Blow up points and the Morse indices of solutions to the Liouville equation in two-dimension

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Abstract. We consider the Liouville equation

$$-\Delta u = \lambda e^u \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega,$$

on a smooth bounded domain Ω in $\mathbb{R}^2$, where $\lambda > 0$ is a parameter. Let $\{u_n\}$ be an *m-point blowing up solution* sequence of the problem for $\lambda = \lambda_n \downarrow 0$, which satisfies

$$\lambda_n \int_\Omega e^{u_n} dx \to 8\pi m \quad \text{as } n \to \infty$$

for $m \in \mathbb{N}$. We prove that the number of blow up points m is less than or equal to the Morse index of u_n for n sufficiently large.

As a corollary, we show that if a solution u_n of Morse index one has the property that $\lambda_n \int_\Omega e^{u_n} dx = O(1)$, then the number of blow up points of the sequence is exactly one. Note that in the last result, we do not need any geometrical assumption such as the convexity of the domain.

Keywords: Morse index, blow up point, Liouville equation.

2010 Mathematics Subject Classifications: 35B40, 35J25.

1. Introduction and Results.

In this note we consider the problem

$$\begin{cases} -\Delta u = \lambda e^u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^2$ and $\lambda > 0$. The maximum principle implies that any solution is positive in Ω.

This kind of problem with exponential nonlinearity appears in many fields of mathematics, such as the study of the prescribed Gauss curvature equation ([4] [5]), Chern-Simons gauge theories ([16] [3]), the vortex theory for the turbulent Euler flow ([2]), and so on, and it has attracted many authors for more than decades.

Let $\{\lambda_n\}$ be a sequence of positive numbers satisfying $\lambda_n \to 0$ as $n \to \infty$. We are interested in a solution sequence $\{u_n\}$ for $\lambda = \lambda_n$, which satisfies

$$\limsup_{n\to\infty} \lambda_n \int_\Omega e^{u_n} dx \leq C \qquad (1.2)$$

for some $C < +\infty$. For the solution sequence $\{u_n\}$ satisfying (1.2), Nagasaki and Suzuki [15] showed that there exists a subsequence (which will be denoted by $\{u_n\}$ again) such that

$$\lambda_n \int_\Omega e^{u_n} dx \to 8\pi m \quad (n \to \infty) \quad \text{for some } m \in \mathbb{N} \cup \{0\}. \qquad (1.3)$$

Furthermore, if $m = 0$, then $u_n \to 0$ uniformly on Ω, and if $m \in \mathbb{N}$, then there exists a set of m-points $\mathcal{S} = \{a_1, \cdots, a_m\}$ such that each a_i is an interior point of Ω, $\|u_n\|_{L^\infty(K)} = O(1)$ for any compact set $K \subset \overline{\Omega} \setminus \mathcal{S}$, $u_n|_{\mathcal{S}} \to +\infty$, and

$$u_n \to 8\pi \sum_{i=1}^m G(\cdot, a_i) \quad \text{in } C^2_{loc}(\overline{\Omega} \setminus \mathcal{S})$$

as $n \to \infty$. Here, $G = G(x, y)$ is the Green function of $-\Delta$ under the Dirichlet boundary condition with a pole $y \in \Omega$. See also [1] and [14] for another proof of this fact. In the above sense, we call $\mathcal{S}$ the *blow up set* for the solution sequence $\{u_n\}$ and each $a_i \in \mathcal{S}$ a *blow up point* of $\{u_n\}$.

In the following, let $i_M(u)$ denote the Morse index of a solution u of (1.1), that is, the number of negative eigenvalues of the linearized operator $L_u = -\Delta - \lambda e^u \cdot$ acting on $H_0^1(\Omega)$.

Now, main result in this note is as follows.

Theorem 1 *Let $\{u_n\}$ be a solution sequence of (1.1) for $\lambda = \lambda_n$ satisfying (1.3). Then $m \leq i_M(u_n)$ for n sufficiently large.*

As a corollary, we obtain the following assertion.

Corollary 2 *Let $\{u_n\}$ be a solution sequence of (1.1) for $\lambda = \lambda_n$ satisfying (1.2). Assume the Morse index $i_M(u_n) = 1$ for any n large. Then we have*

$$\lambda_n \int_\Omega e^{u_n} dx \to 8\pi, \quad as \ n \to \infty,$$

that is, the number of blow up points of $\{u_n\}$ is exactly 1.

We remark here that, the number of blow up points of *any* blowing-up solution sequence to (1.1) is exactly 1 on *convex* domains. This nonexistence of multiple blow up points on convex domains holds true for a wider class of semilinear problems with blowing-up or concentration phenomena; see [10]. Here, in Corollary 2, we do not need any geometrical assumptions such as the convexity of the domain.

As for the Morse index of solutions, Gladiali and Grossi [9] proved that if $\{u_n\}$ is a solution sequence to (1.1) with $\lambda_n \int_\Omega e^{u_n} dx \to 8\pi$, then $i_M(u_n)$ is 1 or 2 on any smooth bounded domain Ω. If Ω is convex, then $i_M(u_n) = 1$ for n sufficiently large: see Corollary 2.8 in [9].

Related results can be seen in the papers by El Mehdi and Pacella [7] for higher-dimensional cases and [17] for another problem in two dimension.

2. Proof of Theorem 1.

In this section, we prove Theorem 1. Let u_n be a solution to (1.1) for $\lambda = \lambda_n$ with the assumption (1.3). If $m = 0$, we have nothing to prove.

In the following, we treat the case $m \in \mathbb{N}$. By the result of [15], we have the blow up set $\mathcal{S} = \{a_1, \cdots, a_m\}$, each a_i an interior point of Ω. Then we have a sufficiently small $\rho > 0$ and m sequences of points $\{x_n^i\}$ such that for each $a_i \in \mathcal{S}$,

$$u_n(x_n^i) = \max_{B_\rho(x_n^i)} u_n(x) \to \infty, \quad x_n^i \to a_i \ (i = 1, \cdots, m),$$

as $n \to \infty$. See, for example, Li-Shafrir [12].

Let x_n^i be the above local maximum point of u_n around $a_i \in \mathcal{S}$. Define the positive number δ_n^i and the scaled function $\tilde{u}_n^i$ by

$$(\delta_n^i)^2 \lambda_n e^{u_n(x_n^i)} = 1, \tag{2.1}$$

$$\tilde{u}_n^i(y) = u_n(\delta_n^i y + x_n^i) - u_n(x_n^i), \quad y \in B_{\rho/\delta_n^i}(0) \tag{2.2}$$

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for $i \in \{1, \cdots, m\}$.

Now we recall the $\sup + \inf$ *type estimate* for the blowing-up solutions to (1.1): For a fixed $\rho \in (0,1)$, there exists a constant $C > 0$ independent of $i = 1, \cdots, m$ and $\lambda_n > 0$ such that

$$\left| u_n(x) - \log \frac{e^{u_n(x_n^i)}}{\left(1 + \frac{\lambda_n}{8} e^{u_n(x_n^i)} |x - x_n^i|^2\right)^2} \right| \leq C \quad \text{for } x \in B_\rho(x_n^i) \qquad (2.3)$$

holds true. See YanYan Li [11], and also [13] for an alternative proof.

We claim that $\delta_n^i = o(1)$ for any $i = \{1, \cdots, m\}$ as $n \to \infty$. Indeed, since u_n is uniformly bounded for $x \in \partial B_R(x_n^i)$ for small $R > 0$, we have

$$\left| \log \frac{e^{u_n(x_n^i)}}{\left(1 + \frac{\lambda_n}{8} e^{u_n(x_n^i)} R^2\right)^2} \right| = O(1)$$

by the $\sup + \inf$ estimate (2.3). Since

$$\frac{e^{u_n(x_n^i)}}{\left(1 + \frac{\lambda_n}{8} e^{u_n(x_n^i)} R^2\right)^2} = \frac{1}{\left(e^{-\frac{1}{2} u_n(x_n^i)} + \frac{\lambda_n^{1/2}}{8} (\delta_n^i)^{-1} R^2\right)^2}$$

by the definition (2.1), this implies there exist constants $c, C > 0$ such that $c \leq \lambda_n^{1/2} (\delta_n^i)^{-1} \leq C$. Thus we have the claim.

Note that $\tilde{u}_n^i$ satisfies

$$\begin{cases} -\Delta \tilde{u}_n^i = e^{\tilde{u}_n^i} & \text{in } B_{\rho/\delta_n^i}(0), \\ \tilde{u}_n^i(0) = 0, \ \tilde{u}_n^i(x) \leq 0, & \forall x \in B_{\rho/\delta_n^i}(0), \\ \int_{B_{\rho/\delta_n^i}(0)} e^{\tilde{u}_n^i} dy = O(1), & (n \to \infty). \end{cases} \qquad (2.4)$$

Here, we have used the assumption (1.3).

Next, we claim that

$$\tilde{u}_n^i \to U(y) = -2 \log \left(1 + \frac{|y|^2}{8}\right)$$

in $C_{loc}^1(\mathbb{R}^2)$ as $n \to \infty$ (without choosing a subsequence).

In fact, let $\{\tilde{u}_{n(k)}^i\}$, denoted by $\{\tilde{u}_k^i\}$, be any subsequence of $\{\tilde{u}_n^i\}$. A result of Brezis-Merle ([1]:Theorem 3) implies that there exists a subsequence of $\{\tilde{u}_k^i\}$, still denoted by the same symbol, satisfying one of the following alternatives as $k \to \infty$:

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(1) $\tilde{u}_k^i$ is bounded in $L_{loc}^\infty(B_{\rho/\delta_k^i}(0))$, or

(2) $\tilde{u}_k^i \to -\infty$ uniformly on compact sets in $B_{\rho/\delta_k^i}(0)$, or

(3) there exists a sequence of points $y_k^i \in B_{\rho/\delta_k^i}(0)$ such that $\tilde{u}_n^i(y_k^i) \to +\infty$.

However, by the fact $\tilde{u}_k^i(0) = \max\limits_{B_{\rho/\delta_k^i}(0)} \tilde{u}_k^i(x) = 0$, (2) or (3) in the above alternatives cannot happen. Therefore $\{\tilde{u}_k^i\}$ is bounded in $L_{loc}^\infty(\mathbb{R}^2)$ along a subsequence, and standard elliptic regularity yields that there exists a subsequence $\{\tilde{u}_{k(l)}^i\}$, denoted by $\{\tilde{u}_l^i\}$, and a function U such that $\tilde{u}_l^i \to U$ as $l \to \infty$ in $C_{loc}^1(\mathbb{R}^2)$. Passing to a limit in (2.4) and using Fatou's lemma, we see that U is a solution of

$$\begin{cases} -\Delta U = e^U & \text{in } \mathbb{R}^2, \\ U(0) = 0, \ U(y) \le 0, & \forall y \in \mathbb{R}^2, \\ \int_{\mathbb{R}^2} e^U \, dy < +\infty. \end{cases}$$

Thus by a result of Chen and Li [6], we have $U(y) = -2\log\left(1 + \frac{|y|^2}{8}\right)$, and this uniqueness of the limit proves the claim. Note that in terms of $\tilde{u}_n^i$ in (2.2), the estimate (2.3) can be written as

$$|\tilde{u}_n^i(y) - U(y)| \le C \quad \text{for } y \in B_{\frac{\rho}{\delta_n^i}}(0), \quad i = 1, \cdots, m.$$

Here, we define two elliptic operators

$$L_n := -\Delta_x - \lambda_n e^{u_n} \cdot : H_0^1(B_\rho(x_n^i)) \to H^{-1}(B_\rho(x_n^i)), \tag{2.5}$$

$$\tilde{L}_n^i := -\Delta_y - e^{\tilde{u}_n^i} \cdot : H_0^1(B_{\rho/\delta_n^i}(0)) \to H^{-1}(B_{\rho/\delta_n^i}(0)). \tag{2.6}$$

The operators (2.5) and (2.6) are related to each other by the formula

$$(\delta_n^i)^2 L_n \Big|_{u_n(x) = \tilde{u}_n^i(y) + u_n(x_n^i)} = \tilde{L}_n^i,$$

where $x = \delta_n^i y + x_n^i$ for $x \in B_\rho(x_n^i)$ and $y \in B_{\rho/\delta_n^i}(0)$. Also for a domain $D \subset B_\rho(x_n^i)$, we have

$$(\delta_n^i)^2 \lambda_j(L_n, D) = \lambda_j(\tilde{L}_n^i, D_n^i), \quad D_n^i = \frac{D - x_n^i}{\delta_n^i}, \tag{2.7}$$

where $\lambda_j(L_n, D), \lambda_j(\tilde{L}_n^i, D_n^i)$ $(j \in \mathbb{N})$ denote the j-th eigenvalue of elliptic operators $L_n, \tilde{L}_n^i$ acting on $H_0^1(D), H_0^1(D_n^i)$ respectively.

We prove

Lemma 3 *There exists $R > 0$ such that $\lambda_1(L_n, B(x_n^i, \delta_n^i R)) < 0$ for n large and for any $i \in \{1, \cdots, m\}$. Here $B(x_n^i, \delta_n^i R))$ denotes an open ball with center x_n^i and radius $\delta_n^i R$. Furthermore, these m balls are disjoint for n large.*

Proof. Define

$$w_n^i(x) = (x - x_n^i) \cdot \nabla u_n(x) + 2, \quad x \in B_\rho(x_n^i).$$

Then w_n^i satisfies the equation $-\Delta w_n^i(x) = \lambda_n e^{u_n} w_n^i$, $x \in B_\rho(x_n^i)$, and if we define $\tilde{w}_n^i$ as

$$\tilde{w}_n^i(y) = w_n^i(\delta_n^i y + x_n^i) = y \cdot \nabla \tilde{u}_n(y) + 2, \quad y \in B_{\rho/\delta_n^i}(0), \qquad (2.8)$$

then $\tilde{w}_n^i$ satisfies $-\Delta \tilde{w}_n^i(y) = e^{\tilde{u}_n^i} \tilde{w}_n^i(y)$ for $y \in B_{\rho/\delta_n^i}(0)$. Since $\tilde{u}_n^i \to U$ in $C_{loc}^1(\mathbb{R}^2)$, we have

$$\tilde{w}_n^i(y) \to y \cdot \nabla U + 2 = 2\frac{8 - |y|^2}{8 + |y|^2} < 0 \quad \text{if } |y| = R > 2\sqrt{2}$$

as $n \to \infty$. Set $A_n^i = \{y \in B(0, R) : \tilde{w}_n^i(y) > 0\}$, $A_n^i \neq \phi$, and define

$$\overline{w}_n^i(y) = \begin{cases} \tilde{w}_n^i(y) & y \in A_n^i, \\ 0 & y \in B(0, R) \setminus \overline{A_n^i}. \end{cases}$$

If we test

$$\lambda_1(\tilde{L}_n^i, B(0, R)) = \inf_{v \in H_0^1(B(0,R))} \frac{\int_{B(0,R)} |\nabla v|^2 dy - \int_{B(0,R)} e^{\tilde{u}_n} v^2 dy}{\int_{B(0,R)} v^2 dy}$$

by $\overline{w}_n^i \in H_0^1(B(0, R))$, we see that $\lambda_1(\tilde{L}_n^i, B(0, R)) \leq 0$. Actually, we have $\lambda_1(\tilde{L}_n^i, B(0, R)) < 0$, since if equality holds, $\overline{w}_n^i$ would be the first eigenfunction of $\tilde{L}_n^i$ on $B(0, R)$, which must be strictly positive. But this contradicts to the fact that $\overline{w}_n^i$ is 0 near $\partial B(0, R)$. Finally, scaling formula (2.7) proves the first half part of the Lemma.

Next, we claim that $B(x_n^i, \delta_n^i R)$ and $B(x_n^j, \delta_n^j R)$ are disjoint for n large, if $i \neq j$. Indeed, since $\tilde{u}_n^i \to U, \tilde{u}_n^j \to U$ in $C_{loc}^1(\mathbb{R}^2)$, $U(y) = -2\log(1 + |y|^2/8)$ is strictly concave and $y \cdot \nabla \tilde{u}_n(y) \to y \cdot \nabla U(y) < 0$ on $B(0, R) \setminus \{0\}$, u_n is also strictly concave and $(x - x_n^i) \cdot \nabla u_n(x) < 0$ on $B(x_n^i, \delta_n^i R) \setminus \{x_n^i\}$ for n large.

In particular, there is not a critical point in $B(x_n^i, \delta_n^i R)$ other than $\{x_n^i\}$. The same holds for u_n on $B(x_n^j, \delta_n^j R) \setminus \{x_n^j\}$, and this concavity property proves the claim. $\square$

Now by Lemma 3, we have m open balls $B^1, \cdots, B^m$, $B^i = B(x_n^i, \delta_n^i R)$, which are disjoint, and

$$\lambda_1(L_n, B^i) < 0 \quad \text{for } i = 1, \cdots, m. \tag{2.9}$$

Furthermore, by a variational characterization of m-th eigenvalue of L_n, we see that

$$\lambda_m(L_n, \Omega) \leq \sum_{i=1}^{m} \lambda_1(L_n, B^i). \tag{2.10}$$

Though this fact is well-known, we give a proof in Appendix for the reader's convenience.

From (2.9) and (2.10), we have $\lambda_m(L_n, \Omega) < 0$. On the other hand, by the definition of the Morse index of u_n, we have $\lambda_{i_M(u_n)+1}(L_n, \Omega) \geq 0$. Therefore $m \leq i_M(u_n)$, and this proves Theorem 1. $\square$

Proof of Corollary. Let $\{u_n\}$ be a solution sequence as in Corollary 2, and choose any subsequence $\{u_{n(k)}\}$ of $\{u_n\}$, denoted by $\{u_k\}$. By a result of [15], there exists a subsequence $\{u_{k(l)}\}$ of $\{u_k\}$, denoted by $\{u_l\}$, such that $\lambda_l \int_\Omega e^{u_l} dx \to 8\pi m$ for some $m \in \mathbb{N} \cup \{0\}$. By Theorem 1, we have $m \leq i_M(u_l) \equiv 1$. If $m = 0$, then u_l coincides with the *minimal solution* for l large, and since the Morse index of the minimal solution is 0, this cannot happen. Thus we have $m = 1$ and $\lambda_l \int_\Omega e^{u_l} dx \to 8\pi$ as $l \to \infty$. Since $\{u_k\}$ is arbitrarily chosen, we have

$$\lambda_n \int_\Omega e^{u_n} dx \to 8\pi \quad \text{as } n \to \infty$$

for the full sequence $\{u_n\}$. $\square$

Appendix.

In this appendix, we prove the following lemma, which gives the validity of (2.10).

Lemma 4 *Let $L = -\Delta - c(x)\cdot$ denote a second-order elliptic differential operator acting on $H_0^1(\Omega)$, where $\Omega \subset \mathbb{R}^N(N \geq 1)$ is a smooth bounded domain and $c \in C(\overline{\Omega})$. If $D_1, \cdots, D_m$ are disjoint smooth subdomains of Ω, then we have*

$$\lambda_m(L, \Omega) \leq \sum_{i=1}^m \lambda_1(L, D_i)$$

where $\lambda_j(L, D)$ denote the j-th eigenvalue of L acting on $H_0^1(D)$ (counted with multiplicities) for $D \subset \Omega$.

Proof. Let $\psi_i \in H_0^1(D_i)$ $(i = 1, \cdots, m)$ be the first eigenfunction of L on D_i:

$$\begin{cases} L\psi_i = \lambda_1(L, D_i)\psi_i & \text{in } D_i, \\ \psi_i = 0 & \text{on } \partial D_i, \end{cases}$$

normalized as $\|\psi_i\|_{L^2(D_i)} = 1$. We can extend ψ_i as 0 outside D_i and obtain new function in $H_0^1(\Omega)$, which will be denoted again by ψ_i. Set $u = \sum_{i=1}^m \alpha_i\psi_i$ for $\alpha_i \in \mathbb{R}$, $u \in H_0^1(\Omega)$. Since

$$\|u\|_{L^2(\Omega)}^2 = \sum_{i=1}^m \alpha_i^2\|\psi_i\|_{L^2(D_i)}^2 = \sum_{i=1}^m \alpha_i^2,$$

$$(u, \phi)_{L^2(\Omega)} = \sum_{i=1}^m \alpha_i(\psi_i, \phi)_{L^2(D_i)}$$

for any $\phi \in H_0^1(\Omega)$, we have $(\alpha_1, \cdots, \alpha_m) \neq (0, \cdots, 0)$ such that

$$\|u\|_{L^2(\Omega)}^2 = 1, \quad \text{and} \quad (u, \phi_j)_{L^2(\Omega)} = 0 \quad (j = 1, \cdots, m-1)$$

holds true, where ϕ_j denotes the j-th eigenfunction corresponding to $\lambda_j(L, \Omega)$. By a variational characterization of $\lambda_m(L, \Omega)$, we have

$$\lambda_m(L, \Omega) = \inf\{(Lv, v)_{L^2(\Omega)} \mid \|v\|_{L^2(\Omega)} = 1,$$
$$(v, \phi_j)_{L^2(\Omega)} = 0 \ (j = 1, \cdots, m-1)\}.$$

Testing this by $u = \sum_{i=1}^m \alpha_i\psi_i$, we obtain

$$\lambda_m(L, \Omega) \leq (Lu, u)_{L^2(\Omega)} = \left(\sum_{i=1}^m \alpha_i L\psi_i, \sum_{i=1}^m \alpha_i\psi_i\right)_{L^2(\Omega)}$$

$$= \sum_{i=1}^m \alpha_i^2 \lambda_1(L, D_i)\|\psi_i\|_{L^2(D_i)}^2 \leq \sum_{i=1}^m \lambda_1(L, D_i).$$

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