Blow up points and the Morse indices of solutions to the Liouville equation: inhomogeneous case

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Abstract.

Let us consider the Liouville equation

$$-\Delta u = \lambda V(x) e^u \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega,$$

where Ω is a smooth bounded domain in $\mathbb{R}^2$, $V(x) > 0$ is a given function in $C^1(\overline{\Omega})$, and $\lambda > 0$ is a constant. Let $\{u_n\}$ be an m-point blowing up solution sequence for $\lambda = \lambda_n \downarrow 0$, in the sense that

$$\lambda_n \int_\Omega V(x) e^{u_n} dx \to 8\pi m \quad \text{as } n \to \infty$$

for $m \in \mathbb{N}$. We prove that the number of blow up points m is less than or equal to the Morse index of u_n for n sufficiently large. This extends the main result of the recent paper [13] to an inhomogeneous ($V \not\equiv 1$) case.

§1. Introduction

In this paper we study the Liouville equation

$$(1) \qquad \begin{cases} -\Delta u = \lambda V(x) e^u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^2$, $V(x) > 0$ is a given function in $C^1(\overline{\Omega})$, and $\lambda > 0$ is a constant.

The purpose of this note is to extend the main result of the recent paper [13], where only $V \equiv 1$ was considered, to the present case.

The Liouville equation appears in several fields of mathematics and physics, and the study of it has a rather long history; see for example, [3], [4], [12], and the references therein.

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Let $\{\lambda_n\}$ be a sequence of positive numbers with $\lambda_n \to 0$ as $n \to \infty$. One of the interesting issues of this problem is the study of asymptotic behavior of solutions as $n \to \infty$. Concerning this, Ma and Wei [10] proved the following fact, which extends the former result by Nagasaki and Suzuki [11] where $V \equiv 1$ was considered.

Theorem 1. *(Ma and Wei [10]) For any solution sequence $\{u_n\}$ of (1) for $\lambda = \lambda_n \downarrow 0$, there exists a subsequence (denoted by u_n again) such that it holds*

$$\lambda_n \int_\Omega V(x) e^{u_n} dx \to 8\pi m, \quad \text{for some } m \in \{0\} \cup \mathbb{N} \cup \{+\infty\},$$

and according to the cases, the solution sequence $\{u_n\}$ behaves as

(i) *uniform convergence to 0: $\|u_n\|_{L^\infty(\Omega)} \to 0$, when $m = 0$,*

(ii) *entire blow-up: $u_n(x) \to +\infty$ as $n \to \infty$ for any $x \in \Omega$ when $m = +\infty$,*

(iii) *m-points blow-up: there exists an m-points set $\mathcal{S} = \{a_1, \cdots, a_m\}$, called blow up set, such that each a_i is an interior point of Ω, $\|u_n\|_{L^\infty(K)} = O(1)$ for any compact set $K \subset \overline{\Omega} \setminus \mathcal{S}$, $u_n|_{\mathcal{S}} \to +\infty$, and*

$$u_n \to 8\pi \sum_{i=1}^m G(\cdot, a_i) \quad \text{in } C^2_{loc}(\overline{\Omega} \setminus \mathcal{S}) \tag{2}$$

as $n \to \infty$ when $m \in \mathbb{N}$. Furthermore, any blow up point $a_i \in \mathcal{S}$ must satisfy the condition

$$\frac{1}{2}\nabla R(a_i) + \sum_{j=1, j\neq i}^m \nabla_x G(a_i, a_j) + \frac{1}{8\pi}\nabla \log V(a_i) = 0$$

for $i = 1, 2, \cdots, m$. Here, $G = G(x,y)$ is the Green function of $-\Delta$ under the Dirichlet boundary condition with a pole $y \in \Omega$, and $R(x) = \left[G(x,y) + \frac{1}{2\pi}\log|x-y|\right]_{y=x}$ denotes the Robin function.

Later, the existence of multiple blowing up solutions with a prescribed blow up set is established; see [6] [7].

Let $i_M(u)$ denote the Morse index of a solution u of (1), that is, the number of negative eigenvalues of the linearized operator $L_u = -\Delta - \lambda V(x)e^u \cdot$ acting on $H^1_0(\Omega)$. In this note, we prove the following, which is an extension of the main theorem in [13] to the inhomogeneous case.

Theorem 2. *Let $\{u_n\}$ be a solution sequence of (1) for $\lambda = \lambda_n$ satisfying*

$$\lambda_n \int_\Omega V(x) e^{u_n} dx \to 8\pi m$$

for some $m \in \mathbb{N}$. Then $m \leq i_M(u_n)$ for n sufficiently large.

In the homogeneous ($V \equiv 1$) case [13], we used the fact that $w(x) = (x-a) \cdot \nabla u_n(x) + 2$ satisfies the equation $-\Delta w = \lambda_n e^{u_n} w$ (except for the boundary condition) for $a \in \mathbb{R}^2$. This is no longer true when V is not a constant, and we need another method. The proof presented here works also for the homogeneous case and the main idea originates from [1].

§2. Proof of Theorem 2

In this section, we prove Theorem 2 along the line of [13].

Let $\{u_n\}$ be a solution sequence to (1) for $\lambda = \lambda_n$ with $\lambda_n \int_\Omega V(x) e^{u_n} dx \to 8\pi m$ for some $m \in \mathbb{N}$. Theorem 1 implies that the existence of the blow up set $\mathcal{S} = \{a_1, \cdots, a_m\} \subset \Omega$. Also we have a sufficiently small $\rho > 0$ and m sequences of local maximum points $\{x_n^i\}$ such that for each $a_i \in \mathcal{S}$,

$$u_n(x_n^i) = \max_{B_\rho(x_n^i)} u_n(x) \to \infty, \quad x_n^i \to a_i \ (i = 1, \cdots, m),$$

as $n \to \infty$.

Now we recall the following local pointwise estimate for the blowing-up solutions to (1) thanks to YanYan Li [8]: For a fixed $\rho \in (0,1)$, there exists a constant $C > 0$ independent of $i = 1, \cdots, m$ and $\lambda_n > 0$ such that

$$\left| u_n(x) - \log \frac{e^{u_n(x_n^i)}}{\left(1 + \frac{\lambda_n}{8} V(x_n^i) e^{u_n(x_n^i)} |x - x_n^i|^2\right)^2} \right| \leq C \text{ for } x \in B_\rho(x_n^i) \tag{3}$$

holds true.

Here we show a proof for the reader's convenience. Define $v_n(x) = u_n(x) + \log \lambda_n$. Then v_n satisfies

$$-\Delta v_n = V(x) e^{v_n} \quad \text{in } \Omega, \quad v_n = \log \lambda_n \quad \text{on } \partial\Omega.$$

Furthermore, by the assumption $\lambda_n \int_\Omega V(x) e^{u_n} dx \to 8\pi m$ and $0 < \exists a \leq V(x) \leq \exists b < +\infty$, we see that $\int_\Omega e^{v_n} dx = O(1)$ as $n \to \infty$.

Now, we claim that $v_n(x_n^i) \to +\infty$ as $n \to \infty$ for any $i \in \{1, \cdots, m\}$. Indeed, assume the contrary that there exists $i \in \{1, \cdots, m\}$ and a subsequence (denoted by the same symbol) such that

(i) $v_n(x_n^i) \to -\infty$, or
(ii) $v_n(x_n^i) \to C$ for some $C \in \mathbb{R}$.

When (i) happens, we see

$$\int_{B_\rho(x_n^i)} V(x)e^{v_n(x)}dx \le e^{v_n(x_n^i)} \int_{B_\rho(x_n^i)} V(x)dx \to 0$$

as $n \to \infty$. However, this contradicts the fact that

$$\lim_{n\to\infty} \int_{B_\rho(x_n^i)} V(x)e^{v_n}dx \ge 8\pi,$$

see, for example, Li and Shafrir [9].

Also if (ii) happens, a result of Brezis and Merle ([2]:Theorem 3) implies that $\{v_n\}$ is bounded in $L^\infty_{loc}(\Omega)$. On the other hand, (2) in Theorem 1 implies that $v_n = u_n + \log\lambda_n \to -\infty$ on any compact set in $\overline{\Omega} \setminus \mathcal{S}$. Thus again we have a contradiction and we have proved the claim.

Once we have the claim, we are in the same situation of Theorem 0.3 in [8] (setting that $\Omega = B_\rho(x_n^i), 0 = x_n^i$ there). Note that

$$\max_{\partial B_\rho(x_n^i)} v_n(x) - \min_{\partial B_\rho(x_n^i)} v_n(x) = \max_{\partial B_\rho(x_n^i)} u_n(x) - \min_{\partial B_\rho(x_n^i)} u_n(x) = O(1)$$

as $n \to \infty$. Thus by Theorem 0.3 in [8], we have

$$\left| v_n(x) - \log \frac{e^{v_n(x_n^i)}}{\left(1 + \frac{\lambda_n}{8} V(x_n^i)e^{v_n(x_n^i)}|x - x_n^i|^2\right)^2} \right| \le C \quad \text{for } x \in B_\rho(x_n^i),$$

which is equivalent to (3).

Now, let us define

$$(\delta_n^i)^2\lambda_n e^{u_n(x_n^i)} = 1,$$
$$\tilde{u}_n^i(y) = u_n(\delta_n^i y + x_n^i) - u_n(x_n^i), \quad y \in B_{\rho/\delta_n^i}(0)$$

for $i \in \{1, \cdots, m\}$. By the above pointwise estimate, we easily see that $\delta_n^i = o(1)$ as $n \to \infty$. The scaled function $\tilde{u}_n^i$ satisfies

$$\begin{cases} -\Delta\tilde{u}_n^i = V(\delta_n^i y + x_n^i)e^{\tilde{u}_n^i} \quad \text{in } B_{\rho/\delta_n^i}(0), \\ \tilde{u}_n^i(0) = 0, \ \tilde{u}_n^i(x) \le 0, \quad \forall x \in B_{\rho/\delta_n^i}(0), \\ \int_{B_{\rho/\delta_n^i}(0)} V(\delta_n^i y + x_n^i)e^{\tilde{u}_n^i}dy = O(1), \quad (n \to \infty). \end{cases}$$

Moreover, by an argument in [13], we obtain

$$\tilde{u}_n^i \to U^i(y) = -2\log\left(1+\frac{V(a_i)}{8}|y|^2\right) \quad \text{for } i=1,\cdots,m \tag{4}$$

in $C^1_{loc}(\mathbb{R}^2)$ as $n\to\infty$, where U^i is a unique ([5]) solution of

$$\begin{cases} -\Delta U^i = V(a_i)e^{U^i} \quad \text{in } \mathbb{R}^2, \\ U^i(0)=0,\ U^i(y)\le 0, \quad \forall y\in\mathbb{R}^2, \\ \int_{\mathbb{R}^2} e^{U^i}dy < +\infty. \end{cases}$$

Now, we define two elliptic operators

$$L_n := -\Delta_x - \lambda_n V(x)e^{u_n(x)}\cdot : H_0^1(\Omega)\to H^{-1}(\Omega),$$
$$\tilde{L}_n^i := -\Delta_y - V(\delta_n^i y + x_n^i)e^{\tilde{u}_n^i(y)}\cdot : H_0^1(B_{\rho/\delta_n^i}(0)) \to H^{-1}(B_{\rho/\delta_n^i}(0)).$$

These two operators are related to each other by the formula

$$(\delta_n^i)^2 L_n\Big|_{u_n(x)=\tilde{u}_n^i(y)+u_n(x_n^i)} = \tilde{L}_n^i,$$

where $x=\delta_n^i y + x_n^i$ for $x\in B_\rho(x_n^i)$ and $y\in B_{\rho/\delta_n^i}(0)$. Also for a domain $D\subset B_\rho(x_n^i)$, we have

$$(\delta_n^i)^2\lambda_j(L_n,D) = \lambda_j(\tilde{L}_n^i, D_n^i), \quad D_n^i = \frac{D-x_n^i}{\delta_n^i}, \tag{5}$$

where $\lambda_j(L_n,D), \lambda_j(\tilde{L}_n^i, D_n^i)$ $(j\in\mathbb{N})$ denote the j-th eigenvalue of elliptic operators $L_n, \tilde{L}_n^i$ acting on $H_0^1(D), H_0^1(D_n^i)$ respectively.

We show the following.

Lemma 2.1. *There exists $R>0$ such that $\lambda_1(L_n, B_{\delta_n^i R}(x_n^i)) < 0$ for n large and for any $i\in\{1,\cdots,m\}$. Furthermore, these m balls are disjoint for n large.*

Proof. For $R>0$, we define

$$w_R(y) = 2\log\frac{8+R^2}{8+|y|^2}.$$

Since $w_R=0$ on $\partial B_R(0)$, we see $w_R\in H_0^1(B_R(0))$.

We will prove that $(\tilde{L}_n^i w_R, w_R)_{L^2(B_R)} < 0$ for $R > 0$ sufficiently large and $B_R(0) \subset B_{\rho/\delta_n^i}(0)$. Indeed,

$$\begin{aligned}(\tilde{L}_n^i w_R, w_R)_{L^2(B_R)} &= \int_{B_R(0)} |\nabla w_R|^2 dy - \int_{B_R(0)} V(\delta_n^i y + x_n^i) e^{\tilde{u}_n(y)} w_R^2(y) dy \\ &=: I_1 - I_2.\end{aligned}$$

We observe that

$$I_1 = \int_{B_R(0)} \frac{16|y|^2}{(8+|y|^2)^2} dy = 2\pi \int_0^R \frac{16r^2}{(8+r^2)^2} r dr = 32\pi \left[\log R + o_R(1)\right],$$

where $o_R(1) \to 0$ as $R \to \infty$. On the other hand, we have

$$\begin{aligned}I_2 &= \int_{B_R(0)} V(\delta_n^i y + x_n^i) e^{\tilde{u}_n(y)} w_R^2(y) dy \\ &= V(a_i) \int_{B_R(0)} \frac{1}{\left(1 + \frac{V(a_i)}{8}|y|^2\right)^2} \left\{\log \frac{8+R^2}{8+|y|^2}\right\}^2 dy + o_n(1) \\ &= 2\pi V(a_i) \int_0^R \frac{r}{\left(1 + \frac{V(a_i)}{8} r^2\right)^2} \left\{\log(8+R^2) - \log(8+r^2)\right\}^2 dr + o_n(1) \\ &= 2\pi V(a_i) \cdot 8^2 \left\{\log(8+R^2)\right\}^2 \left[\frac{1}{16V(a_i)} + o_R(1)\right] + o_n(1) \\ &= 32\pi \left(\log R\right)^2 \left[1 + o_R(1)\right] + o_n(1),\end{aligned}$$

where we have used (4) and

$$\int_0^R \frac{r}{(8+cr^2)^2} dr = \int_0^\infty \frac{r}{(8+cr^2)^2} dr + o_R(1) = \frac{1}{16c} + o_R(1)$$

for $c > 0$. Thus we obtain

$$(\tilde{L}_n^i w_R, w_R)_{L^2(B_R)} = I_1 - I_2 = -32\pi \left(\log R\right)^2 \left[1 + o_R(1)\right] < 0$$

by taking n sufficiently large first, and then $R > 0$ large such that $B_R(0) \subset B_{\rho/\delta_n^i}(0)$. This implies that the first eigenvalue of the operator $\tilde{L}_n^i$ on B_R is negative: $\lambda_1(\tilde{L}_n^i, B_R) < 0$. By this and the scaling formula (5) proves the first half part of the Lemma.

The fact that these balls $B_{\delta_n^i R}(x_n^i)$ are disjoint follows from the strict concavity of the limit functions $U^i(y) = -2\log(1 + \frac{V(a_i)}{8}|y|^2)$; see [13]. □

By Lemma 2.1, we have m open balls $B^1, \cdots, B^m$, $B^i = B_{\delta_n^i R}(x_n^i)$, which are disjoint, and

$$\lambda_1(L_n, B^i) < 0 \quad \text{for } i = 1, \cdots, m.$$

On the other hand, it is easy to see that

$$\lambda_m(L_n, \Omega) \leq \sum_{i=1}^{m} \lambda_1(L_n, B^i)$$

holds; see for example, [13]. Combining these inequalities, we have $\lambda_m(L_n, \Omega) < 0$. Therefore by the definition of the Morse index of u_n, we have $m \leq i_M(u_n)$. This proves Theorem 2. □

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