complex curved flat condition*: Finding all closed $\mathfrak{p}^c$-valued one-forms η with $[\eta, \eta] = 0$, where $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is the Cartan decomposition of $\mathfrak{g} = \mathrm{Lie}(G)$ corresponding to the symmetric space S and $\mathfrak{p}^c = \mathfrak{p} \otimes \mathbb{C}$. The general solution of this problem seems to be unknown.

Extended solutions take values in the based loop space $\Omega G = \{\omega : \mathbb{S}^1 \to G;\ \omega(1) = e\}$. One striking application of extended solutions is a reduction of a large class of pluriharmonic maps to the isotropic case: pluriharmonic maps of *finite uniton number*, where Φ is a rational function of $\lambda = e^{-i\theta}$, i.e. its Fourier series in λ is finite. By a theorem of Uhlenbeck and Ohnita-Valli [U, OV], this holds always if M is compact. Burstall and Guest [BG] have shown that Φ can be deformed to an isotropic extended solution Φ^∞ (where the loops $\Phi^\infty(x) : \mathbb{S}^1 \to G$ are group homomorphisms) using the energy flow on ΩG which acts by real parameter shift $\omega \mapsto \omega_t$ with $\omega_t(\lambda) = \omega(t\lambda)$. If $S \subset G$ is a Cartan embedded inner symmetric space, then pluriharmonic maps $f : M \to S$ correspond to extended solutions Φ with $\Phi_{-1} = f$. The theory has to be slightly modified for outer symmetric spaces [EMQ]. In both cases one obtains a classification of pluriharmonic maps f by certain isotropic pluriharmonic maps f^∞ in the closure of the extended dressing orbit of f.[2]

Pluriharmonic maps are useful for submanifold theory in various ways. By a classical theorem of Ruh and Vilms, a surface isometrically immersed in euclidean 3-space has constant mean curvature (CMC) iff its ($\mathbb{S}^2$-valued) Gauss map is harmonic. What are the Kähler submanifolds in euclidean n-space whose (Grassmann-valued) Gauss map is pluriharmonic? They have parallel *pluri-mean curvature*, "PPMC", i.e. $\nabla\alpha^{(1,1)} = 0$, where the pluri-mean curvature $\alpha^{(1,1)}$ collects the $dz_i d\bar{z}_j$-components of the second fundamental form α. Examples are rare. The only examples known so far are CMC surfaces in $\mathbb{R}^3$ or S^3, pluriminimal submanifold (where $\alpha^{(1,1)} = 0$), and extrinsic hermitian symmetric spaces (where $\nabla\alpha = 0$). It can be shown [EKT] that further examples must have high codimension.

CMC surfaces in euclidean 3-space yet enjoy another property: They can be computed from their (harmonic) Gauss map, using the associated family. This was discovered by Bonnet and restated differently by Bobenko, using a result of Sym, and it is often called Sym-Bobenko formula. Can one obtain PPMC submanifolds from their (pluriharmonic) Gauss map by a generalized Sym-Bobenko formula? Unfortunately this is not true. However, there is a generalization of this formula where the sphere $\mathbb{S}^2$ is replaced by any hermitian symmetric space of compact type, but it leads to a new class of Kähler submanifolds [EQ] which share many properties of CMC surfaces; one might call them "pluri-CMC".

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2. Associated families

In a first differential geometry course we learn about isometric deformations of surfaces. A standard example is the deformation of the catenoid into the helicoid. You can approximate this deformation by a strip of paper or a belt, bent to a cylinder representing a neighborhood of the equator of the catenoid. Pulling the end appart up and down, you obtain a left or right helix, depending on which side you move up, and putting the two motions together in reversed order, you come back to the cylinder but turned inside out. Such isometric deformation, called associated family, exists for each minimal surface in euclidean

[2] The action of the loop group has to be extended by the parameter shifts $\lambda \mapsto \mu\lambda$, $\mu \in \mathbb{C}^*$. This is one half of the extension to the corresponding Kac-Moody group.

3-space. During the deformation the metric, the normal vector and the principal curvatures are preserved while the principal curvature vectors rotate.

Let's discuss this construction in more generality. Given an open, simply connected subset $M \subset \mathbb{C} = \mathbb{R}2$ and a smooth map $f : M \to \mathbb{R}^n$, we consider the multiplication by $e^{i\theta}$ on M, i.e. the rotation

$$\mathsf{R}_\theta = I\cos\theta + J\sin\theta \tag{1}$$

where J is the multiplication by $i = \sqrt{-1}$ (90^o-rotation), and we ask ourselves: Does there exist a smooth map $f_\theta : M \to \mathbb{R}^n$ such that

$$df_\theta = df \circ \mathsf{R}_\theta \tag{2}$$

holds? Obviously this is true iff $d(df \circ \mathsf{R}_\theta) = 0$. By 1 it suffices to check $d(df \circ J) = 0$. But

$$\begin{aligned} (df \circ J)e_1 &= df.e_2 = f_y, \\ (df \circ J)e_2 &= -df.e_1 = -f_x, \end{aligned}$$

hence $d(df \circ J) = 0 \iff$

$$0 = d(f_y dx - f_x dy) = (f_{yy} + f_{xx})dy \wedge dx$$

$\iff \Delta f = 0$, i.e. iff f is harmonic. Since a minimal surface in $\mathbb{R}^n$ is a conformal harmonic map, we obtain associated families for minimal surfaces.

The same argument works for a simply connected open subset M in $\mathbb{C}^n$ rather than in $\mathbb{C}$. Then an associated family (2) exists $\iff$ $f_{x_k x_k} + f_{y_k y_k} = 0$ for $k = 1, \dots, n$. Such maps are called *pluriharmonic*.

Now we want to replace M by an arbitrary simply connected Kähler manifold and euclidean space by a Riemannian symmetric space $S = G/K$. Let $f : M \to S$ be a smooth map. When does there exist a family of maps $f_\theta : M \to S$ with $df_\theta = df \circ \mathsf{R}_\theta$? Yet this does not make sense since $(df_\theta)_x$ and df_x would take values in different tangent spaces $T_{f(x)}M$ and $T_{f_\theta(x)}M$. We need an isomorphism $\Phi_\theta(x) : T_{f(x)}M \to T_{f_\theta(x)}M$ in order to identify the two tangent spaces. In euclidean space this is just parallel translation, and for a symmetric space $S = G/K$ our isomorphism is supposed to have similar properties:

1. $\Phi_\theta(x) \in G$ (as acting on TS), for all $x \in M$,
2. $\Phi_\theta \in \mathrm{Hom}(f^*TS, f_\theta^*TS)$ is parallel, $\nabla\Phi_\theta = 0$.

The last equation $\nabla\Phi_\theta = 0$ is equivalent to

$$g^{-1}\Phi_\theta^{-1} d\Phi_\theta g \in \mathfrak{p} \tag{3}$$

where $g \in G$ with $f(x) = gK \in G/K$ and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is the Cartan decomposition corresponding to S.

Definition. An *associated family* for a smooth map $f : M \to S = G/K$ is a circle of pairs (f_θ, Φ_θ), $\theta \in [0, 2\pi]$, with $f_\theta : M \to S$ and

$\Phi_\theta \in \mathrm{Hom}(f^*TS, f_\theta^*TS)$ satisfying properties 1 and 2 above, such that

$$df_\theta = \Phi_\theta \circ df \circ \mathsf{R}_\theta. \tag{4}$$

A smooth map $f : M \to S$ is called *pluriharmonic* if $f|C$ is harmonic for each complex one-dimensional submanifold $C \subset M$, or equivalently, if the $(1,1)$ component of its hessian vanishes, $\nabla df^{(1,1)} = \nabla'' d' f = 0$.[3]

Theorem 2.1. [ET2] *Let S be a compact symmetric space or dual to a compact symmetric space. Then $f : M \to S$ has an associated family if and only if f is pluriharmonic.*

The main idea of the proof can be explained best in a broader context [ET1]. Given a Riemannian manifold M, a vector bundle $E \to M$ and an E-valued one-form $\omega : TM \to E$, we ask: Does there exist a smooth map $f : M \to S$ into some symmetric space S with $df = \omega$? If yes, we must be able to identify f^*TS with E by a bundle isomorphism $\Phi \in \mathrm{Hom}(f^*TS)$. In particular, Φ induces on E the metric, the connection, and the Lie triple product R^S of f^*TS.[4] Hence we may assume that these data are already given on E, and the Lie triple product $R^S \in \mathrm{Hom}(\otimes^3 E, E)$ is parallel with respect to the connection on E.

Theorem 2.2. [ET1] *Given $\omega \in \mathrm{Hom}(TM, E)$, there exists a smooth map $f : M \to S = G/K$ and a parallel isometric isomorphism $\Phi : f^*TS \to E$ with*

$$df = \Phi \circ \omega \tag{5}$$

if and only if the Cartan structure equations hold:[5]

$$d^\nabla \omega = 0, \quad R^\nabla = \omega^* R^S. \tag{6}$$

The solution (f, Φ) of (5) is unique up to motions (elements of G), i.e. any other solution is $(gf, g\Phi)$ for some $g \in G$.

The proof of this theorem is an application of the Frobenius integrability theorem. We want to apply it to $\omega = df \circ \mathsf{R}_\theta$. Then (6) implies pluriharmonicity, but the converse needs a Lemma of Ohnita and Valli using the definiteness of R^S; without this assumption (e.g. in cases of indefinite metrics) the converse direction is unknown.

Lemma 2.3. [OV] *If $f : M \to S$ is pluriharmonic and R^S positive or negative semi-definite, then $R^S(df.X, df.Y) = 0$ for all $X, Y \in TM$.*

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[3]For the second definition of pluriharmonicity we need a Kähler metric on M, but the first definition shows that the notion is independent of the choice of this metric.

[4]The Riemannian curvature tensor $R^S : \otimes^3 TS \to TS$ of a symmetric space S may be considered as a "triple product", a product with 3 factors, on $\mathfrak{p} = T_oS$. It satisfies an additional identity which makes it into a *Lie triple product*: $R^S(X,Y)$ is a derivation of R^S for any two tangent vectors X, Y on S.

[5]We let $d^\nabla \omega(X,Y) = \nabla_X \omega(Y) - \nabla_Y \omega(X) - \omega[X,Y]$ and $R^\nabla(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$.

If (f_θ, Φ_θ) is an associated family of $f = f_0$ and $F : M \to G$ a (local) lift or *framing* of f, i.e. $f = \pi \circ F$ for the projection $\pi : G \to G/K = S$, then Φ_θ is an *extended solution* in the sense of [U, OV] and $F_\theta = \Phi_\theta F$ an *extended framing* in the sense of [BP, DPW, O, DE], see [EQ]. From now on we shall often replace the subscript $\theta \in [0, 2\pi]$ by $\lambda = e^{-i\theta} \in \mathbb{S}^1$.[6]

Remark. For $\lambda = 1$ we put $f_1 = f$, $\Phi_1 = e$. For $\lambda = -1$ $(\theta = \pi)$, we have $\mathsf{R}_{-1} = -I$, and a solution of (4) is $f_{-1} = f$, $\Phi_{-1} = -I$ or $\Phi_{-1}(x) = s_{f(x)} \in G$. More generally, $\mathsf{R}_{-\lambda} = -\mathsf{R}_\lambda$, and a solution for $-\lambda$ is given by $f_{-\lambda} = f_\lambda$ and $\Phi_{-\lambda} = -\Phi_\lambda = \Phi_\lambda s_f$. From the unicity part of Theorem 2.2 we see that an arbitrary solution satisfies

$$f_{-\lambda} = g_\lambda f, \quad \Phi_{-\lambda} = g_\lambda \Phi_\lambda s_f \tag{7}$$

for some $g_\lambda \in G$. This is the twist condition which we have seen at the beginning for the catenoid deformation.

3. Isotropy and Twistors

A pluriharmonic map $f : M \to S$ is called *isotropic* if $f_\lambda = f$ for all $\lambda \in \mathbb{S}^1$. Then (4) becomes

$$df = \Phi_\lambda \circ df \circ \mathsf{R}_\lambda \tag{8}$$

and in particular, the isometry $\Phi_\lambda(x) \in G$ fixes $f(x)$.

Theorem 3.1. [ET2] *If f is isotropic pluriharmonic and full, i.e. its image is not contained in a proper totally geodesic submanifold of S, then $\Phi(x) : \lambda \mapsto \Phi_\lambda(x)$ is a group homomorphism from $\mathbb{S}^1$ into the isotropy group $G_{f(x)}$ with $\Phi_{-1}(x) = s_{f(x)}$. In particular, $S = G/K$ is an inner symmetric space.*

Proof. We decompose the complexified tangent bundle into J-eigenspaces, $T^cM = T'M + T''M$ with $J = i$ on $T'M$ and $J = -i$ on $T''M$. Then $\mathsf{R}_\lambda = \lambda^{-1}$ on $T'M$, and by (8) we have $\Phi_\lambda = \lambda$ on $T'M$ (and likewise $\Phi_\lambda = \bar\lambda = \lambda^{-1}$ on $T''M$). Thus λ is an eigenvalue of $\Phi_\lambda \in \mathrm{End}(f^*TS)$, and the eigenbundle E_λ contains $df(T'M)$. Note that E_λ is parallel since so is Φ_λ, and the real parallel bundle $E_1 := E_\lambda + E_{\bar\lambda}$ contains $df(TM)$. Since f is full, the smallest R^S-invariant parallel subbundle $E \subset f^*TS$ with $E_1 \subset E$ is the full bundle f^*TS. Since Φ_λ preserves R^S, we have $\Phi_\lambda\, R^S_{AB}C = \lambda^3\, R^S_{AB}C$ for all $A, B, C \in E_\lambda$. Similarly we see that all eigenvalues of Φ_λ on E are odd integer powers of λ. In particular we have the group law $\Phi_\lambda \Phi_\mu = \Phi_{\lambda\mu}$. Furthermore Φ_{-1} acts as an odd power of -1 on each eigenspace, hence $\Phi_{-1} = -I$

[6]Extended solutions and extended framings $\Phi_\lambda, F_\lambda : M \to G$ are defined by the λ-dependence of their (1,0) Maurer-Cartan forms $\alpha_\lambda * = F_\lambda^{-1}\partial F_\lambda$ and $\beta_\lambda = \Phi_\lambda^{-1}\partial\Phi_\lambda$

$$\begin{aligned} \alpha'_\lambda &= \alpha'_{\mathfrak{k}} + \lambda^{-1}\alpha'_{\mathfrak{p}}, \\ \beta_\lambda &= (1 - \lambda^{-1}\beta'. \end{aligned}$$

and viewed as a group element, $\Phi_{-1}(x) = s_{f(x)}$. In particular, the symmetry lies in the connected component of K (conjugate to $G_{f(x)}$), thus $S = G/K$ is an inner symmetric space. $\square$

Since any two $\Phi(x_1), \Phi(x_2)$ are conjugate under parallel displacents which are elements of G, all $\Phi(x)$, $x \in M$ are contained in the same conjugacy class, the conjugacy class of some homomorphism $\tau_o : \mathbb{S}^1 \to K$ with $\tau_o(-1) = s_o$. Such homomorphic circles passing through s_o will be called a *twistors*, and their conjugacy classes *twistor spaces*. Twistors have been introduced by R. Penrose in order to understand even dimensional manifolds (e.g. spacetimes) by complex methods. When there is no distinguished complex structure, one has to study the set of *all* (compatible) complex structures, the *twistor space*, which itself is often a complex manifold. In our case, the twistor space is an adjoint orbit (see below), hence a complex manifold, and to any $\tau \in Z$ we assign the group element $\tau(\frac{\pi}{2})$ which is a complex structure on T_pS for the fixed point p of τ, since $\tau(\frac{\pi}{2})^2 = \tau(\pi) = s_p = -I$ on T_pS.

Passing from τ_o to its generator $\xi_o \in \mathfrak{k}$ with $\tau_o(\theta) = \exp(\theta\xi_o)$, we may view the conjugacy class of τ_o as the adjoint orbit $Z = \mathrm{Ad}(G)\xi_o = G/H$ where H is the centralizer of τ_o or ξ_o. We have $H \subset K$ since H commutes with τ and in particular with $\tau_o(-1) = s_o$.[7] Thus we obtain the *twistor fibration* $\pi_Z : Z = G/H \to G/K = S$. It can be viewed also as the evaluation map

$$ev_{-1} : Z \ni \tau \mapsto \tau(-1) = s_p \in S \subset G$$

where S is considered as a subset of G (the conjugacy class of s_o) via the Cartan embedding $S \ni p \mapsto s_p \in G$.

We have $\mathfrak{g}^c = \sum_k \mathfrak{g}_k$ where $\mathfrak{g}_k$ is the eigenspace of $\mathrm{ad}(\xi_o)$ with eigenvalue ki where $i = \sqrt{-1}$. Furthermore, $\mathfrak{g}_k \subset \mathfrak{p}^c$ for odd k and $\mathfrak{g}_k \subset \mathfrak{k}^c$ for even k while $\mathfrak{g}_0 = \mathfrak{h}$. In the proof of Theorem 2.1 we have seen that $d\Phi$ takes values in the so called *superhorizontal* bundle E_1 determined by $\mathfrak{g}_1 + \mathfrak{g}_{-1} \subset \mathfrak{p}$, and moreover, $\mathfrak{p}$ is generated by $\mathfrak{g}_1 + \mathfrak{g}_{-1}$. Such twistors ξ_o are called *canonical*; they are just sums of inverse fundamental roots without repetition [BR].

Theorem 3.2. [ET2] *A full smooth map $f : M \to S = G/K$ is isotropic pluriharmonic if and only if $f = \pi_Z \circ \Phi$ for some canonical twistor fibration π_Z where $\Phi : M \to Z$ is holomorphic and superhorizontal.*

In fact, since $x \mapsto \Phi(x)$ is parallel, it is a horizontal lift of f. Superhorizontality follows from the fact that df takes values in E_1, and holomorphicity follows from (8) for $\lambda = i = \sqrt{-1}$ since on E_1, the complex structure of Z agrees with Φ_i.

[7] In fact, centralizers are connected, thus H is contained in the identity component of K.

Example: Complex Grassmannians $S = G_p(\mathbb{C}^n) = U_n/(U_p \times U_{n-p})$ Here the twistor space Z is a *flag manifold*. A flag in $\mathbb{C}^n$ can be viewed in two different ways: as a chain of subspaces $W_1 \subset W_2 \subset \dots \subset W_r = \mathbb{C}^n$, or else as an orthogonal decomposition $\mathbb{C}^n = E_1 \oplus E_2 \oplus \dots \oplus E_r$, where $W_1 = E_1$, $W_2 = E_1 \oplus E_2$ etc. A flag manifold Z is the set of all flags with given dimensions. Denoting by E_k also the orthogonal projection onto E_k, we may view a flag $(E_1, \dots, E_r)$ as a matrix $i \sum_{k=1}^{r} kE_k \in \mathfrak{u}_n$, and thus the flag manifold becomes an adjoint orbit in $\mathfrak{u}_n$. The twistor fibration is the map $(E_1, \dots, E_r) \mapsto E_1 + E_3 + \dots : Z \to S$ (sum of the spaces with odd indices). A map $\Phi : M \to Z$ is a "moving flag" $W_1(x) \subset W_2(x) \subset \dots$. It is holomorphic iff all the moving spaces (vector bundles) W_k are holomorphic, i.e. W_k is locally spanned by holomorphic functions $M \to \mathbb{C}^n$, and it is superhorizontal if each space W_k differentiates into the next one, $\partial W_k \subset W_{k+1}$. Now the following theorem is easy; it shows that each isotropic pluriharmonic map $f : M \to G_p(\mathbb{C}^n)$ can be obtained from free holomorphic data:

Theorem 3.3. (F. Burstall, cf. [ET3]) *All superhorizontal holomorphic maps $\Phi = (W_1, \dots, W_r) : M \to Z$ arise (locally) as follows. Start with arbitrary holomorphic functions $f_i : M \to \mathbb{C}^n$ and let $W_1(x)$ be the linear span of the vectors $f_i(x)$. Let W_2 be spanned by the functions f_i, their partial derivatives $\partial_j f_i$ and maybe further arbitrary holomorphic functions g_k. Let W_3 be the span of all these functions spanning W_2, their first partial derivatives and maybe further arbitrary holomorphic functions $h_l : M \to \mathbb{C}^n$, and so on.*

By Lemma 2.3, the rank of such maps cannot be two high. The isotropic pluriharmonic maps of maximal rank into complex Grassmannians have been classified in [EK].

There is but one other case where all isotropic pluriharmonic maps can be given explicitly: quaternionic symmetric spaces, cf. [ET3].

4. Finite uniton number

Sometimes, pluriharmonic maps must be isotropic by topological reasons. Theorems of this type for the surface case have been proved long time ago, starting with the work of Calabi on minimal 2-spheres (see references in [ET2, ET3]):

Theorem 4.1. *If M is a surface of genus 0 and S a sphere or complex projective space, then all harmonic maps $f : M \to S$ are isotropic pluriharmonic.*

Unfortunately, no such theorem is known for higher dimensional M. But there exists a theorem for a more general class of pluriharmonic maps, those with *finite uniton number*. Let us assume that our group G is a matrix group, $G \subset U_n$. Then the extended solution $\Phi_\lambda(x)$

can be developed into a matrix valued Fourier series with respect to λ, $\Phi_\lambda(x) = \sum_{k\in\mathbb{Z}} a_k(x)\lambda^k$. The pluriharmonic map f is called of finite uniton number if its extended solution Φ has finite Fourier series, $\Phi_\lambda(x) = \sum_{|k|\le r} a_k(x)\lambda^k$. Clearly this holds if f is isotropic since then $\Phi_\lambda(x)$ is conjugate to $\tau_o(\lambda) = \sum_{|k|\le r} w_k\lambda^k$ where w_k are the (finitely many) weights of the representation $\mathbb{S}^1 \xrightarrow{\tau} G \subset U_n$.

Theorem 4.2. [U, OV] *Let M be compact, simply connected and $f : M \to S$ pluriharmonic. Then f has finite uniton number.*

The following theorem was shown by Burstall and Guest [BG] for inner symmetric spaces S and was extended to outer symmetric spaces in [EMQ]. It shows that in some sense, pluriharmonic maps of finite uniton number can be reduced to isotropic pluriharmonic maps.

Theorem 4.3. *Any pluriharmonic map of finite uniton number can be deformed into an isotropic pluriharmonic map.*

The deformation is done by the gradient flow of the energy function on the space Ω^{alg} of based loops[8] $\omega : \mathbb{S}^1 \to G$ with finite Fourier series ("algebraic loops"). We assume that G is equipped with a bi-invariant metric. Then the energy $E : \Omega^{alg} \to \mathbb{R}$ is defined by $E(\omega) = \int_{\mathbb{S}^1} |\omega'|^2$. By the first variation formula in Riemannian geometry, the critical points of E are the geodesic loops through e. For or a bi-invariant metric these are precisely the homomorphims $\tau : \mathbb{S}^1 \to G$, and thus the gradient flow lines of E end at homomorphisms. In Riemannian geometry, one usually wishes to decrease energy, but then most of the contractible loops would be flowed to the trivial homomorphism $\tau_o = e$. Instead we increase energy, following the flow of ∇E rather than of $-\nabla E$. It is a property of the space of algebraic loops [PS] that each loop lies in a (finite dimensional) domain of attraction ("Bruhat cell") for some critical point of E. In fact, the flow is obtained from the group action of the positive real numbers $(0, \infty)$ acting on Ω^{alg} by

$$(\omega(\lambda), t) \mapsto \omega(t\lambda). \tag{9}$$

We apply this idea to the extended solution $\Phi : M \to \Omega^{alg}$ of a pluriharmonic map $f : M \to S \subset G$ of finite uniton number. Due to the holomorphicity of Φ (cf. next section), its image $\Phi(M)$ lies in a single Bruhat cell, up to some complex subset $\Sigma \subset M$ of codimension ≥ 1. Applying the flow to $\Phi(x)$ for any $x \in M \setminus \Sigma$, we get extended solutions for all times t, and the limit for $t \to \infty$ is an isotropic extended solution which takes values in a critical manifold, the conjugacy class of some twistor τ_o (which can be chosen to be canonical).

[8]"Based" means $\omega(1) = e \in G$ (unit element)

5. Loop groups and the universal twistor space

Let $S = G/K$ be a compact symmetric space where G is a matrix group. Let $s_o \in G$ be the point reflection of S at the base point $o = eK$ and let $\sigma \in \mathrm{Aut}(G)$ be the conjugation by s_o. The *twisted loop group* is the group

$$\Lambda_\sigma := \Lambda_\sigma G := \{\gamma : \mathbb{S}^1 \to G;\ \gamma_{-\lambda} = \sigma\gamma_\lambda\} \tag{10}$$

Here the maps γ are arbitrary with certain regularity conditions we may impose. Now let $f : M \to S$ be a pluriharmonic map with extended solution Φ_λ and extended framing $F_\lambda = \Phi_\lambda F_1$. From the twist condition (7) we have

$$\begin{aligned} F_{-\lambda} &= \Phi_{-\lambda}F_1 \\ &= g_\lambda \Phi_\lambda s_f F_1 \\ &= g_\lambda \Phi_\lambda F_1 F_1^{-1} s_f F_1 \\ &= g_\lambda F_\lambda s_o. \end{aligned}$$

Now we change our extended framing (and our map f) trivially, putting $\tilde F_\lambda = F_\lambda(x_o)^{-1}F_\lambda$ for some $x_o \in M$ with $f(x_o) = o = eK \in S$. Then

$$\tilde F_{-\lambda} = F_{-\lambda}(x_o)^{-1}F_\lambda = s_o F_\lambda(x_o)^{-1} g_\lambda^{-1} g_\lambda F_\lambda s_o = \sigma(\tilde F_\lambda).$$

Thus the loops $\tilde{\mathsf{F}}(x) : \lambda \mapsto F_\lambda(x)$, $x \in M$, are elements of the twisted loop group Λ_σ, and we obtain a (locally defined) map $\tilde{\mathsf{F}} : M \to \Lambda_\sigma$. We will now write F for $\tilde{\mathsf{F}}$.

F still depends on the choice of the initial frame F_1. This can be changed by an arbitrary map $k : M \to K$ since $\pi(F_1k) = \pi(F_1)$. We remove this ambiguity by passing to the quotient space

$$\hat Z = \Lambda_\sigma/K$$

where K denotes the constant loops in Λ_σ (which must be elements of K, by the twist condition $\gamma_{-\lambda} = \sigma(\gamma_\lambda)$). Working modulo K allows to define F globally on M. The projection $\pi : \Lambda_\sigma/K \to G/K$ is the evaluation at $\lambda = 1$.

Theorem 5.1. [O, DE, E1] *A smooth map $f : M \to S$ is pluriharmonic if and only if $f = \pi \circ \mathsf{F}$ where $\mathsf{F} : M \to \hat Z$ is holomorphic and superhorizontal.*

We have to explain these notions. The space $\hat Z = \Lambda_\sigma/K$ carries a complex structure since it can be viewed as a quotient of two complex groups:

$$\Lambda_\sigma = \Lambda_\sigma^c/\Lambda_\sigma^+ \tag{11}$$

where Λ_σ^c is the group of twisted loops in the complexified (matrix) Lie group G^c and Λ_σ^+ the subgroup of those $\gamma \in \Lambda_\sigma^c$ whose Fourier series has no negative λ-powers, $\gamma = \sum_{k\geq 0} a_k\lambda^k \in \Lambda_\sigma^c$ with a_0 invertible. In fact, the real loop group $\Lambda_\sigma \subset \Lambda_\sigma^c$ acts on the complex quotient $\Lambda_\sigma^c/\Lambda_\sigma^+$, and the action is transitive by the Iwasawa decomposition $\Lambda_\sigma^c = \Lambda_\sigma \Lambda_\sigma^+$ of Λ_σ^c, see [PS]. An element $\gamma \in \Lambda_\sigma$ stabilizes the point $e\Lambda_\sigma^+$ iff $\gamma \in \Lambda_\sigma^+ \cap \Lambda_\sigma = K$: observe that $\gamma = \sum_{k\in\mathbb{Z}} a_k \lambda^l$ is real iff $a_{-k} = \overline{a_k}$; hence $a_{-k} = 0$ for all $k \in \mathbb{N}$ implies $\gamma = a_0$, and from the twist condition $\gamma_{-\lambda} = \sigma(\gamma_\lambda)$ we obtain $\gamma \in K$.

The Maurer-Cartan form $\alpha = \mathsf{F}^{-1}d\mathsf{F}$ of the extended framing F satisfies $\alpha' = \lambda^{-1}\alpha'_{\mathfrak{p}} + \alpha'_{\mathfrak{k}}$ and $\alpha'' = \overline{\alpha'} = \lambda\alpha''_{\mathfrak{p}} + \alpha''_{\mathfrak{k}}$. Modulo Λ_σ^+ only the λ^{-1}-term survives, hence $d''\mathsf{F} = \bar\partial\mathsf{F} = 0$ modulo Λ_σ^+ (i.e. $\mathsf{F} : M \to \hat{Z}$ is holomorphic) while $d'\mathsf{F} = \partial\mathsf{F}$ takes values in the "superhorizontal" distibution[9] $\Lambda_\sigma^c(\lambda^{-1}\mathfrak{p}^c)$ on $\hat{Z} = \Lambda_\sigma^c/\Lambda_\sigma^+$.

Due to the similarity of the theorems 2.2 and 4.1, $\hat{Z}$ could be called the *universal twistor space*,

Remark 1. [E1] The universal twistor space, like its finite dimensional analoga, can be viewed as an adjoint orbit

$$\hat{Z} = \mathrm{Ad}(\hat{\Lambda}_\sigma)\delta \tag{12}$$

where $\hat{\Lambda}_\sigma \subset \mathrm{Aut}(\Lambda_\sigma)$ is the semidirect product of Λ_σ with $\mathbb{S}^1$ acting on Λ_σ by parameter shift $(\gamma_\lambda, \mu) \mapsto \gamma_{\mu\lambda}$, and where $\delta = d/dt$ for $\mu = e^{it}$ is the infinitesimal generator of the $\mathbb{S}^1$-factor. In fact,

$$\mathrm{Ad}(\gamma)\delta = \gamma\delta\gamma^{-1} = \delta + \gamma(\gamma^{-1})^{\cdot} = \delta - \dot\gamma\gamma^{-1},$$

and $\mathrm{Ad}(\gamma)\delta = \delta \iff \dot\gamma\gamma^{-1} = 0 \iff \gamma = \mathit{const} \in G \cap \Lambda_\sigma = K$. The extension from Λ_σ to $\hat{\Lambda}_\sigma$ is the first step to the corresponding Kac-Moody-group. Since the second step, a central extension, does not affect the adjoint action, the universal twistor space can be viewed as an adjoint orbit of the twisted Kac-Moody group corresponding to (G, σ).

Remark 2. [DE, E2] Any finite dimensional twistor space $Z = G/H$, being the conjugacy class of a twistor τ, is embedded into the universal twistor space $\hat{Z}$ by projection of the map

$$i : g \mapsto \tau g \tau^{-1} : G \to \Lambda_\sigma$$

via the diagram

$$\begin{tikzcd} G \arrow[r, "i"] \arrow[d, "\pi"] & \Lambda_\sigma \arrow[d, "ev_1"] \\ G/H \arrow[r] & \Lambda_\sigma/K \end{tikzcd}$$

where ev_1 denotes evaluation at $\lambda = 1$.

Remark 3. There is a large group Λ_σ^c acting on the homogeneous space $\hat{Z}$. The action is holomorphic and preserves superhorizontality.

[9] We consider $\lambda^{-1}\mathfrak{p}^c$ as a subspace of $T_{e\Lambda_\sigma^+}\hat{Z}$, which is invariant under the stabilizer Λ_σ^+ and hence determines a Λ_σ^c-invarinant subbundle of $T\hat{Z}$, called $\Lambda_\sigma^c(\lambda^{-1}\mathfrak{p}^c)$.

hence it induces transformations of a pluriharmonic map f, the so called *dressing transformations.* For $g_\lambda \in \Lambda_\sigma$ this corresponds to the congruent associated family $g_\lambda f_\lambda$, but the complex extension Λ_σ^c yields new pluriharmonic maps. The dressing action can still be enlarged by the (arbitrary complex) parameter shifts as in (9).

6. The DPW method

As we have seen, the extended framing F modulo Λ_σ^+ is a holomorphic map $[\mathsf{F}] : M \to \hat{Z}$. Dorfmeister, Pedit and Wu [DPW, O, DE] constructed a holomorphic local representative $\mathsf{H} : M \to \Lambda_\sigma^c$ of $[\mathsf{F}]$. To do this, one needs to find some $\mathsf{V}_+ \in \Lambda_\sigma^+$ such that $\mathsf{H} = \mathsf{F}\mathsf{V}_+$ is holomorphic, i.e. $\eta'' = 0$ where $\eta = \mathsf{H}^{-1}d\mathsf{H}$. Since

$$\eta = \mathsf{V}_+^{-1}(\alpha + d\mathsf{V}_+\mathsf{V}_+^{-1})\mathsf{V}_+ \tag{13}$$

we have $\eta'' = 0$ iff

$$\bar\partial \mathsf{V}_+ = -\alpha''\mathsf{V}_+. \tag{14}$$

This equation has a solution $\tilde{\mathsf{V}}_+ = \mathsf{F}^{-1}$, thus the integrability condition is satisfied, and since $\alpha'' = \alpha_{\mathfrak{k}}'' + \lambda\alpha_{\mathfrak{p}}''$ takes values in the Lie algebra of Λ_σ^+, we find another solution V_+ in Λ_σ^+ as required.

In fact, there are many such η. A distinguished one is the *normalized potential* with only one Fourier summand $\eta_- = \eta_{-1}\lambda^{-1}$. This is obtained from the *Birkhoff decomposition* of an open dense subset ("big cell") of Λ_σ^c,

$$(\Lambda_\sigma^c)_o = \Lambda_\sigma^-\Lambda_\sigma^+ \tag{15}$$

where Λ_σ^- contains only Fourier series with negative indices, $\gamma_\lambda = \sum_{k\le 0} a_k\lambda^k$. Comparing the Birkhoff decompositions $\mathsf{H} = \mathsf{H}_-\mathsf{H}_+$ and $\mathsf{F} = \mathsf{F}_-\mathsf{F}_+$ of H and F we notice $\mathsf{H}_- = \mathsf{F}_-$ (up to a constant matrix), using the unicity of (15) (up to constant matrices), since

$$\mathsf{F}_-\mathsf{F}_+ = \mathsf{F} = \mathsf{H}\mathsf{V}_+^{-1} = \mathsf{H}_-\mathsf{H}_+\mathsf{V}_+,$$

Thus $\eta_- = \mathsf{H}_-^{-1}d\mathsf{H}_-$ takes values in the Lie algebra of Λ_σ^-, but by (13), its Fourier series starts with λ^{-1} (like the one of α), hence only the λ^{-1}-term remains.

Now in the Maurer-Cartan equation $d\eta_- = [\eta_-, \eta_-]$ the left and the right hand sides are multiples of λ^{-1} and λ^{-2}, respectively, hence both sides must vanish and we obtain

$$d\eta_{-1} = 0, \quad [\eta_{-1}, \eta_{-1}] = 0. \tag{16}$$

Theorem 6.1. *The pluriharmonic maps $f : M \to S = G/K$ are in one-to-one correspondence to closed holomorphic one-forms η on M with values in $\mathfrak{p}^c$ and $[\eta, \eta] = 0$.*

In fact, if such η is given, we put $\eta_- = \eta\lambda^{-1}$, obtain a holomorphic map $\mathsf{H}_- = \mathsf{F}_-$ by solving $\partial\mathsf{H}_- = \mathsf{H}_-\eta_-$ and then recieve $\mathsf{F} = (F_\lambda)$ from the Iwasawa decomposition $\mathsf{F}_- = \mathsf{F}\mathsf{W}_+$ of F_-. Finally we put $f = F_1 \mod K$.

While $[\eta, \eta] = 0$ is no condition if M has complex dimension one (surface case), the general solution of (16) is unknown for higher dimensions.

7. Submanifolds with pluriharmonic "Gauss map"

It is well known [RV] that a surface conformally immersed in euclidean 3-space has constant mean curvature (CMC) iff its Gauss map is harmonic. Can we generalize this to higher dimensions? What are the Kähler submanifolds in euclidean n-space with pluriharmonic Gauss map? These are called submanifolds with *parallel pluri-mean curvature* (PPMC), i.e. the (1,1) part of the 2nd fundamental form α is parallel [BEFT]. However, there are only few examples known: CMC-surfaces, pluriminimal submanifolds and extrinsic Kähler symmetric spaces. It is known that any other examples must have high codimension [EKT].

In order to obtain new examples one would like to use an "inverse Gauss map" method which would recover the submanifold from its Gauss map.[10] In the case of CMC surfaces such a method exists, the so called *Sym-Bobenko* formula [S, Bb] which computes the CMC immersion from its harmonic Gauss map. This formula can be extended to higher dimensions [EQ]. However, it does not lead to new PPMC immersions, but to another class of Kähler submanifolds generalizing CMC surfaces.[11]

As was reported in [H], the story of the Sym-Bobenko formula goes back to Bonnet [Bn], 1853. A classical fact known to Bonnet is that CMC surfaces come in pairs of conformal parallel CMC surfaces with a (non-conformal) surface of constant Gaussian curvature in the middle. More precisely, suppose we have a surface immersion $g : M \to \mathbb{R}^3$ with constant Gaussian curvature $K = 1$ in euclidean 3-space and let $h : M \to \mathbb{S}^2 \subset \mathbb{R}^3$ be its Gauss map. Let $1/r_1$, $1/r_2$ be the two principal curvatures; from $K = 1$ we have $r_1 r_2 = 1$. The principal curvatures of the parallel surfaces $f_\pm = g \pm h$ are $1/(r_i \pm 1)$, and the mean curvature of $f_\pm$ is given by

$$2H_\pm = \frac{1}{r_1 \pm 1} + \frac{1}{r_2 \pm 1} = \frac{r_1 + r_2 \pm 2}{1 \pm (r_1 + r_2) + 1} = \pm 1,$$

using $r_1 r_2 = 1$. Furthermore, the induced metrics of f_+ and f_- are conformal since their principal curvatures have the same ratio (up to sign):

$$\frac{(r_1 - 1)/(r_2 - 1)}{(r_1 + 1)/(r_2 + 1)} = \frac{r_1 r_2 + r_1 - r_2 - 1}{r_1 r_2 - r_1 + r_2 - 1} = -1$$

---

[10] A general approach to the "inverse Gauss map" method has been given in [EKMT]

[11] Everything in this section is common work with Peter Quast [EQ].

where $r_1 r_2 = 1$ has been used again.

Maybe it was also known to Bonnet that a conformal surface immersion $f : M \to \mathbb{R}^3$ with Gauss map $h : M \to \mathbb{S}^2 \subset \mathbb{R}^3$ has constant mean curvature $H = -\frac{1}{2}\,\mathrm{trace}\; dh$ iff h is harmonic, i.e. Δh is perpendicular to the tangent space $h^\perp$:

$$\begin{aligned} -2\partial_v H &= \partial_v\,\mathrm{trace}\; dh \\ &= \mathrm{trace}\;\nabla_v dh \\ &= \mathrm{trace}\,\langle \nabla_v dh, df\rangle \\ &= \langle \mathrm{trace}\;\nabla dh, \partial_v f\rangle \\ &= \langle \Delta h, \partial_v f\rangle \end{aligned} \tag{17}$$

where we have used the symmetry of $\langle \nabla dh, df\rangle$ in all three variables (Codazzi equation).

For the inverse problem, only a harmonic map $h : M \to \mathbb{S}^2$ is given, where M is a simply connected Riemann surface. One looks for a conformal immersion $f = f_+ : M \to \mathbb{R}^3$ with Gauss map h and mean curvature $H = \frac{1}{2}$, or equivalently, for a (non-conformal) immersion g with Gauss map h and Gaussian curvature $K = 1$ (where $f = g + h$). Bonnet solved this problem as follows:

$h : M \to \mathbb{S}^2 \subset \mathbb{R}^3$ is harmonic $\iff$

$$0 = h \times \Delta h = h \times (h_{xx} + h_{yy}) = (h \times h_x)_x + (h \times h_y)_y \tag{18}$$

$\iff$ $\gamma := (h \times h_y)dx - (h \times h_x)dy$ is a closed one form, i.e. $\gamma = dg$ for some map $g : M \to \mathbb{R}^3$, and it turns out that this g satisfies $K = 1$ whereever it is an immersion. We can write γ more simply as

$$\gamma = h \times (dh \circ j) = J\, dh\, j \tag{19}$$

where j and J are the the complex structures on M and $\mathbb{S}^2$, respectively.

We extend this idea to higher dimensions as follows. The 2-sphere $\mathbb{S}^2 \subset \mathbb{R}^3 = \mathfrak{so}_3$ is replaced by a Kähler symmetric space (S, J) with its *standard embedding* $S \subset \mathfrak{g}$ which is obtained as follows. A Riemannian manifold S is Kähler iff there is a parallel almost complex structure J on TS. At any point $p \in S$, this is a skew adjoint derivation of the curvature tensor R^S,

$$JR^S_{XY}Z = R^S_{JX,Y}Z + R^S_{X,JY}Z + R^S_{XY}JZ.$$

In fact, from $\nabla(JZ) = J\nabla Z$ we obtain $R^S_{XY}JZ = JR^S_{XY}Z$ which also implies $R^S_{JX,JY} = R^S_{XY}$ and hence $R^S_{JX,Y} + R^S_{X,JY} = 0$. If S is also symmetric, $S = G/K$, then any skew symmetric derivation of R^S at $o \in S$ generates a one-parameter group of isometries fixing o (the central $\mathbb{S}^1 \subset K$). This is an element of the Lie algebra $\mathfrak{g}$ of the isometry group G of S, and the map

$$S \ni p \mapsto J_p \in \mathfrak{g}$$

is called the *standard embedding* of the Kähler symmetric space S. Thus S becomes an adjoint orbit in $\mathfrak{g}$, and $J_p = \mathrm{ad}(p) = [p, .]$ for any $p \in S \subset \mathfrak{g}$. At the base point o, the tangent and normal spaces are $T_oS = \mathfrak{p}$ and $N_oS = \mathfrak{k}$ where $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is the Cartan decomposition corresponding to S.

Theorem 7.1. *Let (M, j) be a complex manifold and $S \subset \mathfrak{g}$ be a standard embedded Kähler symmetric space and $h : M \to S$ a smooth map. Then h is pluriharmonic if and only if the $\mathfrak{g}$-valued one-form (Bonnet form) γ is closed, where*

$$\gamma = J\,dh\,j = [h, dh\,j]. \tag{20}$$

The argument is precisely the one of Bonnet as shown above, (18). As before, if M is simply connected, we obtain a smooth map $g : M \to \mathfrak{g}$ with $\gamma = dg$. Sym[12] and Bobenko [S, Bb] constructed g in a different way, using the extended framing F_θ or the extended solution $\Phi_\theta = F_\theta F_0^{-1}$ of the (pluri-)harmonic map $h : M \to S$. Let $\delta = \frac{\partial}{\partial\theta}|_{\theta=0}$. This map $\tilde{g} := \delta\Phi : M \to \mathfrak{g}$ will be called *Sym map.*

Theorem 7.2. *"The Sym map integrates the Bonnet form". More precisely, let $h : M \to S$ be a pluriharmonic map with Bonnet form $\gamma : TM \to \mathfrak{g}$ and Sym map $\tilde{g} = \delta\Phi_\theta : M \to \mathfrak{g}$. Then*

$$\gamma = d\tilde{g} \tag{21}$$

Proof. Let h_θ be the associated family of h. The group element $\Phi_\theta(x)$, acting on S as $\mathrm{Ad}(\Phi_\theta(x))$, maps $h(x)$ onto $h_\theta(x)$ and $dh_x\mathsf{R}_\theta$ onto $(dh_\theta)_x$, see (4), where $\mathsf{R}_\theta = \cos\theta\, I + \sin\theta\, j$:

$$dh_\theta = \mathrm{Ad}(\Phi_\theta) dh\, \mathsf{R}_\theta, \tag{22}$$
$$h_\theta = \mathrm{Ad}(\Phi_\theta) h. \tag{23}$$

Taking the differential of (23) we obtain a second relation for dh_θ:

$$dh_\theta = \mathrm{Ad}(\Phi_\theta)\left(\mathrm{ad}(\Phi_\theta^{-1} d\Phi_\theta)h + dh\right).$$

Comparing with (22) yields

$$dh(\mathsf{R}_\theta - I) = \mathrm{ad}(\Phi_\theta^{-1} d\Phi_\theta)h.$$

Taking the θ-derivative δ we obtain

$$dh\,j = \mathrm{ad}(d\tilde{g})h = -[h, d\tilde{g}] = -J\,d\tilde{g}$$

where we have used $\delta(\Phi_\theta^{-1} d\Phi_\theta) = \delta d\Phi_\theta = d\tilde{g}$ since $\Phi_0 = e$ and $d\Phi_0 = 0$. Thus $d\tilde{g} = J\,dh\,j = \gamma$. □

Starting from any pluriharmonic map $h : M \to S$, we now have constructed a map $f = g + h$ (in fact two such maps $f_\pm = g \pm h$) with remarkable properties:

[12] Sym [S] used the method first in order to construct surfaces of Gaussian curvature $K = -1$.

- The map f is a Kähler immersion at all regular points, i.e. the induced metric on M is Kähler for the given complex structure j.
- The second fundamental form α of f (at regular points) satisfies
$$\alpha(v,v) + \alpha(jv,jv) = \alpha_h^S(df.v, df.v) \tag{24}$$
for every $v \in TM$, where α_p^S denotes the second fundamental form of $S \subset \mathfrak{g}$ at the point $p \in S$ (here: $p = h(x)$); in fact
$$\alpha_p(A,B) = [J_pA, B] = [[p,A],B].$$
In the special case $S = \mathbb{S}^2$ we have $\alpha_h^S(df.v, df.v) = |df.v|^2$ which shows that f has mean curvature $2H = \text{trace } \alpha = 1$. Therefore we would like to call a Kähler immersion with (24) *pluri-CMC.*
- We have $df(T_xM) \subset T_{h(x)}S$ for all $x \in M$. Thus the second fundamental form α of f splits as $\alpha = \alpha_T + \alpha_N$ where α_T, α_N are the components of α in T_hS, N_hS, respectively.
- The map f allows an isometric one-parameter deformation f_θ, corresponding to the pluriharmonic maps h_θ. The two components of the second fundamental form α_θ of f_θ behave differently: On $T'M \otimes T'M$ we have
$$\alpha_\theta = \lambda^{-1}\alpha_T + \lambda^{-2}\alpha_N \tag{25}$$
where $\lambda = e^{-i\theta}$.

The special case where the pluriharmonic map h is isotropic was treated by P. Quast [Q].

References

[BEFT] F.E. Burstall, J.-H. Eschenburg, M.J. Ferreira, R. Tribuzy: *Kähler submanifolds with parallel pluri-minimal curvature*, Diff. Geom and Appl. 20 (2004), 47-66

[Bb] A. Bobenko: Constant mean curvature surfaces and integrable equations, *Russian Math. Surveys* 46 (1991), 1 - 45.

[Bn] P.O. Bonnet: Notes sur une propriété de maximum relative à la sphère, *Nouvelles Annales de mathématiques* XII (1853), 433 - 438

[BG] F.E. Burstall, M.A. Guest: *Harmonic two-spheres in compact symmetric spaces, revisited.* Math. Ann. 309 (1997), 541 - 572

[BP] F.E. Burstall, F. Pedit: *Harmonic maps via Adler-Kostant-Symes theory.* In: Harmonic Maps and Integrable Systems, ed. A.P. Forgy and J.C. Wood, Vieweg 1994

[BR] F.E. Burstall, J. Rawnsley: *Twistor Theory for symmetric spaces.* Lect. Notes Math. 1424, Springer 1990

[DE] J. Dorfmeister, J.-H. Eschenburg: *Pluriharmonic maps, loop groups and twistor theory*, Ann. Glob. An. Geom. 24 (2003), 301 - 321

[DPW] J. Dorfmeister, F. Pedit, and H. Wu: *Weierstrass type representations of harmonic maps into symmetric spaces*, Comm. Anal. Geom. 6 (1998), 633 - 668

[E1] J.-H. Eschenburg: *Pluriharmonic maps, twisted loops and twistors*, DMV-Jahresberichte 106 (2004), 39 - 48

[E2] J.-H. Eschenburg: *The associated family*, Matematica Contemporanea 31 (2006), 1 - 12

[EFT] J.-H. Eschenburg, M.J. Ferreira, R. Tribuzy: *Isotropic ppmc immersions*, J. Diff. Geom. Appl., 25 (2007), 351 - 355

[EKMT] J.-H. Eschenburg, B.S. Kruglikov, V.S. Matveev, R. Tribuzy: *Compatibility of Gauß maps with metrics*, submitted to J. Diff. Geom. Appl., arXiv:0903.3225

[EK] J.-H. Eschenburg, P. Kobak: *Pluriharmonic maps of maximal rank*, Math. Z. 256 (2007), 279 - 286

[EKT] J.-H. Eschenburg, A. Kollross, R. Tribuzy: *Codimension of immersions with parallel pluri-mean curvature*, to appear in J. Diff. Geom. Appl. (2009)

[EMQ] J.-H. Eschenburg, A.-L. Mare and P. Quast: *Pluriharmonic maps of finite uniton number into outer symmetric spaces*, Preprint 2008, submitted to Math. Ann.

[EQ] J.-H. Eschenburg, P. Quast: *The spectral parameter of pluriharmonic maps*, Preprint 2009, submitted to Lond. Math. Soc.

[ET1] J.-H. Eschenburg, R. Tribuzy: *Existence and uniqueness of maps into affine homogeneous spaces*, Rend. Sem. Mat. Univ. Padova 69 (1993), 11 - 18

[ET2] J.-H. Eschenburg, R. Tribuzy: *Associated families of pluriharmonic maps and isotropy*, manuscripta math. 95 (1998), 295 - 310

[ET3] J.-H. Eschenburg, R. Tribuzy: *Isotropic pluriminimal submanifolds*, Matematica Contemporanea 17 (1999), 171 - 191

[F] D. Ferus: *Symmetric Submanifolds of Euclidean Space*, Math. Ann. 247 (1980), 81 - 93

[H] F. Hélein: *Harmonic Maps, Conservation Laws and Moving Frames*, 2nd ed., Cambridge University Press, Cambridge 2002

[O] Y. Ohnita: *Harmonic maps into symmetric spaces and integrable system theory.* In: Theory of Lie Groups and Manifolds, Sophia Kokyuroku in Mathematics 45 (2002), 77 - 94

[OV] Y. Ohnita and G. Valli: *Pluriharmonic maps into compact Lie groups and factorization into unitons*, Proc. Lond. Math. Soc. 61 (1990), 546 - 570

[PS] A. Presley and G. Segal: *Loop Groups.* Oxford Science Publications, Clarendon Press, 1986

[Q] P. Quast: *Twistor fibrations over hermitian symmetric spaces and isotropic pluriharmonic maps*, J. Diff. Geom. Appl. 27 (2009), 1 - 6

[RV] E. Ruh, J. Vilms: *The tension field of the Gauss map*, Trans. Amer. Math. Soc. 149 (1970), 569 - 573

[S] A. Sym: *Soliton surfaces and their applications (Soliton geometry from spectral problems)*, in: Geometric Aspects of the Einstein Equations and Integrable Systems, *Lect. Notes Phys.* 239, 154 - 231, Springer–Verlag, Berlin 1986

[U] K. Uhlenbeck: *Harmonic maps into Lie groups (classical solutions of the chiral model)*, J. Diff. Geom 30 (1989), 1 - 50

Institut für Mathematik, Universität Augsburg, Germany
E-mail address: eschenburg@math.uni-augsburg.de