ON HOLOMORPHIC SECTIONS OF A CERTAIN KODAIRA SURFACE REVISITED

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Abstract. In [2], we analysed the number of holomorphic sections of the holomorphic family of genus two surfaces whose complex structure was originally studied by Riera [4]. Our aim was to give a precise estimation of the number of holomorphic sections, whose finiteness was already known as the Mordell conjecture in function fields case. In this note we review the present status of our results and discuss more carefully than in [2]. We give a simple proof about the order of a local monodromy around a puncture, which is crucial for our sections to be extended at the puncture.

In this paper, we review our results in [2].

We start with a definition of holomorphic families of closed Riemann surfaces. Let M be a two-dimensional complex manifold and R be a Riemann surface. We assume that a proper holomorphic mapping $\pi : M \to R$ satisfies the following two conditions:

(i) The Jacobian matrix of π has rank one at every point of M.
(ii) The fiber $S_r = \pi^{-1}(r)$ over each point r of R is a closed Riemann surface of genus g_0.

We call such a triple (M, π, R) a *holomorphic family of closed Riemann surfaces* of genus g_0 over R.

A holomorphic mapping $s : R \to M$ is said to be a *holomorphic section* of (M, π, R) if $\pi \circ s$ is the identity mapping on R.

Let $\mathcal{S}$ be the set of all holomorphic sections of (M, π, R). Denote by $\sharp\mathcal{S}$ the number of all elements of $\mathcal{S}$.

In [2], we constructed a certain Kodaira surface whose fibers are branched over a fixed flat torus. We explain briefly its construction as follows.

Take a point τ in the upper half-plane $\mathbf{H}$. Let $\Gamma_{1,\tau}$ be the discrete subgroup of $\mathrm{Aut}(\mathbf{C}_w)$ generated by $w \mapsto w+1$, $w \mapsto w+\tau$. Let $\alpha_1 : \mathbf{C}_w \to \mathbf{C}_w/\Gamma_{1,\tau}$ be the canonical projection. We denote the pair $(\mathbf{C}_w/\Gamma_{1,\tau}, \alpha_1(0))$ by $(\hat{T}, 0)$ and set $T = \hat{T} \setminus \{0\}$.

For any point $t \in T$, we cut $\hat{T}$ along a simple arc from 0 to t. Next we take two replicas of the torus $\hat{T}$ with the cut and call them sheet I and sheet II. The cut on each sheet has two sides, which are labeled + side and − side. We attach the + side of the cut on I to the − side of the cut on II, and attach the − side of the cut on I to the + side of the cut on II. Now we obtain a closed Riemann surface S_t of genus two, which is the two-sheeted branched covering surface $S_t \to \hat{T}$ branched over 0 and t.

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Note that the complex structure on S_t depends not only on the point t but also on the cut locus from 0 to t. Essentially there are four cuts as in Figure 1 which determine different complex structures on S_t. This is an obstruction to construct a holomorphic family whose fibers are S_t over T.

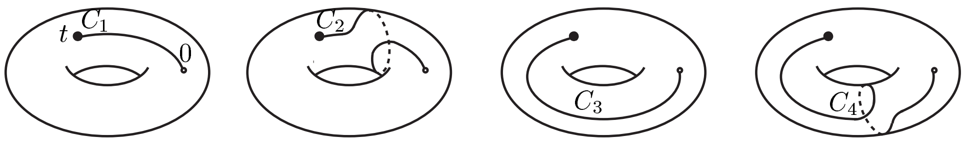


FIGURE 1. Four cuts on $\hat{T}$

To solve this problem, let $\Gamma_{2,2\tau}$ be the discrete subgroup of $\mathrm{Aut}(\mathbf{C}_z)$ generated by $z \mapsto z+2$, $z \mapsto z+2\tau$. Let $\alpha_2 : \mathbf{C}_z \to \mathbf{C}_z/\Gamma_{2,2\tau}$ be the canonical projection and denote the pair $(\mathbf{C}_z/\Gamma_{2,2\tau}, \alpha_2(0))$ by $(\hat{R}, 0)$.

Define $\tilde{\rho} : \mathbf{C}_z \to \mathbf{C}_w$ by $\tilde{\rho}(z) = z$. Then $\tilde{\rho}$ induces a $(\mathbf{Z}/2\mathbf{Z})^2$-unbranched covering $\hat{\rho} : \hat{R} \to \hat{T}$ which corresponds to

$$1 \longrightarrow \pi_1(\hat{R}) \longrightarrow \pi_1(\hat{T}) \longrightarrow (\mathbf{Z}/2\mathbf{Z})^2 \longrightarrow 1.$$

Set $R = \hat{R} \setminus \hat{\rho}^{-1}(0)$ and $\rho = \hat{\rho}|R$.

Proposition 1. *Choose an element of $\hat{\rho}^{-1}(0)$ and denote it by 0_R (we remark that there are four elements in $\hat{\rho}^{-1}(0)$). For any $r \in R$, let C_r be a simple arc from r to 0_R in R whose image $\rho(C_r)$ in T becomes a cut from 0 to $\rho(r)$. Then the complex structure of the genus two surface S_r which is obtained by means of the cut $\rho(C_r)$ of $\hat{T}$, is uniquely determined by $r \in R$, not depending on the choice of C_r.*

Proof. First of all, the two-sheeted branched covering surface $S_t \to \hat{T}$ defined by a cut from 0 to t in $\hat{T}$ corresponds to the index two subgroup of $\pi_1(\hat{T} - \{0, t\})$ with free generators a, b, ε where ε is a cycle around 0 (p300, [4]). Hence the parities of the intersection numbers of the cut with a and b respectively distinguishes the index two subgroup of $\pi_1(\hat{T} - \{0, t\})$ corresponding to the two-sheeted branched covering surface $S_t \to \hat{T}$.

Let C_r and C'_r be simple arcs from r to 0_R in R. Then the composition $C'_r \cdot C_r^{-1}$ is a closed loop from 0_R to itself in $\hat{R}$ and its image $\rho(C'_r \cdot C_r^{-1})$ is a closed loop from 0 to itself in $\hat{T}$, homologous to $ma + nb$ where m, n are even numbers. Therefore two cuts $\rho(C_r)$ and $\rho(C'_r)$ intersect with a and b with the same parity, which implies these two cuts induce the isomorphic two-sheeted branched covering surfaces $S_{\rho(r)} \to \hat{T}$. $\square$

Hence after the choice of $0_R \in \hat{\rho}^{-1}(0)$, $M = \sqcup_{r \in R} S_r \to R$ is well defined as a family of Riemann surfaces.

Following the idea of Riera in [4], we introduced a complex structure on M in [2] such that the two-sheeted branched covering $\Pi : M \to R \times \hat{T}$ defined by $(r, q) \mapsto (r, \beta_r(q))$ is a holomorphic mapping branched over two graphs Γ_0 of 0 and Γ_ρ of ρ in $R \times \hat{T}$, where

β_r is a two-branched covering from S_r to $\hat{T}$ and

$$\Gamma_0 = \{(r, 0) \mid r \in R\} \subset R \times \hat{T},$$
$$\Gamma_\rho = \{(r, \rho(r)) \mid r \in R\} \subset R \times \hat{T}.$$

We define π by the composition $P_R \circ \Pi$ of Π and the first projection $P_R : R \times \hat{T} \to R$, and we define β by $P_{\hat{T}} \circ \Pi$, where $P_{\hat{T}} : R \times \hat{T} \to \hat{T}$ is the second projection. As a consequence, we have the following diagram in Figure 2.

$$\begin{tikzcd}
& M \arrow[dl, "\pi"'] \arrow[d, "\Pi"'] \arrow[dr, "\beta"] & \\
R & R \times \hat{T} \arrow[l, "P_R"'] \arrow[r, "P_{\hat{T}}"] & \hat{T}
\end{tikzcd}$$

FIGURE 2

Thus the triple (M, π, R) is a holomorphic family of closed Riemann surfaces of genus two. For any point $r \in R$, the fiber $S_r = \pi^{-1}(r)$ is a two-sheeted branched covering surface of $\hat{T} \cong \{r\} \times \hat{T}$ in $R \times \hat{T}$ branched at $(r, 0)$ and $(r, \rho(r))$.

Here we should remark that this is a unique complex structure for $\pi : M \to R$.

Proposition 2. *There is at most one complex structure on $M = \sqcup_{r \in R} S_r$ which makes $M \to R$ a holomorphic family of genus two surfaces over R.*

Proof. Suppose that there are two complex structures on $M = \sqcup_{r \in R} S_r$. Then they induce the holomorphic mappings $f_i : \mathbf{H} \to T_2 \quad (i = 1, 2)$ from the universal covering of R to the Teichmüller space of genus two surfaces. They satisfy the equality $\pi_2 \circ f_1 = \pi_2 \circ f_2$ where $\pi_2 : T_2 \to M_2$ is the canonical projection from the Teichmüller space to the moduli space. From this equality, for any $p \in \mathbf{H}$ there exists an element γ_p of the mapping class group Mod_2 of genus two surfaces such that $f_2(p) = \gamma_p(f_1(p))$. Since Mod_2 is a countable group, there exists $\gamma \in Mod_2$ so that $U := \{p \in \mathbf{H} \mid \gamma_p = \gamma\}$ has accumulation points in $\mathbf{H}$. Therefore $f_2 = \gamma \circ f_1$ on U implies $f_2 = \gamma \circ f_1$ on $\mathbf{H}$ by the identity theorem. $\square$

Our aim is to estimate the number $\sharp\mathcal{S}$ of all holomorphic sections of (M, π, R). Every section s of $\mathcal{S}$ induces a holomorphic mapping from R to $\hat{T}$ defined by the composition $\beta \circ s$. Let $\mathrm{Hol}(R, \hat{T})$ be the set of all holomorphic mappings from R to $\hat{T}$. Thus we have the canonical mapping

$$\Phi : \mathcal{S} \to \mathrm{Hol}(R, \hat{T}).$$
$$s \mapsto \beta \circ s$$

If $\Phi(s_1) = \Phi(s_2)$ holds for $s_1, s_2 \in \mathcal{S}$, then $s_1 = j_r \circ s_2$, where $j_r : S_r \to S_r$ is the holomorphic involution with two fixed points and satisfies $\beta_r \circ j_r = \beta_r$. Thus Φ is 2 to 1 except for the trivial sections s_0 and s_ρ defined by $s_0(r) = (r, 0)$ and $s_\rho(r) = (r, \rho(r))$, respectively. So that we have the equality

$$\sharp\mathcal{S} = 2\sharp\Phi(\mathcal{S}) - 2. \tag{1}$$

Now the estimation of $\sharp\mathcal{S}$ reduces to that of $\sharp\Phi(\mathcal{S})$.

Next we consider the set of all non-constant holomorphic mappings from R to $\hat{T}$ and denote it by $\mathrm{Hol}_{\mathrm{n.c.}}(R,\hat{T})$. Then we showed that (Proposition 3.1 in [2])

$$\Phi(\mathcal{S}) \setminus \{0\} \subset \mathrm{Hol}_{\mathrm{n.c.}}(R,\hat{T}).$$

The key idea of our proof is that if $g = \beta \circ s \in \Phi(\mathcal{S})$ is a non-zero constant mapping then the graph Γ_g of g and Γ_ρ must intersect transversely. Then around an intersection point of Γ_g and Γ_ρ, the section s is not univalent, a contradiction. But $\sharp\mathrm{Hol}_{\mathrm{n.c.}}(R,\hat{T})$ is still infinite, our next purpose is to find a finite subset of $\mathrm{Hol}_{\mathrm{n.c.}}(R,\hat{T})$ which contains $\Phi(\mathcal{S})$.

For this purpose, we studied the possibility for $g = \beta \circ s : R \to \hat{T}$ to extend to a holomorphic mapping $\hat{g} : \hat{R} \to \hat{T}$. Since $\hat{g}$ becomes a covering mapping by means of Riemann-Hurwitz formula, the induced mapping $\tilde{g} : \mathbf{C}_z \to \mathbf{C}_w$ is an affine mapping, which is easy to be handled. The proof consists of two steps (Theorem 3.1 in [2]):

The first step is to show that the section $s : R \to M$ can be extended to a holomorphic section $\hat{s} : \hat{R} \to \hat{M}$ where $\hat{M} \to \hat{R}$ is the degenerate family which is a completion of the holomorphic family $M \to R$. To do this, we need to show that the local monodromy around a puncture of R is of infinite order to apply the first author's result (Theorem 3.2 in [2]). In the next section, we will give a simple proof of infinite order of the local monodromy due to Hideki Miyachi.

The second step is to show that $g : R \to \hat{T}$ can be extended to a holomorphic mapping $\hat{g} : \hat{R} \to \hat{T}$. From the result of the first step, we can prove that $g^{-1}(0)$ is a finite set. Then $g : R \setminus g^{-1}(0) \to \hat{T} \setminus \{0\}$ is a holomorphic mapping between hyperbolic Riemann surfaces, hence g can be extended to $\hat{g} : \hat{R} \to \hat{T}$ by Royden's theorem (Theorem 3.3 in [2]).

Now we know that $\hat{g} : \hat{R} \to \hat{T}$ is a covering mapping whose lift $\tilde{g} : \mathbf{C}_z \to \mathbf{C}_w$ of $\hat{g}$ is an affine mapping. Then we could prove that the graph Γ_g intersects with neither Γ_ρ nor Γ_0 in $R \times \hat{T}$ (Proposition 3.2 in [2]). Hence we showed

$$\Phi(\mathcal{S}) \subset \mathrm{Hol}_{\mathrm{dis}}(R,\hat{T}) \cup \{0,\rho\},$$

where $\mathrm{Hol}_{\mathrm{dis}}(R,\hat{T})$ is the set of all holomorphic mappings $g : R \to \hat{T}$ which extend to the mappings $\hat{g} : \hat{R} \to \hat{T}$ and satisfy $\Gamma_g \cap \Gamma_0 = \emptyset$ and $\Gamma_g \cap \Gamma_\rho = \emptyset$ in $R \times \hat{T}$. Fortunately $\mathrm{Hol}_{\mathrm{dis}}(R,\hat{T})$ becomes a finite set and we had the main inequality:

$$2 = \sharp\{s_0, s_\rho\} \le \sharp\Phi(\mathcal{S}) \le \sharp\mathrm{Hol}_{\mathrm{dis}}(R,\hat{T}) + 2. \tag{2}$$

We determine $\mathrm{Hol}_{\mathrm{dis}}(R,\hat{T})$ explicitly. In practice we count the lift $\tilde{g}$ of $\hat{g}$. In the last section we will explain the algorithm to count all elements of $\mathrm{Hol}_{\mathrm{dis}}(R,\hat{T})$.

Here we restate our main results of [2]. First we fix a fundamental domain of $SL(2,\mathbf{Z})$ acting on $\mathbf{H}$ as the domain F of $\mathbf{H}$ defined by the following conditions : (i) $-1/2 \le \mathrm{Re}(\tau) < 1/2$, (ii) $|\tau| \ge 1$, (iii) $\mathrm{Re}(\tau) \le 0$ if $|\tau| = 1$. Any flat torus is biholomorphically equivalent to $\mathbf{C}/\Gamma_{1,\tau}$ for some $\tau \in F$. We also set $\tau_1 = i$ and $\tau_2 = e^{2\pi i/3}$ and put $\hat{T}_j = \mathbf{C}_z/\Gamma_{1,\tau_j}, j = 1, 2$. Our result on $\sharp\mathrm{Hol}_{\mathrm{dis}}(R,\hat{T})$ is

Theorem 1.

(i) *If $\hat{T}$ is not biholomorphically equivalent to either $\hat{T}_1$ or $\hat{T}_2$, then* $\sharp \mathrm{Hol}_{\mathrm{dis}}(R, \hat{T}) = 4$.
(ii) *If $\hat{T}$ is biholomorphically equivalent to either $\hat{T}_1$ or $\hat{T}_2$, then* $\sharp \mathrm{Hol}_{\mathrm{dis}}(R, \hat{T}) = 12$.

Consequently the number $\sharp \mathcal{S}$ can be estimated as follows:

Theorem 2.
(i) *If $\hat{T}$ is not biholomorphically equivalent to either $\hat{T}_1$ or $\hat{T}_2$, then $\sharp \mathcal{S} = 2, 4, \ldots, 8$, or 10.*
(ii) *If hat$T$ is biholomorphically equivalent to either $\hat{T}_1$ or $\hat{T}_2$, then $\sharp \mathcal{S} = 2, 4, \ldots, 24$, or 26.*

1. LOCAL MONODROMY

As explained in the previous section, the next claim is crucial for $g = \beta \circ s : R \to \hat{T}$ to be able to extend to a holomorphic mapping $\hat{g} : \hat{R} \to \hat{T}$:

Proposition 3 (p13, Claim 1 in [2]). *At any puncture $p \in \hat{\rho}^{-1}(0)$ of R, the local monodromy M_p around p is of infinite order.*

We believe that our proof in [2] should be true, but we didn't give a proof that the M_p is the twice product of a Dehn twist. Here we show a simple and clear proof for this claim: The idea is due to Hideki Miyachi.

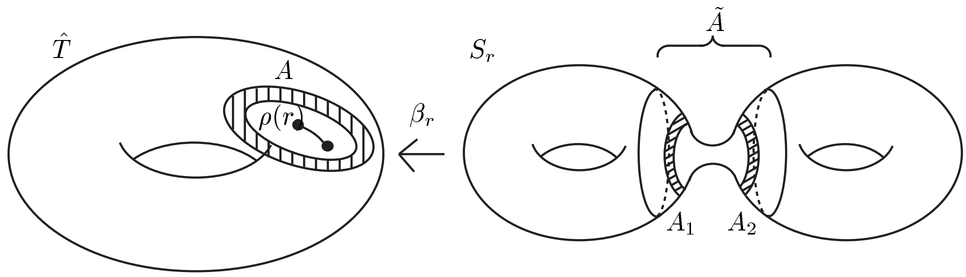
Proof. We assume that $p = 0_R$; the same argument works for the rest cases. Choose $r \in R$ closed to 0_R. Then we can take a cut C from 0 to $\rho(r)$ in T as a short segment (with respect to the Euclidean metric on $\hat{T}$). Draw an annulus A on $\hat{T}$ such that the bounded component of $\hat{T} \setminus A$ contains the cut C. Since C is short, our annulus A may have a big modulus $m(A)$. Consider the branched double covering $\beta_r : S_r \to \hat{T}$ and the inverse image of A under β_r. It consists of disjoint two annuli A_1 and A_2 of S_r homotopic to each other and their moduli satisfy

$$m(A_1) = m(A_2) = m(A),$$

since the modulus is a conformal invariant.

Let $\tilde{A}$ be an annulus of S_r homotopic to A_1 and A_2 and contains them. Then the modulus inequality tells us that

$$m(A_1) + m(A_2) \leq m(\tilde{A}).$$



Therefore if r converses to 0_R, we may assume that $m(A) \to \infty$, which implies $m(\tilde{A}) \to \infty$, so that the extremal length of $\tilde{A}$ goes to 0. Now Maskit's comparison theorem (p.383, Corollary 2 in [3]) guarantees that the hyperbolic length of the simple closed geodesic of S_r homotopic to $\tilde{A}$ also goes to 0, which implies that S_r diverges to the boundary of the Teichmüller space when $r \to 0_R$. Hence M_p must be of infinite order; otherwise S_r converges to the point in the Teichmüller space when $r \to 0_R$ (p.285, Theorem 2 in [1]). □

2. $\mathrm{Hol}_{\mathrm{dis}}(R, \hat{T})$

In this section we will determine all elements of $\mathrm{Hol}_{\mathrm{dis}}(R, \hat{T})$. We recall that $\mathrm{Hol}_{\mathrm{dis}}(R, \hat{T})$ is the set of all holomorphic mappings $g : R \to \hat{T}$ which extend to the mappings $\hat{g} : \hat{R} \to \hat{T}$ and satisfy $\Gamma_g \cap \Gamma_0 = \emptyset$ and $\Gamma_g \cap \Gamma_\rho = \emptyset$ in $R \times \hat{T}$. From the definition of $\mathrm{Hol}_{\mathrm{dis}}(R, \hat{T})$, every element g of $\mathrm{Hol}_{\mathrm{dis}}(R, \hat{T})$ has a holomorphic extension $\hat{g} : \hat{R} \to \hat{T}$ which is a covering mapping of degree less than or equal to 4 since $\sharp \hat{\rho}^{-1}(0) = 4$. Hence a lift $\tilde{g} : \mathbf{C}_z \to \mathbf{C}_w$ of $\hat{g}$ is an affine mapping

$$\tilde{g}(z) = Az + B,$$

where A is uniquely determined by g, while B is determined modulo $A \cdot \Gamma_{2,2\tau} + \Gamma_{1,\tau}$.

Now we may assume that $A \neq 1$ (Lemma 3.1 in [2]). Moreover $\tilde{g}$ can be written as $\tilde{g}(z) = A(z + \omega)$ where $\omega = 0, 1, \tau$ and $1 + \tau$ (Lemma 3.2 in [2]).

Next we determine A. To do this, we may assume $\tilde{g}(z) = Az$. Since $\tilde{g}(\Gamma_{2,2\tau}) \subset \Gamma_{1,\tau}$, we have

$$2A = p + q\tau, \tag{3}$$

$$2A\tau = u + v\tau, \tag{4}$$

for some $p, q, u, v \in \mathbf{Z}$. The Euclidean areas of $\hat{R}$ and $\hat{T}$, and $deg(\hat{g}) \leq 4$ implies that

$$1 \leq pv - qu \leq 4 \tag{5}$$

and

$$|2A| = |p + q\tau| \leq 2. \tag{6}$$

By (3) and (4), we get the quadratic equation of τ:

$$q\tau^2 + (p - v)\tau - u = 0. \tag{7}$$

Since $\mathrm{Im}(\tau) > 0$ implies that the discriminant is negative, we have

$$(p + v)^2 < 4(pv - qu). \tag{8}$$

The root τ of (7) with $\mathrm{Im}(\tau) > 0$ is given by

$$\tau = \begin{cases} \dfrac{v - p + \sqrt{4(pv - qu) - (p + v)^2}\ i}{2q}, & \text{if } q > 0, \\ \dfrac{v - p - \sqrt{4(pv - qu) - (p + v)^2}\ i}{2q}, & \text{if } q < 0. \end{cases} \tag{9}$$

By (5) and (8), we have

$$|p + v| < 4 \tag{10}$$

To determine A, we will find four integers p, q, u and v which satisfy the conditions (5), (6) and (10).

Lemma 1. *$q = 0, \pm 1$, or ± 2.*

Proof. Suppose $|q| \geq 3$. We take the point $\tau_0 = e^{2\pi i/3}$ in F whose imaginary part is the least in F. Then the imaginary part of the number $3\tau_0$ satisfies

$$\mathrm{Im}(3\tau_0) = \frac{3\sqrt{3}}{2} > 2.$$

Hence for any integers p and q, we have

$$\begin{aligned} |p + q\tau| &\geq |\mathrm{Im}(q\tau)| \\ &\geq \mathrm{Im}(3\tau) \\ &\geq \mathrm{Im}(3\tau_0) \\ &> 2 \end{aligned}$$

This contradicts the condition (6) □

Hence we may consider the cases $|q| \leq 2$.

Lemma 2. (i) *If $q = 0$, then $p = \pm 1, \pm 2$.*
(ii) *If $q = \pm 1$, then $|p| \leq 2$.*
(iii) *If $q = \pm 2$, then $|p| \leq 2$.*

Proof. If q is equal to 0, then the relation $0 < |p + q\tau| \leq 2$ shows (i). By the same argument as in the proof of Lemma 1, we have (ii) and (iii). □

By (5) and (10), both u and v are determined. Hence we have four integers p, q, u and v such that they satisfy the conditions (5), (6) and (10).

Next, finding quadruplets (p, q, u, v) in these p, q, u, v such that τ represented in (9) is an element of F, we have the complete list of such a p, q, u, v, τ and the fixed point of $\tilde{g}$, in the following Table 1 and 2.

In these Tables, when some lift $\tilde{g}$ has a fixed point which is not contained in $\Gamma_{1,\tau}$, we see that Γ_g intersects Γ_ρ, a contradiction.

Next when $(p, q, u, v) = (1, -1, 1, 2), (1, -1, 2, 2), (2, 1, -1, 1), (2, 1, -2, 1)$, we see that Γ_g intersects Γ_0, a contradiction. Finally when $(p, q, u, v) = (2, 0, 0, 2)$, $\tilde{g}$ is a lift of ρ, a contradiction.

Hence for every $\tau \in F$, we see that $(p, q, u, v) = (1, 0, 0, 1)$ satisfies (5), (6) and (10). In this case A is equal to $1/2$, and $\tilde{g}(z)$ can be written as

$$\tilde{g}(z) = \frac{1}{2}(z + \omega),$$

where $\omega = 0, 1, \tau$ and $1 + \tau$. Thus any $\tau \in F$ with $\tau \neq i$ and $\tau \neq e^{2\pi i/3}$, we have

$$\sharp\mathrm{Hol}_{\mathrm{dis}}(R, \hat{T}) = 4.$$

Moreover, if τ is equal to i or $e^{2\pi/3}$, we have other choices of $\tilde{g}(z)$ as

$$\tilde{g}(z) = \frac{1-i}{2}(z + \omega),\ \frac{1+i}{2}(z + \omega),$$

or

$$\tilde{g}(z) = \frac{1-\sqrt{3}i}{2}(z+\omega),\ \frac{1+\sqrt{3}i}{2}(z+\omega).$$

Hence

$$\sharp \mathrm{Hol}_{\mathrm{dis}}(R, \hat{T}) = 3 \times 4 = 12.$$

Thus we have proved Theorem 1.

The authors wish to thank Professor Yoshihiro Ohnita for his encouragement. Many thanks are also due to Professor Hideki Miyachi for his valuable suggestions about the local monodromy. Finally the authors would like to express their gratitude to referee for valuable suggestions on the improvement of the paper.

3. APPENDIX

In the referee report, the referee suggested us an algebraic-geometric construction of the holomorphic family $\pi\ M \to R$ appeared in this paper, and he (or she) kindly permitted us to show his (or her) construction in the appendix, hence the following idea is basically due to the referee.

Let $g_\omega : \hat{R} \to \hat{T}$ be the map induced by the affine map between the universal coverings of $\hat{R}$ and $\hat{T}$ defined by $w = \frac{1}{2}(z+\omega)$ where $\omega = 0, 1, \tau$ and $1+\tau$. We consider the product of Riemann surfaces $\hat{R} \times \hat{T}$ and denote the graphs of maps $0, \rho$ and g_ω $(\omega = 0, 1, \tau, 1+\tau)$ from $\hat{R}$ to $\hat{T}$ by Γ_0, Γ_ρ and Γ_{g_ω} $(\omega = 0, 1, \tau, 1+\tau)$ respectively. We remark that Γ_0 and Γ_ρ intersect transversely at four points $p_\omega := ([\omega], [0]) \in \hat{R} \times \hat{T}$ $(\omega = 0, 1, \tau, 1+\tau)$. Define $F_\omega := P_{\hat{R}}^{-1}([\omega])$ where $P_{\hat{R}}$ is the projection map form $\hat{R} \times \hat{T}$ to $\hat{R}$. Then divisors $\Gamma_0 + \Gamma_\rho$ is linearly equivalent to $2\Gamma_{g_\omega} + 2F_\omega$

$$\Gamma_0 + \Gamma_\rho \sim 2\Gamma_{g_\omega} + 2F_\omega,$$

because the meromorphic function $f_\omega(z, w)$ on $\hat{R} \times \hat{T}$ defined by

$$f_\omega(z, w) := \mathcal{P}\left(w - \frac{1}{2}(z+\omega)\right) - \mathcal{P}\left(\frac{1}{2}(z+\omega)\right) \quad (\omega = 0, 1, \tau, 1+\tau)$$

has simple zeros at $\Gamma_0 + \Gamma_\rho$ and double poles at $\Gamma_{g_\omega} + F_\omega$ respectively, where $\mathcal{P}(z)$ is the Weierstrass $\mathcal{P}$-function:

$$\mathcal{P}(z) = \frac{1}{z^2} + \sum_{\omega \in \mathbf{Z} + \mathbf{Z}\tau - \{\mathbf{0}\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right).$$

Let $\sigma : \Sigma \to \hat{R} \times \hat{T}$ be the blow-up of $\hat{R} \times \hat{T}$ at four points p_ω and $E_\omega := \sigma^{-1}(p_\omega)$ be exceptional divisors $(\omega = 0, 1, \tau, 1+\tau)$. We also denote the proper transforms of divisors $\Gamma_0, \Gamma_\rho, \Gamma_{g_\omega}$ and F_ω by $\tilde{\Gamma}_0, \tilde{\Gamma}_\rho, \tilde{\Gamma}_{g_\omega}$ and $\tilde{F}_\omega$ respectively. Now $\tilde{\Gamma}_0$ and $\tilde{\Gamma}_\rho$ are disjoint on Σ. Considering the meromorphic function $f_0 \circ \sigma$ on Σ for simplicity, we have

$$\tilde{\Gamma}_0 + \tilde{\Gamma}_\rho \sim 2(\tilde{\Gamma}_{g_0} + \tilde{F}_0 + E_0 - \sum_{\omega \neq 0} E_\omega).$$

p	q	u	v	τ	fixed point
0	1	-1	0	i	$(4+2i)/5$
0	1	-2	0	$\sqrt{2}i$	$(2+\sqrt{2}i)/3$
0	1	-3	0	$\sqrt{3}i$	$(2+\sqrt{3}i)/7$
0	1	-4	0	$2i$	$(1+i)/2$
0	1	-1	-1	$e^{2\pi i/3}$	$(5+\sqrt{3}i)/7$
0	1	-2	-1	$(-1+\sqrt{7}i)/2$	$(5+\sqrt{7}i)/8$
0	1	-3	-1	$(-1+\sqrt{11}i)/2$	$(5+\sqrt{11}i)/9$
0	1	-4	-1	$(-1+\sqrt{15}i)/2$	$(5+\sqrt{15}i)/10$
0	-1	1	0	i	$2(1+2i)/5$
0	-1	2	0	$\sqrt{2}i$	$2(1+\sqrt{2}i)/3$
0	-1	3	0	$\sqrt{3}i$	$2(3+2\sqrt{3}i)/7$
0	-1	4	0	$2i$	$(1+\sqrt{3}i)/2$
0	-1	1	1	$e^{2\pi i/3}$	$(3-\sqrt{3}i)/3$
0	-1	2	1	$(-1+\sqrt{7}i)/2$	$(5+\sqrt{7}i)/4$
0	-1	3	1	$(-1+\sqrt{11}i)/2$	$(3-\sqrt{11}i)/5$
0	-1	4	1	$(-1+\sqrt{15}i)/2$	$(3-\sqrt{15}i)/6$
0	2	-2	0	i	$(1+i)/2$
0	2	-2	-1	$(-1+\sqrt{15}i)/4$	$(5+\sqrt{15}i)/10$
0	2	-2	-2	$e^{2\pi i/3}$	$\sqrt{3}i/3$
0	-2	2	0	i	$(1+i)/2$
0	-2	2	1	$(-1+\sqrt{15}i)/4$	$(3-\sqrt{15}i)/6$
0	-2	2	2	$e^{2\pi i/3}$	lattice point
1	0	0	1	*any*	lattice point
1	1	-1	0	$e^{2\pi i/3}$	$(3+\sqrt{3}i)/3$
1	1	-2	0	$(-1+\sqrt{7}i)/2$	$(3+\sqrt{7}i)/4$
1	1	-3	0	$(-1+\sqrt{11}i)/2$	$(5+\sqrt{11}i)/5$
1	1	-4	0	$(-1+\sqrt{15}i)/2$	$(3+\sqrt{15}i)/6$
1	1	-1	1	i	lattice point
1	1	-2	1	$\sqrt{2}i$	$(1+\sqrt{2}i)/3$
1	1	-3	1	$\sqrt{3}i$	$(1+\sqrt{3}i)/2$
1	-1	1	1	i	lattice point
1	-1	2	1	$\sqrt{2}i$	$2(1-\sqrt{2}i)/3$
1	-1	3	1	$\sqrt{3}i$	$(1-\sqrt{3}i)/2$
1	-1	1	2	$e^{2\pi i/3}$	lattice point
1	-1	2	2	$(-1+\sqrt{7}i)/2$	lattice point
1	2	-2	-1	$e^{2\pi i/3}$	$2(2+\sqrt{3}i)/7$
1	2	-2	0	$(-1+\sqrt{15}i)/4$	$(3+\sqrt{15}i)/6$

TABLE 1. $p = 0, 1$

p	q	u	v	τ	fixed point
-1	0	0	-1	any	$2(1+\tau)/3$
-1	1	-1	-2	$e^{2\pi i/3}$	$(7+\sqrt{3}i)/13$
-1	1	-2	-2	$(-1+\sqrt{7}i)/2$	$(7+\sqrt{7}i)/14$
-1	1	-1	-1	i	$(3+i)/5$
-1	1	-2	-1	$\sqrt{2}i$	$2(3+\sqrt{2}i)/11$
-1	1	-3	-1	$\sqrt{3}i$	$(3+\sqrt{3}i)/6$
-1	-1	1	-1	i	$2(2+i)/5$
-1	-1	2	-1	$\sqrt{2}i$	$2(2+3\sqrt{2}i)/11$
-1	-1	3	-1	$\sqrt{3}i$	$(1+\sqrt{3}i)/2$
-1	-1	1	0	$e^{2\pi i/3}$	$(5-\sqrt{3}i)/7$
-1	-1	2	0	$(-1+\sqrt{7}i)/2$	$(5-\sqrt{7}i)/8$
-1	-1	3	0	$(-1+\sqrt{11}i)/2$	$(5-\sqrt{11}i)/9$
-1	-1	4	0	$(-1+\sqrt{15}i)/2$	$(5-\sqrt{15}i)/10$
-1	-2	2	0	$(-1+\sqrt{15}i)/4$	$(5-\sqrt{15}i)/10$
-1	-2	2	1	$e^{2\pi i/3}$	$2(2-\sqrt{3}i)/7$
2	0	0	2	any	lattice point
2	1	-1	1	$e^{2\pi i/3}$	lattice point
2	1	-2	1	$(-1+\sqrt{7}i)/2$	lattice point
2	2	-2	0	$e^{2\pi i/3}$	lattice point
-2	0	0	-2	any	$1/2$
-2	-1	1	-1	$e^{2\pi i/3}$	$(7-\sqrt{3}i)/13$
-2	-1	2	-1	$(-1+\sqrt{7}i)/2$	$(7-\sqrt{7}i)/14$
-2	-2	2	0	$e^{2\pi i/3}$	$(3+\sqrt{3}i)/6$

TABLE 2. $p=-1,\pm 2$

Hence the line bundle $\mathcal{O}(\tilde{\Gamma}_{g_0}+\tilde{F}_0+E_0-\sum_{\omega\neq 0}E_\omega)$ is a square root of the line bundle $\mathcal{O}(\tilde{\Gamma}_0+\tilde{\Gamma}_\rho)$:

$$\mathcal{O}(\tilde{\Gamma}_{g_0}+\tilde{F}_0+E_0-\sum_{\omega\neq 0}E_\omega)\to\mathcal{O}(\tilde{\Gamma}_{g_0}+\tilde{F}_0+E_0-\sum_{\omega\neq 0}E_\omega)^{\otimes 2}\cong\mathcal{O}(\tilde{\Gamma}_0+\tilde{\Gamma}_\rho).$$

We denote this map by $\chi:\mathcal{O}(\tilde{\Gamma}_{g_0}+\tilde{F}_0+E_0-\sum_{\omega\neq 0}E_\omega)\to\mathcal{O}(\tilde{\Gamma}_0+\tilde{\Gamma}_\rho)$, and the bundle map from $\mathcal{O}(\tilde{\Gamma}_0+\tilde{\Gamma}_\rho)$ to Σ by $q:\mathcal{O}(\tilde{\Gamma}_0+\tilde{\Gamma}_\rho)\to\Sigma$. By means of $f_0\circ\sigma$, we have a holomorphic section $s:\Sigma\to\mathcal{O}(\tilde{\Gamma}_0+\tilde{\Gamma}_\rho)$ whose zero locus is $\tilde{\Gamma}_0+\tilde{\Gamma}_\rho$. Then $\tilde{M}:=\chi^{-1}(s(\Sigma))$ is a complex surface in $\mathcal{O}(\tilde{\Gamma}_{g_0}+\tilde{F}_0+E_0-\sum_{\omega\neq 0}E_\omega)$ and $q\circ\chi:\ \tilde{M}\to\Sigma$ is a double branched covering of Σ branched at $\tilde{\Gamma}_0+\tilde{\Gamma}_\rho$. Finally

$$\hat{\pi}:=P_{\hat{R}}\circ\sigma\circ q\circ\chi:\ \tilde{M}\to\hat{R}$$

is a degenerate family of Riemann surfaces of genus two over $\hat{R}$.

$$\begin{tikzcd}
 & \mathcal{O}(\tilde{\Gamma}_{g_0} + \tilde{F}_0 + E_0 - \sum_{\omega \neq 0} E_\omega) \arrow[d, "\chi"'] & \supset \tilde{M} \arrow[d] \\
 & \mathcal{O}(\tilde{\Gamma}_0 + \tilde{\Gamma}_\rho) \arrow[d, "q"'] & \supset s(\Sigma) \\
\hat{R} \times \hat{T} \arrow[d, "P_{\hat{R}}"'] & \Sigma \arrow[l, "\sigma"'] \arrow[ur] & \\
\hat{R} & &
\end{tikzcd}$$

In practice except $[\omega]$ $(\omega = 0, 1, \tau, 1 + \tau)$, $\hat{\pi}^{-1}([z])$ is a genus two surface which is a double branched covering of $\hat{T}$ branched at $[0]$ and $\rho([z])$. We can also show that $\hat{\pi}^{-1}([\omega])$ $(\omega = 0, 1, \tau, 1 + \tau)$ are singular fibers; $\hat{\pi}^{-1}([0])$ consists of two disjoint tori and one rational curve which intersects two tori transversely, while $\hat{\pi}^{-1}([\omega])$ $(\omega = 1, \tau, 1 + \tau)$ consists of one torus and one rational curve which intersect each other at two points transversely (See Figure 3). Restricting our degenerate family $\hat{\pi}$ $\tilde{M} \to \hat{R}$ to R, we have a holomorphic family π $M \to R$ appeared in this paper.

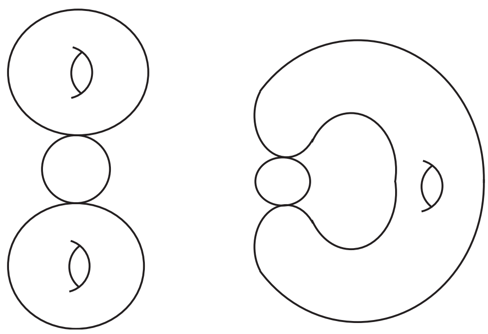


FIGURE 3

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