Boundedness of some integral operators and commutators on generalized Herz spaces with variable exponents *

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Abstract

The generalized Herz spaces with variable exponents $p(\cdot)$, $\alpha(\cdot)$, $q(\cdot)$ are defined. Our aim is to prove boundedness of some operators on those Herz spaces.
Keywords. Herz space, variable exponent, BMO space, commutator, singular integral, fractional integral.

1 Introduction

The class of the Herz spaces is arising from the study on characterization of multipliers on the classical Hardy spaces. Compared with the usual Lebesgue space, we see that the Herz space has an interesting norm in terms of real analysis which represents markedly both global and local properties of functions. Boundedness of some important operators on the Herz spaces obtained by many authors [18, 24, 25, 26, 27] are well known now.

Function spaces with variable exponent are being watched with keen interest not in real analysis but also in partial differential equations and in applied mathematics because they are applicable to the modeling for electrorheological fluids and image restoration. The theory of function spaces with variable exponent has rapidly made progress in the past twenty years since some elementary properties were established by Kováčik–Rákosník [22]. One of the main problems on the theory is the boundedness of the Hardy–Littlewood maximal operator on variable Lebesgue spaces. By virtue of the fine works [3, 4, 5, 7, 8, 9, 21, 23, 28, 29], some important conditions on variable exponent, for example, the log-Hölder conditions and the Muckenhoupt type condition, have been obtained.

Motivated by the study on the Herz spaces and on the variable Lebesgue spaces, the first author [11] has defined the Herz spaces with variable exponent $p(\cdot)$. Later he has given basic lemmas on the Muckenhoupt properties for variable exponent and on generalization of the BMO norm to get boundedness of some integral operators and commutators on the Herz spaces with variable exponent and some characterizations of those Herz spaces (cf. [12, 13, 14, 15, 16]).

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In the present paper we generalize the Herz spaces to the scale of variable exponents $p(\cdot)$, $q(\cdot)$, $\alpha(\cdot)$ based on the idea of the mixed Lebesgue sequence spaces (cf. [1, 10, 20]). We will prove the boundedness of four classes of operators, commutators of BMO function and singular integral, sublinear operators with the proper decay conditions, the fractional integral, and commutators of BMO function and the fractional integral, on the generalized Herz spaces with three variable exponents. Our main results are stated only in the case of the non-homogeneous spaces. We note that the boundedness of those operators on the homogeneous spaces is an open problem.

2 Preliminaries

In this section we define some function spaces with variable exponents and give basic properties and useful lemmas. Throughout this paper we will use the following notation:

1. Given a measurable set E, $|E|$ denotes the Lebesgue measure of E.
2. Given a measurable function f and a measurable set E with $|E| > 0$, f_E means the mean value of f on E, namely $f_E := \frac{1}{|E|}\int_E f(x)\,\mathrm{d}x$.
3. The symbol χ_E means the characteristic function for a measurable set E.
4. Given a measurable function $p(\cdot) : E \to (1,\infty)$, $p'(\cdot)$ means the conjugate exponent function, namely $1/p(x) + 1/p'(x) \equiv 1$ holds.
5. The symbol $\mathbb{N}_0$ is the set of all non-negative integers.
6. We write $B_l := \{x \in \mathbb{R}^n : |x| \le 2^l\}$ for $l \in \mathbb{Z}$.

We also note that all cubes are assumed to have their sides parallel to the coordinate axes.

2.1 Lebesgue spaces with variable exponent

Let $\Omega \subset \mathbb{R}^n$ be an open set such that $|\Omega| > 0$. Given a measurable function $p(\cdot) : \Omega \to (0,\infty)$ with $0 < \operatorname{ess\,inf}_{x\in\Omega} p(x)$, we define the variable Lebesgue space $L^{p(\cdot)}(\Omega)$ by

$$L^{p(\cdot)}(\Omega) := \{f \text{ is measurable on } \Omega : \rho_p(f) < \infty\},$$

where

$$\rho_p(f) := \int_{\{p(x)<\infty\}} |f(x)|^{p(x)}\,\mathrm{d}x + \|f\|_{L^\infty(\{p(x)=\infty\})}.$$

The set $L^{p(\cdot)}(\Omega)$ becomes a quasi Banach space when it is equipped with the Luxemburg–Nakano norm

$$||f||_{L^{p(\cdot)}} := \inf\{\lambda > 0 : \rho_p(f/\lambda) \le 1\}. \tag{1}$$

Below we define some classes of variable exponents.

Definition 2.1.

1. Given a measurable function $p(\,\cdot\,)$ defined on Ω, we write

$$p_- := \operatorname*{ess\,inf}_{x\in\Omega} p(x), \quad p_+ := \operatorname*{ess\,sup}_{x\in\Omega} p(x).$$

2. The set $\mathcal{P}_0(\Omega)$ consists of all measurable functions $p(\cdot) : \Omega \to (0,\infty)$ such that $0 < p_- \le p_+ < \infty$.

3. The set $\mathcal{P}(\Omega)$ consists of all measurable functions $p(\cdot) : \Omega \to (1,\infty)$ such that $1 < p_- \le p_+ < \infty$.

4. We write $C^{\log}(\Omega)$ for the set of all measurable functions $p : \Omega \to (0,\infty)$ satisfying following conditions (2) and (3):

$$|p(x) - p(y)| \lesssim \frac{1}{-\log(|x-y|)} \quad (|x-y| \le 1/2), \tag{2}$$

$$|p(x) - p_\infty| \lesssim \frac{1}{\log(e+|x|)} \quad (x \in \Omega), \tag{3}$$

   where p_∞ is a constant independent of x.

5. Given a function $f \in L^1_{\mathrm{loc}}(\Omega)$, the Hardy–Littlewood maximal operator M is defined by

$$Mf(x) := \sup_{Q\ni x} \frac{1}{|Q|} \int_Q |f(y)|\,\mathrm{d}y \quad (x \in \Omega),$$

   where the supremum is taken over all cubes $Q \subset \Omega$ containing x.

6. The set $\mathcal{B}(\Omega)$ consists of all $p(\,\cdot\,) \in \mathcal{P}(\Omega)$ satisfying that the Hardy–Littlewood maximal operator M is bounded on $L^{p(\,\cdot\,)}(\Omega)$.

Remark 2.2. The Hardy–Littlewood maximal operator M is not always bounded on Lebesgue spaces with variable exponent since Pick–Růžička [29] gives a counter example. But some sufficient conditions for the boundedness of M are known now. If $1 < p_- \le p_+ \le \infty$ and $1/p(\cdot) \in C^{\log}(\mathbb{R}^n)$, then M is bounded on $L^{p(\,\cdot\,)}(\mathbb{R}^n)$. This fact has initially proved by Cruz-Uribe–Fiorenza–Neugebauer [5], Diening [7] and Nekvinda [28] in the case of $p_+ < \infty$. Later Cruz-Uribe–Diening–Fiorenza [3] and Diening–Harjulehto–Hästö–Mizuta–Shimomura [9] have proved it for $p_+ = \infty$. On the other hand, Kopaliani [21] has proved the following: If $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ equals to a constant outside a ball and satisfies the Muckenhoupt type condition

$$\sup_{Q:\mathrm{cube}} \frac{1}{|Q|} \|\chi_Q\|_{L^{p(\cdot)}} \|\chi_Q\|_{L^{p'(\cdot)}} < \infty,$$

then $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ holds. Lerner [23] has given another proof of Kopaliani's result.

The next lemma is due to Diening (Lemmas 3.2, 5.3 and 5.5 in [8]).

Lemma 2.3. *If $p(\,\cdot\,) \in \mathcal{B}(\mathbb{R}^n)$, then there exists a constant $0 < \delta_1 < 1$ such that for all $0 < \delta \le \delta_1$, all families of pairwise disjoint cubes Y, all $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ with $|f|_Q > 0$ ($Q \in Y$) and all positive sequence $\{t_Q\}_{Q\in Y} \subset (0,\infty)$,*

$$\left\| \sum_{Q\in Y} t_Q \left|\frac{f}{f_Q}\right|^\delta \chi_Q \right\|_{L^{p(\,\cdot\,)}} \lesssim \left\| \sum_{Q\in Y} t_Q \chi_Q \right\|_{L^{p(\,\cdot\,)}}.$$

In particular

$$\left\| |f|^\delta \chi_Q \right\|_{L^{p(\,\cdot\,)}} \lesssim (|f|_Q)^\delta \, \|\chi_Q\|_{L^{p(\,\cdot\,)}}$$

holds.

As a consequence of Lemma 2.3, we obtain the following lemma (cf. [13]).

Lemma 2.4. *If $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, then there exists a positive constant δ_1 such that*

$$\frac{||\chi_S||_{L^{p(\cdot)}}}{||\chi_B||_{L^{p(\cdot)}}} \lesssim \left(\frac{|S|}{|B|}\right)^{\delta_1} \tag{4}$$

for all balls $B \subset \mathbb{R}^n$ and all measurable subsets $S \subset B$.

Remark 2.5.

1. Diening (Theorem 8.1 in [8]) has proved that $p'(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ whenever $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. Thus if $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, then we can also take a constant $\delta_2 > 0$ so that for all balls $B \subset \mathbb{R}^n$ and all measurable subsets $S \subset B$,

$$\frac{||\chi_S||_{L^{p'(\cdot)}}}{||\chi_B||_{L^{p'(\cdot)}}} \lesssim \left(\frac{|S|}{|B|}\right)^{\delta_2} \tag{5}$$

   holds.

2. If $p_2(\cdot) \in \mathcal{P}(\mathbb{R}^n) \cap C^{\log}(\mathbb{R}^n)$, then we see that $p_2'(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. Hence we can take a constant $r \in (0, 1/(p_2')_+)$ so that

$$\frac{||\chi_S||_{L^{p_2'(\cdot)}}}{||\chi_B||_{L^{p_2'(\cdot)}}} \lesssim \left(\frac{|S|}{|B|}\right)^{\delta_2} \tag{6}$$

   holds.

The next lemma is known as the generalized Hölder inequality on Lebesgue spaces with variable exponent (cf. [22]).

Lemma 2.6. *Suppose $p(\,\cdot\,) \in \mathcal{P}(\Omega)$. Then we have that for all $f \in L^{p(\cdot)}(\Omega)$ and $g \in L^{p'(\cdot)}(\Omega)$,*

$$\int_\Omega |f(x)g(x)|\,\mathrm{d}x \le \left(1 + \frac{1}{p_-} - \frac{1}{p_+}\right) \|f\|_{L^{p(\cdot)}} \|g\|_{L^{p'(\cdot)}}.$$

We will use the following simple inequality which takes the place of Jensen's inequality.

Lemma 2.7. *If $a_k \ge 0$ and $1 \le p_k < \infty$ $(k \in \mathbb{N}_0)$, then*

$$\sum_{k=0}^{\infty} a_k^{p_k} \le \left(\sum_{k=0}^{\infty} a_k\right)^{p_*}$$

holds, where

$$p_* := \begin{cases} \min_{k\in\mathbb{N}_0} p_k & \text{if } \sum_{k=0}^\infty a_k \le 1, \\ \max_{k\in\mathbb{N}_0} p_k & \text{if } \sum_{k=0}^\infty a_k > 1. \end{cases}$$

Proposition 2.8. *Let $p(\cdot)$ be a measurable function on $\mathbb{R}^n$ with range in $[\alpha, \beta]$, where $\alpha > 0$. Let $q(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$. If $p(\cdot)q(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$ and $g(\cdot) \in L^{p(\cdot)q(\cdot)}(\mathbb{R}^n)$, then we have*

$$\min\left(\|g(\cdot)\|_{L^{p(\cdot)q(\cdot)}}^\beta, \|g(\cdot)\|_{L^{p(\cdot)q(\cdot)}}^\alpha\right) \le \left\| |g(\cdot)|^{p(\cdot)} \right\|_{L^{q(\cdot)}} \le \max\left(\|g(\cdot)\|_{L^{p(\cdot)q(\cdot)}}^\beta, \|g(\cdot)\|_{L^{p(\cdot)q(\cdot)}}^\alpha\right).$$

Proof. Let $p(\cdot)$ be a measurable function on $\mathbb{R}^n$ with range in $[\alpha, \beta]$. Let $q(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$ and $g(\cdot) \in L^{p(\cdot)q(\cdot)}$. If $||g(\cdot)||_{L^{p(\cdot)q(\cdot)}} \geq 1$, then we have

$$\int_{\mathbb{R}^n} \left(\frac{|g(x)|^{p(x)}}{||g(\cdot)||^{\beta}_{L^{p(\cdot)q(\cdot)}}} \right)^{q(x)} \mathrm{d}x = \int_{\mathbb{R}^n} \left(\frac{|g(x)|}{||g(\cdot)||^{\frac{\beta}{p(x)}}_{L^{p(\cdot)q(\cdot)}}} \right)^{p(x)q(x)} \mathrm{d}x$$
$$\leq \int_{\mathbb{R}^n} \left(\frac{|g(x)|}{||g(\cdot)||_{L^{p(\cdot)q(\cdot)}}} \right)^{p(x)q(x)} \mathrm{d}x = 1$$

by $1 \leq \beta/p(x)$ for almost all $x \in \mathbb{R}^n$. By $1 \geq \alpha/p(x)$ and the same calculation, we have

$$\int_{\mathbb{R}^n} \left(\frac{|g(x)|^{p(x)}}{||g(\cdot)||^{\alpha}_{L^{p(\cdot)q(\cdot)}}} \right)^{q(x)} \mathrm{d}x = \int_{\mathbb{R}^n} \left(\frac{|g(x)|}{||g(\cdot)||^{\frac{\alpha}{p(x)}}_{L^{p(\cdot)q(\cdot)}}} \right)^{p(x)q(x)} \mathrm{d}x$$
$$\geq \int_{\mathbb{R}^n} \left(\frac{|g(x)|}{||g(\cdot)||_{L^{p(\cdot)q(\cdot)}}} \right)^{p(x)q(x)} \mathrm{d}x = 1.$$

These imply that $||g(\cdot)||^{\alpha}_{L^{p(\cdot)q(\cdot)}} \leq || \, |g(\cdot)|^{p(\cdot)} ||_{L^{q(\cdot)}} \leq ||g(\cdot)||^{\beta}_{L^{p(\cdot)q(\cdot)}}$.

A similar argument yields for the case $||g(\cdot)||_{L^{p(\cdot)q(\cdot)}} < 1$. □

2.2 Remarks on the BMO norm

The BMO space and the BMO norm are defined respectively as follows:

$$\mathrm{BMO}(\mathbb{R}^n) := \left\{ b \in L^1_{\mathrm{loc}}(\mathbb{R}^n) \, : \, \|b\|_{\mathrm{BMO}} < \infty \right\},$$
$$\|b\|_{\mathrm{BMO}} := \sup_{Q:\mathrm{cube}} \frac{1}{|Q|} \int_Q |b(x) - b_Q| \, \mathrm{d}x.$$

Applying Lemma 2.3, the first author (Lemma 3 in [14]) has proved the next result.

Lemma 2.9. *Let k be a positive integer and suppose $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. Then we have that for all $b \in \mathrm{BMO}(\mathbb{R}^n)$ and all j, $l \in \mathbb{Z}$ with $j > l$,*

$$\|b\|^k_{\mathrm{BMO}} \simeq \sup_{B:\mathrm{ball}} \frac{1}{\|\chi_B\|_{L^{p(\cdot)}}} \|(b - b_B)^k \chi_B\|_{L^{p(\cdot)}}, \tag{7}$$

$$\|(b - b_{B_l})^k \chi_{B_j}\|_{L^{p(\cdot)}} \lesssim (j - l)^k \|b\|^k_{\mathrm{BMO}} \|\chi_{B_j}\|_{L^{p(\cdot)}}. \tag{8}$$

Remark 2.10. We note that (7) implies a generalization of the BMO norm in terms of the variable exponent. In the case of that $p(\cdot)$ equals to a constant, this is a well-known fact obtained by an argument applying the John–Nirenberg inequality. Recently a corresponding result to the case $p_- = 1$ has proved in [17]: If $1 = p_- \leq p_+ < \infty$ and $p(\cdot) \in C^{\log}(\mathbb{R}^n)$, then

$$\|b\|_{\mathrm{BMO}} \simeq \sup_{B:\mathrm{ball}} \frac{1}{\|\chi_B\|_{L^{p(\cdot)}}} \|(b - b_B)\chi_B\|_{L^{p(\cdot)}}$$

holds for all $b \in \mathrm{BMO}(\mathbb{R}^n)$.

2.3 Mixed Lebesgue sequence spaces and Herz spaces

Below we will use the following notation in order to define Herz spaces:

$$\begin{aligned} R_l &:= \{x \in \mathbb{R}^n \,:\, 2^{l-1} < |x| \le 2^l\} = B_l \setminus B_{l-1} \text{ if } l \in \mathbb{N}, \\ R_0 &:= \{x \in \mathbb{R}^n \,:\, |x| \le 1\} = B_0, \\ \chi_l &:= \chi_{R_l} \text{ for } l \in \mathbb{N}_0. \end{aligned}$$

We first define the mixed Lebesgue sequence space $\ell^{q(\cdot)}(L^{p(\cdot)})$. Let $p(\cdot),\, q(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$. The space $\ell^{q(\cdot)}(L^{p(\cdot)})$ is the collection of all sequences $\{f_j\}_{j=0}^{\infty}$ of measurable functions on $\mathbb{R}^n$ such that

$$||\{f_j\}_{j=0}^{\infty}||_{\ell^{q(\cdot)}(L^{p(\cdot)})} := \inf\left\{\mu > 0 \,:\, \varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}\left(\left\{\frac{f_j}{\mu}\right\}_{j=0}^{\infty}\right) \le 1\right\} < \infty,$$

where

$$\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}\left(\{f_j\}_{j=0}^{\infty}\right) := \sum_{j=0}^{\infty} \inf\left\{\lambda_j \,:\, \int_{\mathbb{R}^n}\left(\frac{|f_j(x)|}{\lambda_j^{\frac{1}{q(x)}}}\right)^{p(x)} \mathrm{d}x \le 1\right\}.$$

Since we assume that $q_+ < \infty$, we have

$$\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}\left(\{f_j\}_{j=0}^{\infty}\right) = \sum_{j=0}^{\infty}\left|\left||f_j|^{q(\cdot)}\right|\right|_{L^{\frac{p(\cdot)}{q(\cdot)}}}. \tag{9}$$

If $\{g_j\}_{j=0}^{N}$ is a finite sequence of measurable functions on $\mathbb{R}^n$, then we define that an infinite sequence $\{g'_j\}_{j=0}^{\infty}$ of measurable functions on $\mathbb{R}^n$,

$$g'_j := \begin{cases} g_j & \text{for } j = 0, 1, \cdots, N \\ 0 & \text{for } j > N. \end{cases}$$

and that

$$\left|\left|\{g_j\}_{j=0}^{N}\right|\right|_{\ell^{q(\cdot)}(L^{p(\cdot)})} := ||\{g'_j\}_{j=0}^{\infty}||_{\ell^{q(\cdot)}(L^{p(\cdot)})} = \inf\left\{\mu > 0 \,:\, \varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}\left(\left\{\frac{g'_j}{\mu}\right\}_{j=0}^{\infty}\right) \le 1\right\}.$$

Remark 2.11. Almeida–Hästö [1] has proved that $||\cdot||_{\ell^{q(\cdot)}(L^{p(\cdot)})}$ is a quasi-norm for all $p(\cdot), q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and that $||\cdot||_{\ell^{q(\cdot)}(L^{p(\cdot)})}$ is a norm when $\frac{1}{p(\cdot)} + \frac{1}{q(\cdot)} \le 1$. On the other hand, Kempka–Vybíral [20] has proved that $||\cdot||_{\ell^{q(\cdot)}(L^{p(\cdot)})}$ is a norm if $p(\cdot), q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfy either $1 \le q(x) \le p(x) \le \infty$ or $\frac{1}{p(x)} + \frac{1}{q(x)} \le 1$ for almost every $x \in \mathbb{R}^n$. Furthermore, it is also proved in [20] that there exist $p(\cdot), q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ with $\min\left\{\inf_{x\in\mathbb{R}^n} p(x),\ \inf_{x\in\mathbb{R}^n} q(x)\right\} \ge 1$ such that the triangle inequality does not hold for $||\cdot||_{\ell^{q(\cdot)}(L^{p(\cdot)})}$. This means that $||\cdot||_{\ell^{q(\cdot)}(L^{p(\cdot)})}$ does not always become a norm even if $p(\cdot)$ and $q(\cdot)$ satisfy $\min\{p_-,\, q_-\} \ge 1$.

Definition 2.12. Let $p(\cdot),\, q(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$ and $\alpha(\cdot) : \mathbb{R}^n \to \mathbb{R}$ such that $-\infty < \alpha_- \le \alpha_+ < \infty$. Given a measurable set $E \subset \mathbb{R}^n$, $|E| > 0$, the space $L^{p(\cdot)}_{\mathrm{loc}}(E)$ is defined by

$$L^{p(\cdot)}_{\mathrm{loc}}(E) := \left\{f \,:\, f \in L^{p(\cdot)}(K) \text{ for all compact sets } K \subset E\right\}.$$

The non-homogeneous Herz space $K_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)$ is the collection of $f \in L_{\mathrm{loc}}^{p(\cdot)}(\mathbb{R}^n)$ such that

$$\begin{aligned} \|f\|_{K_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}} &:= \left\| \left\{ 2^{k\alpha(\cdot)} |f\chi_k| \right\}_{k=0}^{\infty} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \\ &= \inf \left\{ \lambda > 0 \, : \, \sum_{k=0}^{\infty} \left\| \left(\frac{2^{k\alpha(\cdot)}|f\chi_k|}{\lambda} \right)^{q(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}} \le 1 \right\} < \infty. \end{aligned} \tag{10}$$

For any $\lambda > 0$, it is easy to see that

$$\sum_{k=0}^{\infty} \left\| \left(\frac{|f|\chi_k}{\lambda} \right)^{p(\cdot)} \right\|_{L^1} = \sum_{k=0}^{\infty} \int_{\mathbb{R}^n} \left(\frac{|f(x)|\chi_k}{\lambda} \right)^{p(x)} \mathrm{d}x = \int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} \mathrm{d}x. \tag{11}$$

This implies that $K_{p(\cdot)}^{0,p(\cdot)}(\mathbb{R}^n) = L^{p(\cdot)}(\mathbb{R}^n)$ if $p(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$, by (1) and (10).

3 Boundedness of operators on the non-homogeneous Herz spaces

In this section we prove the boundedness of four kinds of operators:

3.1 Higher order commutators of singular integral and BMO function,

3.2 Sublinear operators with the proper size conditions,

3.3 The fractional integral,

3.4 Commutators of the fractional integral and BMO function,

on the non-homogeneous Herz spaces with variable exponents $p(\cdot)$, $q(\cdot)$ and $\alpha(\cdot)$. Boundedness of those operators on the usual Herz spaces with constant exponents is well known (cf. [18, 24, 25, 26, 27]). The first author [12, 13, 14, 15, 16] has obtained some results in the case with constant exponents α, q and variable exponent $p(\cdot)$. Our main results are generalizations of them for the non-homogeneous spaces.

3.1 Higher order commutators of singular integral

Let $k \in \mathbb{N}$, $b \in \mathrm{BMO}(\mathbb{R}^n)$ and f be a locally integrable function on $\mathbb{R}^n$. We define the higher order commutator by

$$T_b^k f(x) := \int_{\mathbb{R}^n} \{b(x) - b(y)\}^k K(x-y) f(y) \mathrm{d}y,$$

where $K(x)$ is a function on $\mathbb{R}^n \setminus \{0\}$ satisfying the following.

(1) K is locally integrable on $\mathbb{R}^n \setminus \{0\}$.

(2) The Fourier transform of K is bounded.

(3) For all $x \in \mathbb{R}^n \setminus \{0\}$, $|K(x)| \lesssim |x|^{-n}$ and $|\nabla K(x)| \lesssim |x|^{-n-1}$ hold.

Then we have the following theorem.

Theorem 3.1. *Let $k \in \mathbb{N}_0$, $b \in \mathrm{BMO}(\mathbb{R}^n)$, $1 < r < \infty$, $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, $q_1(\cdot)$, $q_2(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$ and $\alpha(\cdot)$ be a real-valued function. Suppose that $(q_1)_+ \le (q_2)_-$, $-n\delta_1 < \alpha_- \le \alpha_+ < n\delta_2$, where δ_1, $\delta_2 > 0$ are the constants appearing in* (4) *and* (5). *Then we have that for all $\{f_h\}_{h\in\mathbb{N}}$ such that $\|\|\{f_h\}_h\|_{\ell^r}\|_{K_{p(\cdot)}^{\alpha_+,q_1(\cdot)}} < \infty$,*

$$\left\| \left(\sum_{h=1}^{\infty} |T_b^k(f_h)|^r \right)^{\frac{1}{r}} \right\|_{K_{p(\cdot)}^{\alpha(\cdot),q_2(\cdot)}} \lesssim \|b\|_{\mathrm{BMO}}^k \left\| \left(\sum_{h=1}^{\infty} |f_h|^r \right)^{\frac{1}{r}} \right\|_{K_{p(\cdot)}^{\alpha_+,q_1(\cdot)}}.$$

To prove the Theorem 3.1, we apply the next theorem. It is initially proved by Karlovich–Lerner [19] for the scalar-valued case. Independently Cruz-Uribe–Fiorenza–Martell–Pérez [4] has proved it by virtue of the extrapolation theorem.

Theorem 3.2. *Let $k \in \mathbb{N}_0$, $b \in \mathrm{BMO}(\mathbb{R}^n)$, $1 < r < \infty$ and $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. Then we have the vector-valued inequality*

$$\left\| \|\{T_b^k(f_h)\}_h\|_{\ell^r} \right\|_{L^{p(\cdot)}} \lesssim \|b\|_{\mathrm{BMO}}^k \, \|\|\{f_h\}_h\|_{\ell^r}\|_{L^{p(\cdot)}}$$

for all sequences of functions $\{f_h\}_{h\in\mathbb{N}}$ satisfying $\|\|\{f_h\}_h\|_{\ell^r}\|_{L^{p(\cdot)}} < \infty$.

Proof of Theorem 3.1. Let $\{f_h\}_{h\in\mathbb{N}}$ satisfy $\| \|\{f_h\}_h\|_{\ell^r}\|_{K_{p(\cdot)}^{\alpha_+,q_1(\cdot)}} < \infty$. For any $j \in \mathbb{N}_0$, we consider the norm

$$\left\| \left(\frac{2^{j\alpha(\cdot)} \|\{T_b^k(f_h)\}_h\|_{\ell^r} \chi_j}{\lambda} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}}$$

because

$$\begin{aligned} &\left\| \|\{T_b^k(f_h)\}_h\|_{\ell^r} \right\|_{K_{p(\cdot)}^{\alpha(\cdot),q_2(\cdot)}} \\ &= \inf \left\{ \lambda > 0 : \sum_{j=0}^{\infty} \left\| \left(\frac{2^{j\alpha(\cdot)} \|\{T_b^k(f_h)\}_h\|_{\ell^r} \chi_j}{\lambda} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} \le 1 \right\}. \end{aligned} \tag{12}$$

Let

$$\lambda_1 := \left\| \left\{ 2^{j\alpha(\cdot)} \left\| \left\{ \sum_{l=0}^{j-2} T_b^k(f_h \chi_l) \right\}_h \right\|_{\ell^r} \chi_j \right\}_j \right\|_{\ell^{q_2(\cdot)}(L^{p(\cdot)})}, \tag{13}$$

$$\lambda_2 := \left\| \left\{ 2^{j\alpha(\cdot)} \left\| \left\{ \sum_{l=j-1}^{j+1} T_b^k(f_h \chi_l) \right\}_h \right\|_{\ell^r} \chi_j \right\}_j \right\|_{\ell^{q_2(\cdot)}(L^{p(\cdot)})}, \tag{14}$$

$$\lambda_3 := \left\| \left\{ 2^{j\alpha(\cdot)} \left\| \left\{ \sum_{l=j+2}^{\infty} T_b^k(f_h \chi_l) \right\}_h \right\|_{\ell^r} \chi_j \right\}_j \right\|_{\ell^{q_2(\cdot)}(L^{p(\cdot)})}. \tag{15}$$

Then we see that

$$
\begin{aligned}
\left\|\left(\frac{2^{j\alpha(\cdot)}\|\{T_b^k(f_h)\}_h\|_{\ell^r}\chi_j}{\lambda}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} &\le \left\|\left(\frac{2^{j\alpha(\cdot)}\|\{\sum_{l=0}^{\infty}T_b^k(f_h\chi_l)\}_h\|_{\ell^r}\chi_j}{\lambda}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} \\
&\le \left\|\left(\frac{2^{j\alpha(\cdot)}\|\{\sum_{l=0}^{j-2}T_b^k(f_h\chi_l)\}_h\|_{\ell^r}\chi_j}{\lambda_1}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} \\
&\quad + \left\|\left(\frac{2^{j\alpha(\cdot)}\|\{\sum_{l=j-1}^{j+1}T_b^k(f_h\chi_l)\}_h\|_{\ell^r}\chi_j}{\lambda_2}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} \\
&\quad + \left\|\left(\frac{2^{j\alpha(\cdot)}\|\{\sum_{l=j+2}^{\infty}T_b^k(f_h\chi_l)\}_h\|_{\ell^r}\chi_j}{\lambda_3}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}},
\end{aligned}
$$

where we put $\lambda := \lambda_1 + \lambda_2 + \lambda_3$. Hence we have

$$
\sum_{j=0}^{\infty}\left\|\left(\frac{2^{j\alpha(\cdot)}\|\{T_b^k(f_h)\}_h\|_{\ell^r}\chi_j}{\lambda}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} \lesssim 1
$$

by (13), (14) and (15). This implies that

$$
\big\| \|\{T_b^k(f_h)\}_h\|_{\ell^r}\big\|_{K_{p(\cdot)}^{\alpha(\cdot),q_2(\cdot)}} \lesssim \lambda_1 + \lambda_2 + \lambda_3
$$

by (12). Hence it suffices to estimate λ_1, λ_2 and λ_3. Let $\mu := \big\| \|\{f_h\}\|_{\ell^r}\big\|_{K_{p(\cdot)}^{\alpha(\cdot),q_1(\cdot)}}$.

Step 1. We estimate λ_2. For each $j \in \mathbb{N}_0$ we define

$$
(q_{2*})_j := \begin{cases} (q_2)_+ & \text{if } \left\|\left(\frac{2^{j\alpha(\cdot)}\|\{\sum_{l=j-1}^{j+1}T_b^k(f_h\chi_l)\}_h\|_{\ell^r}\chi_j}{\mu\|b\|_{\mathrm{BMO}}^k}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} \ge 1, \\ (q_2)_- & \text{otherwise.} \end{cases}
$$

Letting $D := (\max\{2^{-\alpha_-}, 2^{\alpha_+}\})^{(q_2)_+}$, then we have

$$
\begin{aligned}
&\sum_{j=0}^{\infty}\left\|\left(\frac{2^{j\alpha(\cdot)}\|\{T_b^k(\sum_{l=j-1}^{j+1}f_h\chi_l)\}_h\|_{\ell^r}\chi_j}{\mu\|b\|_{\mathrm{BMO}}^k}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} \\
&\quad \le \sum_{j=0}^{\infty}\left\|\frac{\|\{T_b^k(\sum_{l=j-1}^{j+1}2^{(j-l)\alpha(\cdot)}2^{l\alpha_+}f_h\chi_l)\}_h\|_{\ell^r}\chi_j}{\mu\|b\|_{\mathrm{BMO}}^k}\right\|_{L^{p(\cdot)}}^{(q_{2*})_j} \\
&\quad \le D\sum_{j=0}^{\infty}\left(\sum_{l=j-1}^{j+1}\left\|\frac{\|\{T_b^k(2^{l\alpha_+}f_h\chi_l)\}_h\|_{\ell^r}\chi_j}{\mu\|b\|_{\mathrm{BMO}}^k}\right\|_{L^{p(\cdot)}}\right)^{(q_{2*})_j},
\end{aligned}
$$

where we have used Proposition 2.8. By Theorem 3.2, we see that

$$\sum_{j=0}^{\infty}\left\|\left(\frac{2^{j\alpha(\cdot)}\|\{T_b^k(\sum_{l=j-1}^{j+1} f_h\chi_l)\}_h\|_{\ell^r}\chi_j}{\mu\|b\|_{\mathrm{BMO}}^k}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}}$$

$$\lesssim D\sum_{j=0}^{\infty}\left(\sum_{l=j-1}^{j+1}\left\|\left\|\left\{\frac{2^{l\alpha_+}f_h\chi_l}{\mu}\right\}_h\right\|_{\ell^r}\right\|_{L^{p(\cdot)}}\right)^{(q_{2*})_j}$$

$$\lesssim D\max\{1,2^{2\{(q_2)_+-1\}}\}\sum_{j=0}^{\infty}\sum_{l=j-1}^{j+1}\left\|\left\|\left\{\frac{2^{l\alpha_+}f_h\chi_l}{\mu}\right\}_h\right\|_{\ell^r}\right\|_{L^{p(\cdot)}}^{(q_{2*})_j}$$

$$\lesssim D\max\{1,2^{2\{(q_2)_+-1\}}\}\sum_{j=0}^{\infty}\left\|2^{j\alpha_+}\left\|\left\{\frac{f_h}{\mu}\right\}_h\right\|_{\ell^r}\chi_j\right\|_{L^{p(\cdot)}}^{(q_{2*})_j}.$$

Hence, by Proposition 2.8 and Lemma 2.7, we have

$$\sum_{j=0}^{\infty}\left\|\left(\frac{2^{j\alpha(\cdot)}\|\{T_b^k(\sum_{l=j-1}^{j+1} f_h\chi_l)\}_h\|_{\ell^r}\chi_j}{\mu\|b\|_{\mathrm{BMO}}^k}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}}$$

$$\lesssim D\max\{1,2^{2\{(q_2)_+-1\}}\}\sum_{j=0}^{\infty}\left\|\left(2^{j\alpha_+}\left\|\left\{\frac{f_h}{\mu}\right\}_h\right\|_{\ell^r}\chi_j\right)^{q_1(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_1(\cdot)}}}^{\frac{(q_{2*})_j}{(q_1)_+}}$$

$$\lesssim D\max\{1,2^{2\{(q_2)_+-1\}}\}\left\{\sum_{j=0}^{\infty}\left\|\left(2^{j\alpha_+}\left\|\left\{\frac{f_h}{\mu}\right\}_h\right\|_{\ell^r}\chi_j\right)^{q_1(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_1(\cdot)}}}\right\}^{q_*}$$

$$\lesssim 1,$$

where $q_* \geq 1$ is a constant number as in Lemma 2.7. This implies that

$$\lambda_2 \lesssim \|b\|_{\mathrm{BMO}}^k\| \, \|\{f_h\}\|_{\ell^r}\|_{K_{p(\cdot)}^{\alpha_+,q_1(\cdot)}}.$$

Step 2. Next we estimate λ_1. For each $j\in\mathbb{N}_0$, $l\leq j-2$ and a. e. $x\in R_j$,

$$\begin{aligned}
\left\|\left\{T_b^k\left(\sum_{l=0}^{j-2}f_h\chi_l\right)(x)\right\}\right\|_{\ell^r} &\lesssim \left\|\left\{\int_{B_{j-2}}|b(x)-b(y)|^k\frac{|f_h(y)|}{|x-y|^n}\mathrm{d}y\right\}_h\right\|_{\ell^r}\\
&\lesssim 2^{-jn}\left\|\left\{\int_{B_{j-2}}|b(x)-b(y)|^k|f_h(y)|\mathrm{d}y\right\}_h\right\|_{\ell^r}\\
&\lesssim 2^{-jn}\int_{B_{j-2}}|b(x)-b(y)|^k\,\|\{f_h(y)\}_h\|_{\ell^r}\,\mathrm{d}y\\
&\lesssim 2^{-jn}|b(x)-b_{B_l}|^k\int_{B_{j-2}}\|\{f_h(y)\}_h\|_{\ell^r}\,\mathrm{d}y\\
&\quad+2^{-jn}\int_{B_{j-2}}|b_{B_l}-b(y)|^k\,\|\{f_h(y)\}_h\|_{\ell^r}\,\mathrm{d}y\\
&\lesssim 2^{-jn}\left\|\sum_{l=0}^{j-2}\|\{f_h(\cdot)\}_h\|_{\ell^r}\chi_l\right\|_{L^{p(\cdot)}}\|b\|_{\mathrm{BMO}}^k\\
&\quad\times\left\{|b(x)-b_{B_l}|^k\left\|\sum_{l=0}^{j-2}\chi_l\right\|_{L^{p'(\cdot)}}+\left\|(b_{B_l}-b(\cdot))^k\sum_{l=0}^{j-2}\chi_l\right\|_{L^{p'(\cdot)}}\right\}
\end{aligned}$$

by the generalized Hölder inequality and the Minkowski inequality. Then, for each $j \in \mathbb{N}_0$, we see that

$$
\begin{aligned}
&\left\| \left(\frac{2^{j\alpha(\cdot)} ||\{T_b^k(\sum_{l=0}^{j-2} f_h\chi_l)\}_h||_{\ell^r}\chi_j}{\mu ||b||_{\mathrm{BMO}}^k} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} \\
&\le \left\| \left\| \frac{\{T_b^k(\sum_{l=0}^{j-2} 2^{(j-l)\alpha_+} 2^{l\alpha_+} f_h\chi_l)\}_h}{\mu ||b||_{\mathrm{BMO}}^k} \right\|_{\ell^r} \chi_j \right\|_{L^{p(\cdot)}}^{(q_2**)_j} \\
&\le \Bigg\| 2^{-jn} \left\| \sum_{l=0}^{j-2} \left\| \frac{\{2^{(j-l)\alpha_+} 2^{l\alpha_+} f_h\}_h}{\mu} \right\|_{\ell^r} \chi_l \right\|_{L^{p(\cdot)}} \\
&\qquad \times \left\{ |b(x) - b_{B_l}|^k \chi_j \left\| \sum_{l=0}^{j-2} \chi_l \right\|_{L^{p'(\cdot)}} + \chi_j \left\| (b_{B_l} - b(\cdot))^k \sum_{l=0}^{j-2} \chi_l \right\|_{L^{p'(\cdot)}} \right\} \Bigg\|_{L^{p(\cdot)}}^{(q_2**)_j} \\
&\le \left\| \sum_{l=0}^{j-2} 2^{(j-l)\alpha_+ - jn} 2^{l\alpha_+} \left\| \frac{\{f_h\}_h}{\mu} \right\|_{\ell^r} \chi_l \right\|_{L^{p(\cdot)}}^{(q_2**)_j} \\
&\qquad \times \left\| |b(x) - b_{B_l}|^k \chi_j \left\| \sum_{l=0}^{j-2} \chi_l \right\|_{L^{p'(\cdot)}} + \chi_j \left\| (b_{B_l} - b(\cdot))^k \sum_{l=0}^{j-2} \chi_l \right\|_{L^{p'(\cdot)}} \right\|_{L^{p(\cdot)}}^{(q_2**)_j},
\end{aligned}
$$

where

$$
(q_{2**})_j := \begin{cases} (q_2)_+ & \text{if } \left\| \left(\frac{2^{j\alpha(\cdot)} ||\{\sum_{l=0}^{j-2} T_b^k(f_h\chi_l)\}_h||_{\ell^r}\chi_j}{\mu ||b||_{\mathrm{BMO}}^k} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} \ge 1, \\ (q_2)_- & \text{otherwise.} \end{cases}
$$

Hence we have

$$
\begin{aligned}
&\left\| |b(x) - b_{B_l}|^k \chi_j \left\| \sum_{l=0}^{j-2} \chi_l \right\|_{L^{p'(\cdot)}} + \chi_j \left\| (b_{B_l} - b(\cdot))^k \sum_{l=0}^{j-2} \chi_l \right\|_{L^{p'(\cdot)}} \right\|_{L^{p(\cdot)}}^{(q_2**)_j} \\
&\le \left(\left\| |b(x) - b_{B_l}|^k \chi_j \right\|_{L^{p(\cdot)}} \left\| \sum_{l=0}^{j-2} \chi_l \right\|_{L^{p'(\cdot)}} + \|\chi_j\|_{L^{p(\cdot)}} \left\| (b_{B_l} - b(\cdot))^k \sum_{l=0}^{j-2} \chi_l \right\|_{L^{p'(\cdot)}} \right)^{(q_2*)_j} \\
&\le \left((j-l)^k ||\chi_{B_j}||_{L^{p(\cdot)}} ||\chi_{B_l}||_{L^{p'(\cdot)}} \right)^{(q_{2**})_j}.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
&\left\| \left(\frac{2^{j\alpha(\cdot)} ||\{T_b^k(\sum_{l=0}^{j-2} f_h\chi_l)\}_h||_{\ell^r}\chi_j}{\mu ||b||_{\mathrm{BMO}}^k} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} \\
&\lesssim \left(\sum_{l=0}^{j-2} 2^{(j-l)\alpha_+ - jn} \left\| 2^{l\alpha_+} \left\| \left\{ \frac{f_h(\cdot)}{\mu} \right\}_h \right\|_{\ell^r} \chi_l \right\|_{L^{p(\cdot)}} (j-l)^k ||\chi_{B_j}||_{L^{p(\cdot)}} ||\chi_{B_l}||_{L^{p'(\cdot)}} \right)^{(q_{2**})_j} \\
&\lesssim \left(\sum_{l=0}^{j-2} 2^{(j-l)(\alpha_+ - n\delta_2)} \left\| 2^{l\alpha_+} \left\| \left\{ \frac{f_h(\cdot)}{\mu} \right\}_h \right\|_{\ell^r} \chi_l \right\|_{L^{p(\cdot)}} (j-l)^k \right)^{(q_{2**})_j}.
\end{aligned}
$$

This implies that

$$\sum_{j=0}^{\infty}\left\|\left(\frac{2^{j\alpha(\cdot)}\|\{\sum_{l=0}^{j-2}T_b^k(f_h\chi_l)\}_h\|_{\ell^r}\chi_j}{\mu\|b\|_{\mathrm{BMO}}^k}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}}$$
$$\lesssim \sum_{j=0}^{\infty}\left(\sum_{l=0}^{j-2}2^{(j-l)(\alpha_+-n\delta_2)}\left\|\left(\frac{2^{l\alpha_+}\|\{f_h(\cdot)\}_h\|_{\ell^r}\chi_l}{\mu}\right)^{q_1(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_1(\cdot)}}}^{\frac{1}{(q_1)_+}}(j-l)^k\right)^{(q_{2**})_j}.$$

If $(q_1)_+\le 1$, by Proposition 2.8 and Lemma 2.7, then we see that

$$\sum_{j=0}^{\infty}\left\|\left(\frac{2^{j\alpha(\cdot)}\|\{\sum_{l=0}^{j-2}T_b^k(f_h\chi_l)\}_h\|_{\ell^r}\chi_j}{\mu\|b\|_{\mathrm{BMO}}^k}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}}$$
$$\lesssim \left(\sum_{j=0}^{\infty}\sum_{l=0}^{j-2}2^{(q_1)_+(j-l)(\alpha_+-n\delta_2)}\left\|\left(\frac{2^{l\alpha_+}\|\{f_h(\cdot)\}_h\|_{\ell^r}\chi_l}{\mu}\right)^{q_1(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_1(\cdot)}}}(j-l)^{(q_1)_+k}\right)^{\frac{(q_2)_-}{(q_1)_+}}$$
$$\lesssim 1,$$

where $q_*\ge 1$ is a constant number as in Lemma 2.7.

If $(q_1)_+>1$, then we define $s:=((q_1)_+)'$. By using the Hölder inequality and Lemma 2.7, we have

$$\sum_{j=0}^{\infty}\left\|\left(\frac{2^{j\alpha(\cdot)}\|\{\sum_{l=0}^{j-2}T_b^k(f_h\chi_l)\}_h\|_{\ell^r}\chi_j}{\mu\|b\|_{\mathrm{BMO}}^k}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}}$$
$$\lesssim \sum_{j=0}^{\infty}\Bigg\{\left(\sum_{l=0}^{j-2}2^{(q_1)_+(j-l)(\alpha_+-n\delta_2)/2}\left\|\left(\frac{2^{l\alpha_+}\|\{f_h(\cdot)\}_h\|_{\ell^r}\chi_l}{\mu}\right)^{q_1(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_1(\cdot)}}}\right)^{\frac{1}{(q_1)_+}}$$
$$\times\left(\sum_{l=0}^{j-2}2^{s(j-l)(\alpha_+-n\delta_2)/2}(j-l)^{ks}\right)^{\frac{1}{s}}\Bigg\}^{(q_{2**})_j}$$
$$\lesssim \left(\sum_{j=0}^{\infty}\sum_{l=0}^{j-2}2^{(q_1)_+(j-l)(\alpha_+-n\delta_2)/2}\left\|\left(\frac{2^{l\alpha_+}\|\{f_h(\cdot)\}_h\|_{\ell^r}\chi_l}{\mu}\right)^{q_1(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_1(\cdot)}}}\right)^{q_*}$$
$$\lesssim 1.$$

Hence we see that $\lambda_1\lesssim\|b\|_{\mathrm{BMO}}^k\|\,\|\{f_h\}\|_{\ell^r}\|_{K_{p(\cdot)}^{\alpha_+,q_1(\cdot)}}$.

Step 3. Finally we estimate λ_3. For any $j\in\mathbb{N}_0$, $l\ge j+2$ and a. e. $x\in R_j$, by the same argument in Step 2, we see that

$$\begin{aligned}\left\|\{T_b^k(f_h\chi_l)(x)\}\right\|_{\ell^r}&\lesssim 2^{-ln}\left\|\left\{\int_{R_l}|b(x)-b(y)|^k|f_h(y)|\mathrm{d}y\right\}_h\right\|_{\ell^r}\\&\lesssim 2^{-ln}\int_{R_l}|b(x)-b(y)|^k\,\|\{f_h(y)\}_h\|_{\ell^r}\,\mathrm{d}y\\&\lesssim 2^{-ln}\left\|\,\|\{f_h(\cdot)\}_h\|_{\ell^r}\chi_l\right\|_{L^{p(\cdot)}}\|b\|_{\mathrm{BMO}}^k\\&\quad\times\left\{|b(x)-b_{B_l}|^k\|\chi_{B_l}\|_{L^{p'(\cdot)}}+\left\|(b_{B_l}-b(\cdot))^k\chi_l\right\|_{L^{p'(\cdot)}}\right\}\end{aligned}$$

and

$$
\begin{aligned}
&\left\|\left(\frac{2^{j\alpha(\cdot)}||\{\sum_{l=j+2}^{\infty}T_b^k(f_h\chi_l)\}_h||_{\ell^r}\chi_j}{\mu||b||_{\mathrm{BMO}}^k}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} \\
&\lesssim \left(\sum_{l=j+2}^{\infty}2^{(j-l)\alpha_-}\left\|\left\|\left\{T_b^k(\frac{2^{l(\alpha(y)+\alpha_+-\alpha_-)}f_h\chi_l}{\mu||b||_{\mathrm{BMO}}^k})\right\}_h\right\|_{\ell^r}\chi_j\right\|_{L^{p(\cdot)}}\right)^{(q_{2***})_j} \\
&\lesssim \Big(\sum_{l=j+2}^{\infty}2^{(j-l)\alpha_--ln}\left\|2^{l\alpha_+}\left\|\left\{\frac{f_h(\cdot)}{\mu}\right\}_h\right\|_{\ell^r}\chi_l\right\|_{L^{p(\cdot)}} \\
&\qquad\times\left\{||(b(\cdot)-b_{B_l})^k\chi_j||_{L^{p(\cdot)}}||\chi_{B_l}||_{L^{p'(\cdot)}}+\left||(b_{B_l}-b(\cdot))^k\chi_l||_{L^{p'(\cdot)}}||\chi_j\right||_{L^{p(\cdot)}}\right\}\Big)^{(q_{2***})_j} \\
&\lesssim \left(\sum_{l=j+2}^{\infty}2^{(j-l)\alpha_--ln}\left\|2^{l\alpha_+}\left\|\left\{\frac{f_h(\cdot)}{\mu}\right\}_h\right\|_{\ell^r}\chi_l\right\|_{L^{p(\cdot)}}(j-l)^k||\chi_{B_j}||_{L^{p(\cdot)}}||\chi_{B_l}||_{L^{p'(\cdot)}}\right)^{(q_{2***})_j} \\
&\lesssim \left(\sum_{l=j+2}^{\infty}2^{(j-l)(\alpha_-+n\delta_1)}\left\|2^{l\alpha_+}\left\|\left\{\frac{f_h(\cdot)}{\mu}\right\}_h\right\|_{\ell^r}\chi_l\right\|_{L^{p(\cdot)}}(j-l)^k\right)^{(q_{2***})_j},
\end{aligned}
$$

where

$$
(q_{2***})_j := \begin{cases} (q_2)_+ & \text{if } \left\|\left(\frac{2^{j\alpha(\cdot)}||\{\sum_{l=j+2}^{\infty}T_b^k(f_h\chi_l)\}_h||_{\ell^r}\chi_j}{\mu||b||_{\mathrm{BMO}}^k}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}}\geq 1, \\ (q_2)_- & \text{otherwise.}\end{cases}
$$

Hence we have $\lambda_3 \lesssim ||b||_{\mathrm{BMO}}^k||\,||\{f_h\}||_{\ell^r}||_{K_{p(\cdot)}^{\alpha_+,q_1(\cdot)}}$ by the same argument in Step 2. This completes the proof of Theorem 3.1. □

Remark 3.3. Later we will give Theorem 3.4 for the scalar-valued case. It is also true for the vector-valued case because the statement of the proof above is valid for this theorem.

3.2 Sublinear operators with the proper size conditions

We have the following theorem.

Theorem 3.4. *Let $p(\cdot)\in\mathcal{B}(\mathbb{R}^n)$, $q_1(\cdot)$, $q_2(\cdot)\in\mathcal{P}_0(\mathbb{R}^n)$ with $(q_1)_+\leq(q_2)_-$, $\alpha(\cdot)$ satisfy $-n\delta_1<\alpha_-\leq\alpha_+<n\delta_2$, where δ_1, $\delta_2>0$ are the constants appearing in (4) and (5), and T be a sublinear operator satisfying the size conditions*

$$|Tf(x)|\lesssim \|f\|_{L^1}\,|x|^{-n} \tag{16}$$

when supp $f\subset R_k$ *and* $|x|\geq 2^{k+1}$ *with* $k\in\mathbb{N}_0$, *and*

$$|Tf(x)|\lesssim 2^{-kn}\,\|f\|_{L^1} \tag{17}$$

when supp $f\subset R_k$ *and* $|x|\leq 2^{k-2}$ *with* $k\in\mathbb{N}_0$. *If T is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$, then we have that for all $f\in K_{p(\cdot)}^{\alpha_+,q_1(\cdot)}(\mathbb{R}^n)$,*

$$\|Tf\|_{K_{p(\cdot)}^{\alpha(\cdot),q_2(\cdot)}}\lesssim\|f\|_{K_{p(\cdot)}^{\alpha_+,q_1(\cdot)}}.$$

Remark 3.5. We note that many important sublinear operators, including the Hardy–Littlewood maximal operator and singular integrals, satisfy the assumptions above (cf. [4]), and therefore Theorem 3.4 is applicable to justify the boundedness of those operators on the Herz spaces.

Proof of Theorem 3.4. Without loss of generality, we can postulate $||f||_{K^{\alpha_+, q_1(\cdot)}_{p(\cdot)}} = 1$. We divide this proof into 3 parts as below:

$$\begin{aligned}\sum_{j=0}^{\infty}\left\|\left(2^{j\alpha(\cdot)}|Tf|\chi_j\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} &\lesssim \sum_{j=0}^{\infty}\left\|\left(\left|T\left(\sum_{l=0}^{j-2}2^{(j-l)\alpha(\cdot)}2^{l\alpha_+}f\chi_l\right)\right|\chi_j\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}}\\ &+\sum_{j=0}^{\infty}\left\|\left(\left|T\left(\sum_{l=j-1}^{j+1}2^{(j-l)\alpha(\cdot)}2^{l\alpha_+}f\chi_l\right)\right|\chi_j\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}}\\ &+\sum_{j=0}^{\infty}\left\|\left(\left|T\left(\sum_{l=j+2}^{\infty}2^{(j-l)\alpha(\cdot)}2^{l\alpha_+}f\chi_l\right)\right|\chi_j\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}}\\ &=: I_1+I_2+I_3.\end{aligned}$$

Hence it is suffice to prove $I_1, I_2, I_3 \lesssim 1$. By using the same argument in the proof of Theorem 3.1, we have $I_2 \lesssim 1$.

Step 1 We estimate I_1. By (16) and the generalized Hölder inequality, for any $l \le k-2$ and $x \in R_k$, we have

$$|T(f\chi_l)(x)| \lesssim 2^{-kn}\|f\chi_l\|_{L^1} \lesssim 2^{-kn}\|f\chi_l\|_{L^{p(\cdot)}}\|\chi_l\|_{L^{p'(\cdot)}}. \tag{18}$$

By using (18) and

$$2^{-kn}\|\chi_k\|_{L^{p(\cdot)}}\|\chi_l\|_{L^{p'(\cdot)}} \le 2^{n\delta_2(l-k)},$$

we see that

$$\begin{aligned}I_1 &\le \sum_{k=0}^{\infty}\left\|\sum_{l=0}^{k-2}2^{(k-l)\alpha_+}\left|T(2^{l\alpha_+}f\chi_l)\chi_k\right|\right\|_{L^{p(\cdot)}}^{(q_2)_-}\\ &\le \sum_{k=0}^{\infty}\left\|\sum_{l=0}^{k-2}\left\|2^{l\alpha_+}f\chi_l\right\|_{L^{p(\cdot)}}\|\chi_l\|_{L^{p'(\cdot)}}2^{(k-l)\alpha_+}2^{-kn}\chi_k\right\|_{L^{p(\cdot)}}^{(q_2)_-}\\ &\le \sum_{k=0}^{\infty}\left(\sum_{l=0}^{k-2}\left\|2^{l\alpha_+}f\chi_l\right\|_{L^{p(\cdot)}}\|\chi_l\|_{L^{p'(\cdot)}}2^{(k-l)\alpha_+}2^{-kn}\|\chi_k\|_{L^{p(\cdot)}}\right)^{(q_2)_-}\\ &\le \sum_{k=0}^{\infty}\left(\sum_{l=0}^{k-2}\left\|2^{l\alpha_+}f\chi_l\right\|_{L^{p(\cdot)}}2^{(k-l)(\alpha_+-n\delta_2)}\right)^{(q_2)_-}\\ &\lesssim \sum_{k=0}^{\infty}\sum_{l=0}^{k-2}\left\|\left(2^{l\alpha_+}f\chi_l\right)^{q_1(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_1(\cdot)}}}^{\frac{(q_2)_-}{(q_1)_+}}2^{(k-l)(\alpha_+-n\delta_2)\frac{(q_2)_-}{2}}\\ &\lesssim \left(\sum_{l=0}^{k-2}\left\|\left(2^{l\alpha_+}f\chi_l\right)^{q_1(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_1(\cdot)}}}\right)^{\frac{(q_2)_-}{(q_1)_+}} \le 1.\end{aligned}$$

Hence we have $I_1 \lesssim 1$.

Step 2 Finally we estimate I_3. For every $k \in \mathbb{N}_0$, $l \geq k+2$ and a. e. $x \in R_k$, we have

$$|T(f\chi_l)(x)| \lesssim 2^{-ln} \|f\chi_l\|_{L^1} \lesssim 2^{-ln} \|f\chi_l\|_{L^{p(\cdot)}} \|\chi_l\|_{L^{p'(\cdot)}}. \tag{19}$$

By using (19) and

$$2^{-ln} \|\chi_k\|_{L^{p(\cdot)}} \|\chi_l\|_{L^{p'(\cdot)}} \leq 2^{n\delta_1(k-l)},$$

and the same arguments in Step 1, we have $I_3 \lesssim 1$.

These complete the proof. □

3.3 The fractional integral

The fractional integral I^β is defined by

$$I^\beta f(x) := \frac{1}{\gamma(\beta)} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\beta}} \mathrm{d}y,$$

where $0 < \beta < n$ and $\gamma(\beta) := \frac{\pi^{n/2} 2^\beta \Gamma(\beta/2)}{\Gamma((n-\beta)/2)}$.

We use the following result on the boundedness of the fractional integral on variable Lebesgue spaces proved by Capone–Cruz-Uribe–Fiorenza [2]. Diening [6] has initially proved it when the variable exponent equals to a constant outside a ball.

Theorem 3.6. *Suppose that $p_1(\cdot) \in \mathcal{P}(\mathbb{R}^n) \cap C^{\log}(\mathbb{R}^n)$. Let $0 < \beta < n/(p_1)_+$. Define the variable exponent $p_2(\cdot)$ by*

$$\frac{1}{p_1(x)} - \frac{1}{p_2(x)} = \frac{\beta}{n}.$$

Then we have that for all $f \in L^{p_1(\cdot)}(\mathbb{R}^n)$,

$$\left|\left|I^\beta f\right|\right|_{L^{p_2(\cdot)}} \lesssim ||f||_{L^{p_1(\cdot)}}.$$

Then we have the following theorem.

Theorem 3.7. *Suppose that $p_2(\cdot) \in \mathcal{P}(\mathbb{R}^n) \cap C^{\log}(\mathbb{R}^n)$ and take a constant $0 < r < 1/(p_2')_+$ so that (6) holds. Let $0 < \beta < nr$, $\alpha(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$ with $\alpha_+ < nr - \beta$ and $q_1(\cdot)$, $q_2(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$ with $(q_1)_+ \leq (q_2)_-$. Define the variable exponent $p_1(\cdot)$ by*

$$\frac{1}{p_1(x)} - \frac{1}{p_2(x)} = \frac{\beta}{n}.$$

Then we have

$$\left\|I^\beta f\right\|_{K^{\alpha(\cdot), q_2(\cdot)}_{p_2(\cdot)}} \lesssim ||f||_{K^{\alpha_+, q_1(\cdot)}_{p_1(\cdot)}}$$

for all $f \in K^{\alpha_+, q_1(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)$.

Proof. Let $f \in K^{\alpha_+, q_1(\cdot)}_{p_1(\cdot)}$. We can assume that $||f||_{K^{\alpha_+, q_1(\cdot)}_{p_1(\cdot)}} = 1$. Then we see that

$$\begin{aligned}
\sum_{k=0}^{\infty} \left\| \left\{ 2^{k\alpha(\cdot)} |I^\beta f| \chi_k \right\}^{q_2(\cdot)} \right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}} &\le \sum_{k=0}^{\infty} \left\| \left\{ \sum_{j=0}^{\infty} 2^{k\alpha(\cdot)} |I^\beta (f_j)| \chi_k \right\}^{q_2(\cdot)} \right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}} \\
&\lesssim \sum_{k=0}^{\infty} \left\| \left\{ \sum_{j=0}^{k-2} 2^{k\alpha(\cdot)} |I^\beta (f_j)| \chi_k \right\}^{q_2(\cdot)} \right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}} \\
&\quad + \sum_{k=0}^{\infty} \left\| \left\{ \sum_{j=k-1}^{\infty} 2^{k\alpha(\cdot)} |I^\beta (f_j)| \chi_k \right\}^{q_2(\cdot)} \right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}} \\
&\lesssim \sum_{k=0}^{\infty} \left\| \left\{ \sum_{j=0}^{k-2} 2^{(k-j)\alpha_+} |I^\beta (2^{j\alpha_+} f_j)| \chi_k \right\}^{q_2(\cdot)} \right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}} \\
&+ \sum_{k=0}^{\infty} \left\| \left\{ \sum_{j=k-1}^{\infty} 2^{(k-j)\alpha_-} |I^\beta (2^{j\alpha_+} f_j)| \chi_k \right\}^{q_2(\cdot)} \right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}} \\
&=: U_1 + U_2. \qquad (20)
\end{aligned}$$

First we estimate U_1. By the same argument as the proof of [16, Theorem 2], we obtain

$$\left\| I^\beta (2^{j\alpha_+} f_j) \chi_k \right\|_{L^{p_2(\cdot)}} \lesssim 2^{(\beta - nr)(k-j)} \left\| I^\beta (2^{j\alpha_+} f_j) \chi_k \right\|_{L^{p_1(\cdot)}}.$$

Then, by taking a positive number ϵ so that $\beta - nr + \alpha_+ + \epsilon < 0$, we have

$$\begin{aligned}
U_1 &\lesssim \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k-2} \left\| 2^{j\alpha_+} f_j \right\|_{L^{p_1(\cdot)}} 2^{(\beta - nr + \alpha_+)(k-j)} \right)^{(q_2)_-} \\
&\lesssim \sum_{k=0}^{\infty} \sum_{j=0}^{k-2} \left\| 2^{j\alpha_+} f_j \right\|_{L^{p_1(\cdot)}}^{(q_2)_-} 2^{(\beta - nr + \alpha_+ + \epsilon)(k-j)(q_2)_-} \\
&\lesssim \sum_{k=0}^{\infty} \sum_{j=0}^{k-2} \left\| \left(2^{j\alpha_+} f_j \right)^{q_1(\cdot)} \right\|_{L^{\frac{p_1(\cdot)}{q_1(\cdot)}}}^{\frac{(q_2)_-}{(q_1)_+}} 2^{(\beta - nr + \alpha_+ + \epsilon)(k-j)(q_2)_-} \\
&= \sum_{j=0}^{\infty} \sum_{k'=2}^{\infty} \left\| \left(2^{j\alpha_+} f_j \right)^{q_1(\cdot)} \right\|_{L^{\frac{p_1(\cdot)}{q_1(\cdot)}}}^{\frac{(q_2)_-}{(q_1)_+}} 2^{(\beta - nr + \alpha_+ + \epsilon)k'(q_2)_-} \\
&\lesssim \sum_{j=0}^{\infty} \left\| \left(2^{j\alpha_+} f_j \right)^{q_1(\cdot)} \right\|_{L^{\frac{p_1(\cdot)}{q_1(\cdot)}}}^{\frac{(q_2)_-}{(q_1)_+}} \\
&\lesssim 1.
\end{aligned}$$

Finally we estimate U_2. By Theorem 3.6 we see that

$$
\begin{aligned}
U_2 &= \sum_{k=0}^{\infty} \left\| \left\{ \sum_{j=k-1}^{\infty} 2^{(k-j)\alpha_-} |I^\beta(2^{j\alpha_+} f_j)| \chi_k \right\}^{q_2(\cdot)} \right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}} \\
&\le \sum_{k=0}^{\infty} \left\| \sum_{j=k-1}^{\infty} 2^{(k-j)\alpha_-} |I^\beta(2^{j\alpha_+} f_j)| \chi_k \right\|_{L^{p_2(\cdot)}}^{(q_2)_-} \\
&\le \sum_{k=0}^{\infty} \left(\sum_{j=k-1}^{\infty} 2^{(k-j)\alpha_-} \left\| |I^\beta(2^{j\alpha_+} f_j)| \chi_k \right\|_{L^{p_2(\cdot)}} \right)^{(q_2)_-} \\
&\lesssim \sum_{k=0}^{\infty} \left(\sum_{j=k-1}^{\infty} 2^{(k-j)\alpha_-} \left\| 2^{j\alpha_+} f_j \right\|_{L^{p_1(\cdot)}} \right)^{(q_2)_-} \\
&\lesssim \sum_{k=0}^{\infty} \left(\sum_{j=k-1}^{\infty} 2^{(k-j)\alpha_-} \left\| \left(2^{j\alpha_+} |f_j|\right)^{q_1(\cdot)} \right\|_{L^{\frac{p_1(\cdot)}{q_1(\cdot)}}}^{\frac{1}{(q_1)_+}} \right)^{(q_2)_-}.
\end{aligned}
$$

If $(q_1)_+ \le 1$, then we have

$$
\begin{aligned}
U_2 &\le \sum_{k=0}^{\infty} \left(\sum_{j=k-1}^{\infty} 2^{(q_1)_+(k-j)\alpha_-} \left\| \left(2^{j\alpha_+} |f_j|\right)^{q_1(\cdot)} \right\|_{L^{\frac{p_1(\cdot)}{q_1(\cdot)}}} \right)^{\frac{(q_2)_-}{(q_1)_+}} \\
&\le \left(\sum_{k=1}^{\infty} \sum_{j=k-1}^{\infty} 2^{(q_1)_+(k-j)\alpha_-} \left\| \left(2^{j\alpha_+} |f_j|\right)^{q_1(\cdot)} \right\|_{L^{\frac{p_1(\cdot)}{q_1(\cdot)}}} \right)^{\frac{(q_2)_-}{(q_1)_+}} \lesssim 1 \qquad (21)
\end{aligned}
$$

by virtue of Lemma 2.7. If $(q_1)_+ > 1$, then we obtain by writing $s := ((q_1)_+)'$,

$$
\begin{aligned}
U_2 &\le \sum_{k=0}^{\infty} \left(\sum_{j=k-1}^{\infty} 2^{(q_1)_+(k-j)\alpha_-/2} \left\| \left(2^{j\alpha_+} |f_j|\right)^{q_1(\cdot)} \right\|_{L^{\frac{p_1(\cdot)}{q_1(\cdot)}}} \right)^{\frac{(q_2)_-}{(q_1)_+}} \left(\sum_{j=k-1}^{\infty} 2^{s(k-j)\alpha_-/2} \right)^{\frac{(q_2)_-}{s}} \\
&\lesssim \sum_{k=0}^{\infty} \left(\sum_{j=k-1}^{\infty} 2^{(q_1)_+(k-j)\alpha_-/2} \left\| \left(2^{j\alpha_+} |f_j|\right)^{q_1(\cdot)} \right\|_{L^{\frac{p_1(\cdot)}{q_1(\cdot)}}} \right)^{\frac{(q_2)_-}{(q_1)_+}} \\
&\le \left(\sum_{k=0}^{\infty} \sum_{j=k-1}^{\infty} 2^{(q_1)_+(k-j)\alpha_-/2} \left\| \left(2^{j\alpha_+} |f_j|\right)^{q_1(\cdot)} \right\|_{L^{\frac{p_1(\cdot)}{q_1(\cdot)}}} \right)^{\frac{(q_2)_-}{(q_1)_+}} \\
&\lesssim 1, \qquad (22)
\end{aligned}
$$

where we have used the Hölder inequality and Lemma 2.7. Hence by (21) and (22), we obtain $U_2 \lesssim 1$. Therefore we have

$$
\sum_{k=0}^{\infty} \left\| \left\{ 2^{k\alpha(\cdot)} |I^\beta f| \chi_k \right\}^{q_2(\cdot)} \right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}} \lesssim 1
$$

by (20). This completes the proof of Theorem 3.7. □

3.4 Commutators of the fractional integral

The commutator of the fractional integral I^β $(0<\beta<n)$ and $b \in \mathrm{BMO}(\mathbb{R}^n)$ is defined by

$$[b, I^\beta]f(x) := b(x)I^\beta f(x) - I^\beta(bf)(x).$$

We use the following theorem which is proved in [15].

Theorem 3.8. *Suppose that $p_1(\cdot) \in C^{\log}(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$. Let $0<\beta<n/(p_1)_+$. Define the variable exponent $p_2(\cdot)$ by*

$$\frac{1}{p_1(x)} - \frac{1}{p_2(x)} = \frac{\beta}{n}.$$

Then we have that for all $f \in L^{p_1(\cdot)}(\mathbb{R}^n)$ and $b \in \mathrm{BMO}(\mathbb{R}^n)$,

$$\left|\left|[b, I^\beta]f\right|\right|_{L^{p_2(\cdot)}} \lesssim ||b||_{\mathrm{BMO}} ||f||_{L^{p_1(\cdot)}}.$$

Theorem 3.9. *Let $\alpha(\cdot)$, $q_1(\cdot)$, $q_2(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$ and $p_2(\cdot) \in C^{\log}(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$. Take a constant $r \in (0, 1/(p_2')_+)$ so that (6) holds. Suppose that $0<\beta<nr$, $\alpha_+ < nr-\beta$ and $(q_1)_+ \le (q_2)_-$. Define the variable exponent $p_1(\cdot)$ by*

$$\frac{1}{p_1(x)} - \frac{1}{p_2(x)} = \frac{\beta}{n}.$$

Then, for all $f \in K_{p_1(\cdot)}^{\alpha_+, q_1(\cdot)}(\mathbb{R}^n)$ and all $b \in \mathrm{BMO}(\mathbb{R}^n)$ we have

$$\left|\left|[b, I^\beta]f\right|\right|_{K_{p_2(\cdot)}^{\alpha(\cdot), q_2(\cdot)}} \lesssim ||b||_{\mathrm{BMO}} ||f||_{K_{p_1(\cdot)}^{\alpha_+, q_1(\cdot)}}.$$

Proof. Take $f \in K_{p_1(\cdot)}^{\alpha_+, q_1(\cdot)}(\mathbb{R}^n)$ and $b \in \mathrm{BMO}(\mathbb{R}^n)$ arbitrarily. Let

$$\lambda_1 := \left|\left|\left\{ 2^{k\alpha(\cdot)} \left|[b, I^\beta]\left(\sum_{j=0}^{k-2} f_j\right)\chi_k\right|\right\}_{k=0}^{\infty}\right|\right|_{\ell^{q_2(\cdot)}(L^{p_2(\cdot)})},$$

$$\lambda_2 := \left|\left|\left\{ 2^{k\alpha(\cdot)} \left|[b, I^\beta]\left(\sum_{j=k-1}^{\infty} f_j\right)\chi_k\right|\right\}_{k=0}^{\infty}\right|\right|_{\ell^{q_2(\cdot)}(L^{p_2(\cdot)})}$$

and $\lambda := \lambda_1 + \lambda_2$. Then we have

$$\begin{aligned}\sum_{k=0}^{\infty}\left|\left|\left(\frac{2^{k\alpha(\cdot)}|[b, I^\beta](f)\chi_k|}{\lambda}\right)^{q_2(\cdot)}\right|\right|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}} &\lesssim \sum_{k=0}^{\infty}\left|\left|\left(\frac{2^{k\alpha(\cdot)}|[b, I^\beta](\sum_{j=0}^{k-2} f_j)\chi_k|}{\lambda_1}\right)^{q_2(\cdot)}\right|\right|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}} \\ &\quad + \sum_{k=0}^{\infty}\left|\left|\left(\frac{2^{k\alpha(\cdot)}|[b, I^\beta](\sum_{j=k-1}^{\infty} f_j)\chi_k|}{\lambda_2}\right)^{q_2(\cdot)}\right|\right|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}} \\ &\lesssim 1.\end{aligned}$$

This implies that $||[b, I^\beta]f||_{K_{p_2(\cdot)}^{\alpha(\cdot), q_2(\cdot)}} \lesssim \lambda_1 + \lambda_2$. It suffices to prove that

$$\lambda_1,\ \lambda_2 \lesssim ||f||_{K_{p_1(\cdot)}^{\alpha_+, q_1(\cdot)}}.$$

Below we will estimate λ_1 and λ_2 respectively. Let $\mu := ||f||_{K^{\alpha_+, q_1(\cdot)}_{p_1(\cdot)}}$. We write $g := \frac{f}{\mu}$, $g_j := \frac{f\chi_j}{\mu}$ and $\tilde{\chi_j} := \frac{\chi_j}{||b||_{\mathrm{BMO}}}$ for every $j \in \mathbb{N}_0$. Let

$$\mu_k := \left(\sum_{j=0}^{k-2} \left|\left| \left(2^{j\alpha_+} g_j\right)^{q_1(\cdot)} \right|\right|_{L^{\frac{p_1(\cdot)}{q_1(\cdot)}}}^{\frac{1}{(q_1)_+}} (k-j) 2^{(k-j)(\alpha_+ + \beta - nr)} \right)^{(q_2)_{*k}},$$

where

$$(q_2)_{*k} := \begin{cases} (q_2)_+ & \text{if } \sum_{j=0}^{k-2} \left|\left| \left(2^{j\alpha_+} g_j\right)^{q_1(\cdot)} \right|\right|_{L^{\frac{p_1(\cdot)}{q_1(\cdot)}}}^{\frac{1}{(q_1)_+}} (k-j) 2^{(k-j)(\alpha_+ + \beta - nr)} \geq 1, \\ (q_2)_- & \text{otherwise.} \end{cases}$$

Then we see that

$$\begin{aligned} &\int_{\mathbb{R}^n} \left(\frac{\left(2^{k\alpha(x)} \sum_{j=0}^{k-2} \left| [b, I^\beta]\left(\frac{f_j}{||b||_{\mathrm{BMO}}\mu}\right)(x) \right| \chi_k(x) \right)^{q_2(x)}}{\mu_k} \right)^{\frac{p_2(x)}{q_2(x)}} \mathrm{d}x \\ &\leq \int_{\mathbb{R}^n} \left(\frac{\left(2^{(k-j)\alpha(x)} \sum_{j=0}^{k-2} \left| [b, I^\beta](2^{j\alpha_+} g_j)(x) \right| \tilde{\chi}_k(x) \right)^{q_2(x)}}{\mu_k} \right)^{\frac{p_2(x)}{q_2(x)}} \mathrm{d}x \\ &\leq \int_{\mathbb{R}^n} \left(\frac{2^{(k-j)\alpha(x)} \sum_{j=0}^{k-2} \left| [b, I^\beta](2^{j\alpha_+} g_j)(x) \right| \tilde{\chi}_k(x)}{\sum_{j=0}^{k-2} \left|\left| (2^{j\alpha_+} g_j)^{q_1(\cdot)} \right|\right|_{L^{\frac{p_1(\cdot)}{q_1(\cdot)}}}^{\frac{1}{(q_1)_+}} (k-j) 2^{(k-j)(\alpha_+ + \beta - nr)}} \right)^{p_2(x)} \mathrm{d}x \leq 1. \end{aligned}$$

If $(q_1)_+ \leq 1$, then we have that by Lemma 2.7,

$$\begin{aligned} I_1 &:= \sum_{k=2}^{\infty} \left|\left| \left(\frac{2^{k\alpha} \sum_{j=0}^{k-2} \left| [b, I^\beta](f_j)(\cdot) \right| \chi_k(\cdot)}{||b||_{\mathrm{BMO}}\mu} \right)^{q_2(\cdot)} \right|\right|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}} \\ &\leq \sum_{k=2}^{\infty} \mu_k \\ &= \sum_{k=2}^{\infty} \left(\sum_{j=0}^{k-2} \left|\left| \left(2^{j\alpha_+} g_j\right)^{q_1(\cdot)} \right|\right|_{L^{\frac{p_1(\cdot)}{q_1(\cdot)}}}^{\frac{1}{(q_1)_+}} (k-j) 2^{(k-j)(\alpha_+ + \beta - nr)} \right)^{(q_2)_{*k}} \\ &\leq \sum_{k=2}^{\infty} \left(\sum_{j=0}^{k-2} \left|\left| \left(2^{j\alpha_+} g_j\right)^{q_1(\cdot)} \right|\right|_{L^{\frac{p_1(\cdot)}{q_1(\cdot)}}} (k-j)^{(q_1)_+} 2^{(q_1)_+(k-j)(\alpha_+ + \beta - nr)} \right)^{\frac{(q_2)_{*k}}{(q_1)_+}} \\ &\leq \left(\sum_{k=2}^{\infty} \sum_{j=0}^{k-2} \left|\left| \left(2^{j\alpha_+} g_j\right)^{q_1(\cdot)} \right|\right|_{L^{\frac{p_1(\cdot)}{q_1(\cdot)}}} (k-j)^{(q_1)_+} 2^{(q_1)_+(k-j)(\alpha_+ + \beta - nr)} \right)^{q_*} \lesssim 1, \end{aligned} \tag{23}$$

where $q_* \geq 1$ is a constant number as in Lemma 2.7. If $(q_1)_+ > 1$, then we have

$$\begin{aligned}
I_1 &\leq \sum_{k=2}^{\infty} \mu_k \\
&= \sum_{k=2}^{\infty} \left(\sum_{j=0}^{k-2} \left|\left| \left(2^{j\alpha_+} g_j\right)^{q_1(\cdot)} \right|\right|_{L^{\frac{p_1(\cdot)}{q_1(\cdot)}}}^{\frac{1}{(q_1)_+}} (k-j) 2^{(k-j)(\alpha_+ + \beta - nr)} \right)^{(q_2)_{*k}} \\
&\leq \sum_{k=2}^{\infty} \left\{ \left(\sum_{j=0}^{k-2} \left|\left| \left(2^{j\alpha_+} g_j\right)^{q_1(\cdot)} \right|\right|_{L^{\frac{p_1(\cdot)}{q_1(\cdot)}}} 2^{(q_1)_+(k-j)(\alpha_+ + \beta - nr)/2} \right)^{\frac{(q_2)_{*k}}{(q_1)_+}} \right. \\
&\qquad \left. \times \left(\sum_{j=0}^{k-2} (k-j)^{q_1'} 2^{q_1'(k-j)(\alpha_+ + \beta - nr)/2} \right)^{\frac{(q_2)_{*k}}{q_1'}} \right\} \\
&\lesssim \sum_{k=2}^{\infty} \left(\sum_{j=0}^{k-2} \left|\left| \left(2^{j\alpha_+} g_j\right)^{q_1(\cdot)} \right|\right|_{L^{\frac{p_1(\cdot)}{q_1(\cdot)}}} 2^{(q_1)_+(k-j)(\alpha_+ + \beta - nr)/2} \right)^{\frac{(q_2)_{*k}}{(q_1)_+}} \\
&\leq \left(\sum_{k=2}^{\infty} \sum_{j=0}^{k-2} \left|\left| \left(2^{j\alpha_+} g_j\right)^{q_1(\cdot)} \right|\right|_{L^{\frac{p_1(\cdot)}{q_1(\cdot)}}} 2^{(q_1)_+(k-j)(\alpha_+ + \beta - nr)/2} \right)^{q_*} \lesssim 1 \qquad (24)
\end{aligned}$$

by the Hölder inequality and Lemma 2.7. Hence we have $\lambda_1 \lesssim ||b||_{\mathrm{BMO}} ||f||_{K_{p_1(\cdot)}^{\alpha_+, q_1(\cdot)}}$ by (23) and (24). Finally we estimate λ_2. For any $k \in \mathbb{N}$, we define

$$(q_2)_{**k} := \begin{cases} (q_2)_+ & \text{if } \left|\left| \left(2^{k\alpha(\cdot)} |[b, I^\beta](\sum_{j=k-1}^{\infty} \frac{g_j}{||b||_{\mathrm{BMO}}}) \chi_k|\right)^{q_2(\cdot)} \right|\right|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}} \geq 1, \\ (q_2)_- & \text{otherwise.} \end{cases}$$

By Theorem 3.8, we see that

$$\begin{aligned}
I_2 &:= \sum_{k=1}^{\infty} \left|\left| \left(\frac{2^{k\alpha} \sum_{j=k-1}^{\infty} \left|[b, I^\beta](f_j)(\cdot)\right| \chi_k(\cdot)}{||b||_{\mathrm{BMO}} \mu} \right)^{q_2(\cdot)} \right|\right|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}} \\
&\leq \sum_{k=1}^{\infty} \left|\left| 2^{(k-j)\alpha(\cdot)} \left| [b, I^\beta] \left(\sum_{j=k-1}^{\infty} \frac{2^{j\alpha_+} g_j}{||b||_{\mathrm{BMO}}} \right) \right| \chi_k \right|\right|_{L^{p_2(\cdot)}}^{(q_2)_{**k}} \\
&\leq \sum_{k=1}^{\infty} \left(\sum_{j=k-1}^{\infty} 2^{(k-j)\alpha_-} \left|\left| [b, I^\beta] \left(\frac{2^{j\alpha_+} g_j}{||b||_{\mathrm{BMO}}} \right) \right| \chi_k \right|\right|_{L^{p_2(\cdot)}} \right)^{(q_2)_{**k}} \\
&\lesssim \sum_{k=1}^{\infty} \left(\sum_{j=k-1}^{\infty} 2^{(k-j)\alpha_-} \left|\left| 2^{j\alpha_+} g_j \right|\right|_{L^{p_1(\cdot)}} \right)^{(q_2)_{**k}} \\
&\lesssim \sum_{k=1}^{\infty} \left(\sum_{j=k-1}^{\infty} 2^{(k-j)\alpha_-} \left|\left| (2^{j\alpha_+} |g_j|)^{q_1(\cdot)} \right|\right|_{L^{\frac{p_1(\cdot)}{q_1(\cdot)}}}^{\frac{1}{(q_1)_+}} \right)^{(q_2)_{**k}}. \qquad (25)
\end{aligned}$$

If $(q_1)_+ \le 1$, then we see that by Lemma 2.7 and (25),

$$\begin{aligned} I_2 &\le \sum_{k=1}^{\infty}\left(\sum_{j=k-1}^{\infty} 2^{(q_1)_+(k-j)\alpha_-}\left|\left|(2^{j\alpha_+}|g_j|)^{q_1(\cdot)}\right|\right|_{L^{\frac{p_1(\cdot)}{q_1(\cdot)}}}\right)^{\frac{(q_2)_{**k}}{(q_1)_+}} \\ &\le \left(\sum_{k=1}^{\infty}\sum_{j=k-1}^{\infty} 2^{(q_1)_+(k-j)\alpha_-}\left|\left|(2^{j\alpha_+}|g_j|)^{q_1(\cdot)}\right|\right|_{L^{\frac{p_1(\cdot)}{q_1(\cdot)}}}\right)^{q_*} \lesssim 1, \end{aligned} \tag{26}$$

where $q_* \ge 1$ is a constant number as in Lemma 2.7. If $(q_1)_+ > 1$, then we write $s := ((q_1)_+)'$ to get

$$\begin{aligned} I_2 &\le \sum_{k=1}^{\infty}\left(\sum_{j=k-1}^{\infty} 2^{(q_1)_+(k-j)\alpha_-/2}\left|\left|(2^{j\alpha_+}|g_j|)^{q_1(\cdot)}\right|\right|_{L^{\frac{p_1(\cdot)}{q_1(\cdot)}}}\right)^{\frac{(q_2)_{**k}}{(q_1)_+}}\left(\sum_{j=k-1}^{\infty} 2^{s(k-j)\alpha_-/2}\right)^{\frac{(q_2)_{**k}}{s}} \\ &\lesssim \sum_{k=1}^{\infty}\left(\sum_{j=k-1}^{\infty} 2^{(q_1)_+(k-j)\alpha_-/2}\left|\left|(2^{j\alpha_+}|g_j|)^{q_1(\cdot)}\right|\right|_{L^{\frac{p_1(\cdot)}{q_1(\cdot)}}}\right)^{\frac{(q_2)_{**k}}{(q_1)_+}} \\ &\le \left(\sum_{k=1}^{\infty}\sum_{j=k-1}^{\infty} 2^{(q_1)_+(k-j)\alpha_-/2}\left|\left|(2^{j\alpha_+}|g_j|)^{q_1(\cdot)}\right|\right|_{L^{\frac{p_1(\cdot)}{q_1(\cdot)}}}\right)^{q_*} \\ &\lesssim 1 \end{aligned} \tag{27}$$

by the Hölder inequality, Lemma 2.7 and (25). Hence we have $\lambda_2 \lesssim ||b||_{\mathrm{BMO}}||f||_{K_{p_1(\cdot)}^{\alpha_+,q_1(\cdot)}}$ by (26) and (27). This completes the proof of Theorem 3.9. □

4 Boundedness of operators on the homogeneous Herz spaces

We can also define the homogeneous Herz spaces with variable exponents by analogy with the definition of the non-homogeneous case.

Definition 4.1. Let $p(\cdot)$, $q(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$ and $\alpha(\cdot) : \mathbb{R}^n \to \mathbb{R}$ such that $-\infty < \alpha_- \le \alpha_+ < \infty$. The homogeneous Herz space $\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)$ is the collection of $f \in L_{\mathrm{loc}}^{p(\cdot)}(\mathbb{R}^n \setminus \{0\})$ such that

$$||f||_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}} := \inf\left\{\lambda > 0 \,:\, \sum_{k=-\infty}^{\infty}\left|\left|\left(\frac{2^{k\alpha(\cdot)}|f\chi_{B_k\setminus B_{k-1}}|}{\lambda}\right)^{q(\cdot)}\right|\right|_{L^{\frac{p(\cdot)}{q(\cdot)}}} \le 1\right\} < \infty.$$

If the variable exponent $\alpha(\,\cdot\,)$ equals to a constant, then our main results, namely Theorems 3.1, 3.4, 3.7, 3.9, are true for the homogeneous Herz spaces because the proofs of those theorems are directly applicable. But it remains an open problem on the boundedness in the case of general variable exponent $\alpha(\,\cdot\,)$.

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References

[1] A. Almeida and P. Hästö, *Besov spaces with variable smoothness and integrability*, J. Funct. Anal. **258** (2010), 1628–1655.

[2] C. Capone, D. Cruz-Uribe, SFO and A. Fiorenza, *The fractional maximal operator and fractional integrals on variable L^p spaces*, Rev. Mat. Iberoamericana **23** (2007), 743–770.

[3] D. Cruz-Uribe, L. Diening and A. Fiorenza, *A new proof of the boundedness of maximal operators on variable Lebesgue spaces*, Bull. Unione mat. Ital. (9) **2 (1)** (2009), 151–173.

[4] D. Cruz-Uribe, SFO, A. Fiorenza, J.M. Martell and C. Pérez, *The boundedness of classical operators on variable L^p spaces*, Ann. Acad. Sci. Fenn. Math. **31** (2006), 239–264.

[5] D. Cruz-Uribe, A. Fiorenza and C.J. Neugebauer, *The maximal function on variable L^p spaces*, Ann. Acad. Sci. Fenn. Math. **28** (2003), 223–238, and **29** (2004), 247–249.

[6] L. Diening, *Riesz potential and Sobolev embeddings on generalized Lebesgue and Sobolev spaces $L^{p(\cdot)}$ and $W^{k,p(\cdot)}$*, Math. Nachr. **268** (2004), 31–43.

[7] L. Diening, *Maximal functions on generalized Lebesgue spaces $L^{p(\cdot)}$*, Math. Inequal. Appl. **7** (2004), 245–253.

[8] L. Diening, *Maximal function on Musielak–Orilcz spaces and generalized Lebesgue spaces*, Bull. Sci. Math. **129** (2005), 657–700.

[9] L. Diening, P. Harjulehto, P. Hästö, Y. Mizuta and T. Shimomura, *Maximal functions in variable exponent spaces: limiting cases of the exponent*, Ann. Acad. Sci. Fenn. Math. **34** (2009), 503–522.

[10] L. Diening, P. Hästö and S. Roudenko, *Function spaces of variable smoothness and integrability*, J. Funct. Anal. **256** (2009), 1731–1768.

[11] M. Izuki, *Herz and amalgam spaces with variable exponent, the Haar wavelets and greediness of the wavelet system*, East J. Approx. **15** (2009), 87–109.

[12] M. Izuki, *Vector-valued inequalities on Herz spaces and characterizations of Herz–Sobolev spaces with variable exponent*, Glasnik Mat. **45** (2010), 475–503.

[13] M. Izuki, *Boundedness of sublinear operators on Herz spaces with variable exponent and application to wavelet characterization*, Anal. Math. **36** (2010), 33–50.

[14] M. Izuki, *Boundedness of commutators on Herz spaces with variable exponent*, Rend. Circ. Mat. Palermo **59** (2010), 199–213.

[15] M. Izuki, *Commutators of fractional integrals on Lebesgue and Herz spaces with variable exponent*, Rend. Circ. Mat. Palermo **59** (2010), 461–472.

[16] M. Izuki *Fractional integrals on Herz–Morrey spaces with variable exponent*, Hiroshima Math. J. **40** (2010), 343–355.

[17] M. Izuki and Y. Sawano, *Variable Lebesgue norm estimates for* BMO *functions*, submitted.

[18] Y. Jiang, L. Tang and D. Yang, *Continuity of commutators on weighted Herz-type spaces*, Kyushu J. Math. **53** (1999), 245–263.

[19] A. Yu. Karlovich and A.K. Lerner, *Commutators of singular integrals on generalized L^p spaces with variable exponent*, Publ. Mat. **49** (2005), 111–125.

[20] H. Kempka and J. Vybíral, *A note on the spaces of variable integrability and summability of Almeida and Hästö*, arXiv. 1102. 1597v1.

[21] T.S. Kopaliani, *Infimal convolution and Muckenhoupt $A_{p(\cdot)}$ condition in variable L_p spaces*, Arch. Math. **89** (2007), 185–192.

[22] O. Kováčik and J. Rákosník, *On spaces $L^{p(x)}$ and $W^{k,p(x)}$*, Czech. Math. J. **41** (1991), 592–618.

[23] A.K. Lerner, *On some questions related to the maximal operator on variable L^p spaces*, Trans. Amer. Math. Soc. **362** (2010), 4229–4242.

[24] X. Li and D. Yang, *Boundedness of some sublinear operators on Herz spaces*, Illinois J. Math. **40** (1996), 484–501.

[25] S. Lu and D. Yang, *The decomposition of the weighted Herz spaces and its application*, Sci. China (Ser. A) **38** (1995), 147–158.

[26] S.Z. Lu and D.C. Yang, *Hardy–Littlewood–Sobolev theorems of fractional integration on Herz-type spaces and its applications*, Can. J. Math. **48** (1996), 363–380.

[27] S. Lu and D. Yang, *The continuity of commutators on Herz-type spaces*, Michigan Math. J. **44** (1997), 255–281.

[28] A. Nekvinda, *Hardy–Littlewood maximal operator on $L^{p(x)}(\mathbb{R}^n)$*, Math. Inequal. Appl. **7** (2004), 255–265.

[29] L. Pick and M. Růžička, *An example of a space $L^{p(\cdot)}$ on which the Hardy–Littlewood maximal operator is not bounded*, Expo. Math. **19** (2001), 369–371.