Asymptotic behavior of least energy solutions for a 2D nonlinear Neumann problem with large exponent

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Abstract. In this paper, we consider the following elliptic problem with the nonlinear Neumann boundary condition:

\[ (E_p) \begin{cases} -\Delta u + u = 0 & \text{on } \Omega, \\ u > 0 & \text{on } \Omega, \\ \frac{\partial u}{\partial \nu} = u^p & \text{on } \partial \Omega, \end{cases} \]

where \( \Omega \) is a smooth bounded domain in \( \mathbb{R}^2 \), \( \nu \) is the outer unit normal vector to \( \partial \Omega \), and \( p > 1 \) is any positive number.

We study the asymptotic behavior of least energy solutions to \((E_p)\) when the nonlinear exponent \( p \) gets large. Following the arguments of X. Ren and J.C. Wei [10], [11], we show that the least energy solutions remain bounded uniformly in \( p \), and it develops one peak on the boundary, the location of which is controlled by the Green function associated to the linear problem.

Keywords: least energy solution, nonlinear Neumann boundary condition, large exponent, concentration.


1. Introduction.

In this paper, we consider the following elliptic problem with the nonlinear Neumann boundary condition:

\[ (E_p) \begin{cases} -\Delta u + u = 0 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = u^p & \text{on } \partial \Omega, \end{cases} \]
where $\Omega$ is a smooth bounded domain in $\mathbb{R}^2$, $\nu$ is the outer unit normal vector to $\partial \Omega$, and $p > 1$ is any positive number. Let $H^1(\Omega)$ be the usual Sobolev space with the norm $\|u\|_{H^1(\Omega)}^2 = \int_\Omega (|\nabla u|^2 + u^2) \, dx$. Since the trace Sobolev embedding $H^1(\Omega) \hookrightarrow L^{p+1}(\partial \Omega)$ is compact for any $p > 1$, we can obtain at least one solution of (1.1) by a standard variational method. In fact, let us consider the constrained minimization problem

$$C_p^2 = \inf \left\{ \int_\Omega (|\nabla u|^2 + u^2) \, dx \mid u \in H^1(\Omega), \int_{\partial \Omega} |u|^{p+1} ds_x = 1 \right\}. \quad (1.2)$$

Standard variational method implies that $C_p^2$ is achieved by a positive function $u_p \in H^1(\Omega)$ and then $u_p = C_p^{2/(p-1)} \bar{u}_p$ solves (1.1). We call $u_p$ a least energy solution to the problem (1.1).

In this paper, we prove the followings:

**Theorem 1** Let $u_p$ be a least energy solution to $(E_p)$. Then it holds

$$1 \leq \liminf_{p \to \infty} \|u_p\|_{L^\infty(\partial \Omega)} \leq \limsup_{p \to \infty} \|u_p\|_{L^\infty(\partial \Omega)} \leq \sqrt{e}.$$ 

To state further results, we set

$$v_p = u_p / \left( \int_{\partial \Omega} u_p^p ds_x \right). \quad (1.3)$$

**Theorem 2** Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded domain. Then for any sequence $v_{p_n}$ of $v_p$ defined in (1.3) with $p_n \to \infty$, there exists a subsequence (still denoted by $v_{p_n}$) and a point $x_0 \in \partial \Omega$ such that the following statements hold true.

1. $$f_n = \frac{u_{p_n}^p}{\int_{\partial \Omega} u_{p_n}^p ds_x} \rightharpoonup \delta_{x_0}$$
   in the sense of Radon measures on $\partial \Omega$.

2. $v_{p_n} \to G(\cdot, x_0)$ in $C^1_{loc}(\overline{\Omega} \setminus \{x_0\})$, $L^1(\Omega)$ and $L^1(\partial \Omega)$ respectively for any $1 \leq t < \infty$, where $G(x,y)$ denotes the Green function of $-\Delta$ for the following Neumann problem:

$$\begin{cases}
-\Delta_x G(x, y) + G(x, y) = 0 & \text{in } \Omega, \\
\frac{\partial G}{\partial \nu_x}(x, y) = \delta_y(x) & \text{on } \partial \Omega.
\end{cases} \quad (1.4)$$
(3) $x_0$ satisfies
\[ \nabla_{\tau(x_0)} R(x_0) = 0, \]
where $\tau(x_0)$ denotes a tangent vector at the point $x_0 \in \partial \Omega$ and $R$ is the Robin function defined by $R(x) = H(x, x)$, where
\[ H(x, y) := G(x, y) - \frac{1}{\pi} \log |x - y|^{-1} \]
denotes the regular part of $G$.

Concerning related results, X. Ren and J.C. Wei [10], [11] first studied the asymptotic behavior of least energy solutions to the semilinear problem
\[
\begin{cases}
-\Delta u = u^p & \text{in } \Omega, \\
u > 0 & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega
\end{cases}
\]
as $p \to \infty$, where $\Omega$ is a bounded smooth domain in $\mathbb{R}^2$. They proved that the least energy solutions remain bounded and bounded away from zero in $L^\infty$-norm uniformly in $p$. As for the shape of solutions, they showed that the least energy solutions must develop one “peak” in the interior of $\Omega$, which must be a critical point of the Robin function associated with the Green function subject to the Dirichlet boundary condition. Later, Adimurthi and Grossi [1] improved their results by showing that, after some scaling, the limit profile of solutions is governed by the Liouville equation
\[ -\Delta U = e^U \quad \text{in } \mathbb{R}^2, \quad \int_{\mathbb{R}^2} e^U \, dx < \infty, \]
and obtained that $\lim_{p \to \infty} \|u_p\|_{L^\infty(\Omega)} = \sqrt{e}$ for least energy solutions $u_p$.

Actual existence of concentrating solutions to (1.1) is recently obtained by H. Castro [4] by a variational reduction procedure, along the line of [7] and [6]. Also in our case, we may conjecture that the limit problem of (1.1) is
\[
\begin{cases}
\Delta U = 0 & \text{in } \mathbb{R}_+^2, \\
\frac{\partial U}{\partial \nu} = e^U & \text{on } \partial \mathbb{R}_+^2, \\
\int_{\partial \mathbb{R}_+^2} e^U \, ds < \infty,
\end{cases}
\]
and $\lim_{p \to \infty} \|u_p\|_{L^\infty(\partial \Omega)} = \sqrt{e}$ holds true at least for least energy solutions $u_p$. Verification of these conjectures remains as the future work.

In this section, we provide some estimates for $C^2_p$ in (1.2) as $p \to \infty$.

**Lemma 3** For any $s \geq 2$, there exists $\tilde{D}_s > 0$ such that for any $u \in H^1(\Omega)$,

$$
\|u\|_{L^s(\partial\Omega)} \leq \tilde{D}_s s^{\frac{1}{2}} \|u\|_{H^1(\Omega)}
$$

holds true. Furthermore, we have

$$
\lim_{s \to \infty} \tilde{D}_s = \left(2\pi e\right)^{-\frac{1}{2}}.
$$

**Proof.** Let $u \in H^1(\Omega)$. By Trudinger-Moser trace inequality, see [5] and the references therein, we have

$$
\int_{\partial\Omega} \exp \left( \frac{\pi |u(x) - u_{\partial\Omega}|^2}{\|\nabla u\|_{L^2(\Omega)}^2} \right) ds_x \leq C(\Omega)
$$

for any $u \in H^1(\Omega)$, where $u_{\partial\Omega} = \frac{1}{|\partial\Omega|} \int_{\partial\Omega} u ds_x$. Thus, by an elementary inequality $\frac{x^s}{\Gamma(s+1)} \leq e^x$ for any $x \geq 0$ and $s \geq 0$, where $\Gamma(s)$ is the Gamma function, we see

$$
\frac{1}{\Gamma((s/2) + 1)} \int_{\partial\Omega} |u - u_{\partial\Omega}|^s ds_x
$$

$$
= \frac{1}{\Gamma((s/2) + 1)} \int_{\partial\Omega} \left( \frac{\pi |u(x) - u_{\partial\Omega}|^2}{\|\nabla u\|_{L^2(\Omega)}^2} \right)^{s/2} ds_x \pi^{-s/2} \|\nabla u\|_{L^2(\Omega)}^s
$$

$$
\leq \int_{\partial\Omega} \exp \left( \frac{\pi |u(x) - u_{\partial\Omega}|^2}{\|\nabla u\|_{L^2(\Omega)}^2} \right) ds_x \pi^{-s/2} \|\nabla u\|_{L^2(\Omega)}^s
$$

$$
\leq C(\Omega) \pi^{-s/2} \|\nabla u\|_{L^2(\Omega)}^s.
$$

Set

$$
D_s := (\Gamma(s/2 + 1))^{1/s} C(\Omega)^{1/s} \pi^{-1/2} s^{-1/2}.
$$

Then we have

$$
\|u - u_{\partial\Omega}\|_{L^s(\partial\Omega)} \leq D_s s^{1/2} \|\nabla u\|_{L^2(\Omega)}.
$$

Stirling’s formula says that $(\Gamma(s/2 + 1))^{1/s} \sim \left(\frac{s}{2\pi e}\right)^{1/2}$ as $s \to \infty$, so we have

$$
\lim_{s \to \infty} D_s = \left(2\pi e\right)^{-\frac{1}{2}}.
$$
On the other hand, by the embedding \( \|u\|_{L^2(\partial \Omega)} \leq C(\Omega) \|u\|_{H^1(\Omega)} \) for any \( u \in H^1(\Omega) \), we see
\[
|u_{\partial \Omega}| \leq \frac{1}{|\partial \Omega|^{1/2}} \left( \int_{\partial \Omega} |u|^2 ds_x \right)^{1/2} \leq \frac{C(\Omega)}{|\partial \Omega|^{1/2}} \|u\|_{H^1(\Omega)}.
\]
Thus,
\[
\|u\|_{L^s(\partial \Omega)} \leq \|u - u_{\partial \Omega}\|_{L^s(\partial \Omega)} + \|u_{\partial \Omega}\|_{L^s(\partial \Omega)} \leq \|u - u_{\partial \Omega}\|_{L^s(\partial \Omega)} + |u_{\partial \Omega}| |\partial \Omega|^{1/s} \leq s^{1/2} \|u\|_{H^1(\Omega)} \left( D(s) + \frac{C(\Omega)|\partial \Omega|^{1/s-1/2}}{s^{1/2}} \right).
\]

Put \( \tilde{D}(s) = D(s) + \frac{C(\Omega)|\partial \Omega|^{1/s-1/2}}{s^{1/2}} \). Then, we have \( \lim_{s \to \infty} \tilde{D}(s) = \lim_{s \to \infty} D(s) = \frac{1}{\sqrt{2\pi e}} \) and
\[
\|u\|_{L^s(\partial \Omega)} \leq \tilde{D}_s s^{1/2} \|u\|_{H^1(\Omega)}
\]
holds.

**Lemma 4** Let \( \Omega \) be a smooth bounded domain in \( \mathbb{R}^2 \). Then we have
\[
\lim_{p \to \infty} pC_p^2 = 2\pi e.
\]

**Proof.** For the estimate from below, we use Lemma 3. By Lemma 3, we have
\[
\|u\|_{L^{p+1}(\partial \Omega)}^2 \leq \tilde{D}_{p+1}^2 (p + 1) \|u\|_{H^1(\Omega)}^2
\]
for any \( u \in H^1(\Omega) \), which leads to \( \tilde{D}_{p+1}^{-2} \left( \frac{p}{p+1} \right) \leq pC_p^2 \). Thus, we have
\[
2\pi e \leq \liminf_{p \to \infty} pC_p^2,
\]
since \( \lim_{p \to \infty} \tilde{D}_{p+1} = (2\pi e)^{-1/2} \).

For the estimate from above, we use the Moser function. Let \( 0 < l < L \).

First, we assume \( \Omega \cap B_L(0) = \Omega \cap B_L^+ \) where \( B_L^+ = B_L(0) \cap \{y = (y_1, y_2) \mid y_2 > 0\} \). Define
\[
m_l(y) = \frac{1}{\sqrt{\pi}} \begin{cases} 
(\log L/l)^{1/2} & \text{if } 0 \leq |y| \leq l, y \in B_L^+, \\
(\log L/|y|)|^{1/2} & \text{if } l \leq |y| \leq L, y \in B_L^+, \\
0 & \text{if } L \leq |y|, y \in B_L^+.
\end{cases}
\]

5
Then \( \| \nabla m_l \|_{L^2(B_L^+)} = 1 \) and since \( m_l \equiv 0 \) on \( \partial B_L^+ \cap \{ y_2 > 0 \} \), we have

\[
\| m_l \|_{L^{p+1}(\partial B_L^+)}^{p+1} = 2 \int_0^L |m_l(y_1)|^{p+1} dy_1 + 2 \int_L^L |m_l(y_1)|^{p+1} dy_1 \\
\geq 2 \int_0^L \left( \frac{1}{\sqrt{\pi}} \sqrt{\log(L/l)} \right)^{p+1} dy_1 = 2l \left( \frac{1}{\pi} \log \left( \frac{L}{l} \right) \right)^{p+1}.
\]

Thus \( \| m_l \|^2_{L^{p+1}(\partial B_L^+)} \geq (2l)^{\frac{2}{p+1}} \frac{1}{\pi} \log(L/l) \). Also,

\[
\| m_l \|^2_{L^2(B_L^+)} = \int_0^\pi \int_0^L |m_l|^2 rdrd\theta \\
= \int_0^\pi \int_0^l |m_l|^2 rdrd\theta + \int_0^\pi \int_l^L |m_l|^2 rdrd\theta \\
=: I_1 + I_2.
\]

We calculate

\[
I_1 = \frac{l^2}{2} \log(L/l),
\]

\[
I_2 = \frac{1}{\log(L/l)} \int_l^L (\log L/r)^2 rdr \\
= -\frac{l^2}{2} - \frac{l^2}{2} \log(L/l) + \frac{1}{\log(L/l)} \frac{L^2 - l^2}{4}.
\]

Thus we have \( \| m_l \|^2_{L^2(B_L^+)} = -\frac{l^2}{2} + \frac{1}{\log(L/l)} \frac{L^2 - l^2}{4} \).

Now, put \( l = Le^{-\frac{L}{2L}} \) and extend \( m_l \) by 0 outside \( B_L^+ \) and consider it as a function in \( H^1(\Omega) \). Then

\[
pC_p^2 \leq p \frac{\| m_l \|^2_{H^1(B_L^+)}}{\| m_l \|^2_{L^{p+1}(\partial B_L^+)}} = \frac{p}{\| m_l \|^2_{L^{p+1}(\partial B_L^+)}} + \frac{p}{\| m_l \|^2_{L^{p+1}(\partial B_L^+)}}.
\]

We estimate

\[
\frac{p}{\| m_l \|^2_{L^{p+1}(\partial B_L^+)}} \leq \frac{p}{(2l)^{\frac{2}{p+1}} \frac{1}{\pi} \log(L/l)} = \left( \frac{p}{p+1} \right) 2\pi e - \frac{1}{(2L)^{\frac{2}{p+1}}} \to 2\pi e,
\]

6
and
\[
\frac{p\|m_1\|^2_{L^2(B^+_L)}}{\|m_1\|^2_{L^{p+1}(\partial B^+_L)}} \leq \frac{p}{(2L)^\frac{2}{p+1}} \left( \frac{L^2}{2} e^{-(p+1)} + \frac{L^2(1-e^{-(p+1)})}{p+1} \right) \rightarrow 0
\]
as \( p \rightarrow \infty \). Therefore, we have obtained \( \limsup_{p \rightarrow \infty} pC_p^2 \leq 2\pi e \) in this case.

In the general case, we introduce a diffeomorphism which flattens the boundary \( \partial \Omega \), see Ni and Takagi [9]. We may assume \( 0 \in \partial \Omega \) and in a neighborhood \( U \) of 0, the boundary \( \partial \Omega \) can be written by the graph of function \( \psi: \partial \Omega \cap U = \{ x = (x_1, x_2) \mid x_2 = \psi(x_1) \} \), with \( \psi(0) = 0 \) and \( \frac{\partial \psi}{\partial x_1}(0) = 0 \).

Define \( x = \Phi(y) = (\Phi_1(y), \Phi_2(y)) \) for \( y = (y_1, y_2) \), where
\[
x_1 = \Phi_1(y) = y_1 - y_2 \frac{\partial \psi}{\partial x_1}(y_1), \quad x_2 = \Phi_2(y) = y_2 + \psi(y_1),
\]
and put \( D_L = \Phi(B^+_L) \). Note that \( \partial D_L \cap \partial \Omega = \Phi(\partial B^+_L \cap \{(y_1, 0)\}) \). Since \( D\Phi(0) = Id \), we obtain there exists \( \Psi = \Phi^{-1} \) in a neighborhood of 0. Finally, define \( \tilde{m}_I \in H^1(\Omega) \) as \( \tilde{m}_I(x) = m_I(\Psi(x)) \) for \( x \in U \cap \Omega \). Then, Lemma A.1 in [9] implies the estimates
\[
\|\nabla \tilde{m}_I\|_{L^2(D_L)}^2 = \|\nabla m_1\|_{L^2(B^+_L)}^2 + O(\frac{1}{p}),
\]
\[
\|\tilde{m}_I\|_{L^2(D_L)}^2 \leq (1 + O(L)) \|m_1\|_{L^2(B^+_L)}^2,
\]
\[
\|\tilde{m}_I\|_{L^{p+1}(\partial D_L \cap \partial \Omega)}^2 \geq \|m_1\|_{L^{p+1}(\partial B^+_L \cap \{(y_1, 0)\})}^2.
\]
The last inequality comes from that, if we put \( J = \{(y_1, 0) \mid L \leq y_1 \leq L\} \subset \partial B^+_L \) and \( J = \Phi(I) \subset \partial \Omega \), then \( ds_x = \sqrt{1 + (\psi'(x_1))^2} dx_1 \) and \( J = \{(x_1, x_2) \mid x_1 = y_1, x_2 = \psi(y_1)\} \). Thus
\[
\int_I |\tilde{m}_I(x)|^{p+1} ds_x = \int_I |m_I(y)|^{p+1} \sqrt{1 + (\psi'(y_1))^2} dy_1 \geq \int_I |m_I(y)|^{p+1} dy_1.
\]
By testing \( C^2_p \) with \( \tilde{m}_I \), again we obtain \( \limsup_{p \rightarrow \infty} pC_p^2 \leq 2\pi e \).

Corollary 5 Let \( u_p \) be a least energy solution to \((E_p)\). Then we have
\[
\lim_{p \rightarrow \infty} p \int_{\partial \Omega} u_p^{p+1} ds_x = 2\pi e, \quad \lim_{p \rightarrow \infty} p \int_{\Omega} (|\nabla u_p|^2 + u_p^2) dx = 2\pi e.
\]
Proof. Since \( u_p \) satisfies
\[
\int_{\Omega} (|\nabla u_p|^2 + u_p^2) \, dx = \int_{\partial\Omega} u_p^{p+1} \, ds_x
\]
and
\[
pC_p^2 = p\int_{\Omega} (|\nabla u_p|^2 + u_p^2) \, dx \left( \int_{\partial\Omega} u_p^{p+1} \, ds_x \right)^{\frac{p+1}{p}} = \left( p \int_{\partial\Omega} u_p^{p+1} \, ds_x \right)^{\frac{p+1}{p+1}} \frac{2}{p+1},
\]
the results follow from Lemma 4.

3. Proof of Theorem 1.

The uniform estimate of \( \|u\|_{L^\infty(\partial\Omega)} \) from below holds true for any solution \( u \) of \((E_p)\), as in [10].

Lemma 6 There exists \( C_1 > 0 \) independent of \( p \) such that
\[
\|u\|_{L^\infty(\partial\Omega)} \geq C_1
\]
holds true for any solution \( u \) to \((E_p)\).

Proof. Let \( \lambda_1 > 0 \) be the first eigenvalue of the eigenvalue problem
\[
\begin{cases}
-\Delta \varphi + \varphi = 0 & \text{in } \Omega, \\
\frac{\partial \varphi}{\partial \nu} = \lambda \varphi & \text{on } \partial\Omega
\end{cases}
\]
and let \( \varphi_1 \) be the corresponding eigenfunction. It is known that \( \lambda_1 \) is simple, isolated, and \( \varphi_1 \) can be chosen positive on \( \overline{\Omega} \). (see, [12]). Then by integration by parts, we have
\[
0 = \int_{\Omega} \{(-\Delta u + u) \varphi_1 - (-\Delta \varphi_1 + \varphi_1) u \} \, dx = \int_{\partial\Omega} \left( \frac{\partial \varphi_1}{\partial \nu} u - \frac{\partial u}{\partial \nu} \varphi_1 \right) \, ds_x
\]
\[
= \int_{\partial\Omega} \varphi_1 u (\lambda_1 - u^{p-1}) \, ds_x.
\]
Since \( \varphi_1 u > 0 \) on \( \partial\Omega \), this implies \( \|u\|_{L^\infty(\partial\Omega)}^{p-1} \geq \lambda_1. \)

\[\square\]
Lemma 7 Let $u_p$ be a least energy solution to $(E_p)$. Then it holds

$$\limsup_{p \to \infty} \|u_p\|_{L^\infty(\partial \Omega)} \leq \sqrt{e}.$$ 

Proof. We follow the argument of [11], which in turn originates from [8], and use Moser's iteration procedure. Let $u$ be a solution to $(E_p)$. For $s \geq 1$, multiplying $u^{2s-1} \in H^1(\Omega)$ to the equation of $(E_p)$ and integrating, we get

$$\left(\frac{2s-1}{s^2}\right)^2 \int_{\Omega} |\nabla (u^s)|^2 dx + \int_{\Omega} u^{2s} dx = \int_{\partial \Omega} u^{2s-1+p} ds_x.$$ 

Since $\frac{2s-1}{s^2} \leq 1$ for $s \geq 1$, we have

$$\left(\frac{2s-1}{s^2}\right) \|u^s\|^2_{H^1(\Omega)} \leq \int_{\partial \Omega} u^{2s-1+p} ds_x. \tag{3.1}$$

Also by Lemma 3 applied to $u^s \in H^1(\Omega)$, we have

$$\left(\int_{\partial \Omega} u^{\nu s} ds_x\right)^{1/\nu} \leq \hat{D}_\nu^2 \|u^s\|^2_{H^1(\Omega)}$$

for any $\nu \geq 2$. Thus by (3.1), we see

$$\left(\int_{\partial \Omega} u^{\nu s} ds_x\right)^{1/\nu} \leq \hat{D}_\nu \nu \left(\frac{s^2}{2s-1}\right)^{1/2} \left(\int_{\partial \Omega} u^{2s-1+p} ds_x\right)^{1/2}.$$ 

Since $\hat{D}_\nu^2 \left(\frac{s}{2s-1}\right) \leq C_1$ for some $C_1 > 0$ independent of $s \geq 1$ and $\nu \geq 2$, we obtain

$$\left(\int_{\partial \Omega} u^{\nu s} ds_x\right)^{2/\nu} \leq C_1 \nu s \int_{\partial \Omega} u^{2s-1+p} ds_x. \tag{3.2}$$

Once the iteration scheme (3.2) is obtained, the rest of the argument is exactly the same as one in [11]. Indeed, by Lemma 3, we have

$$\left(\int_{\partial \Omega} u^\nu ds_x\right)^{1/\nu} \leq (2\pi e)^{-\frac{1}{2}} (1 + o(1)) \nu^{1/2} \|u\|_{H^1(\Omega)}, \tag{3.3}$$

here $o(1) \to 0$ as $\nu \to \infty$. Now, we fix $\alpha > 0$ and $\epsilon > 0$ which will be chosen small later and put $\nu = (1 + \alpha)(p + 1) > 2$ in (3.3). By Corollary 5,
\[ p^{1/2}(2\pi e)^{-1/2}\|u_p\|_{H^1(\Omega)} \to 1 \] as \( p \to \infty \) for a least energy solution \( u_p \). Thus by (3.3), we see there exists \( p_0 > 1 \) such that
\[
\int_{\partial \Omega} u_p^{\nu} ds_x \leq (1 + \alpha + \varepsilon)^{\nu/2} =: M_0
\]
for \( p > p_0 \). Define \( \{ s_j \}_{j=0,1,2,\ldots} \) and \( \{ M_j \}_{j=0,1,2,\ldots} \) such that
\[
\begin{align*}
& p - 1 + 2s_0 = \nu, \\
& p - 1 + 2s_{j+1} = \nu s_j, \quad (j = 0, 1, 2, \ldots),
\end{align*}
\]
and
\[
\begin{align*}
& M_0 = (1 + \alpha + \varepsilon)^{\nu/2}, \\
& M_{j+1} = (C_1 \nu s_j M_j)^{\nu/2}, \quad (j = 0, 1, 2, \ldots).
\end{align*}
\]
We easily see that \( s_0 = \frac{\alpha(p+1)}{2} > 0 \), \( s_j \) is increasing in \( j \), \( s_j \to +\infty \) as \( j \to \infty \), and actually,
\[
s_j = \left( \frac{\nu}{2} \right)^j (s_0 - x) + x \quad \text{where} \quad x = \frac{p - 1}{\nu - 2} > 0.
\]
At this moment, we can follow exactly the same argument in [11] to obtain the estimates
\[
\|u_p\|_{L^{\nu s_j - 1}(\partial \Omega)} \leq M_j^{\nu s_j - 1} \leq \exp(m(\alpha, p, \varepsilon)),
\]
where \( m(\alpha, p, \varepsilon) \) is a constant depending on \( \alpha, p \) and \( \varepsilon \), satisfying
\[
\lim_{p \to \infty} m(\alpha, p, \varepsilon) = \frac{1 + \alpha}{2\alpha} \log(1 + \alpha + \varepsilon).
\]
Letting \( j \to \infty \), \( p \to \infty \) first, we get
\[
\limsup_{p \to \infty} \|u_p\|_{L^\infty(\partial \Omega)} \leq (1 + \alpha + \varepsilon)^{\frac{1+\alpha}{2\alpha}},
\]
and then letting \( \alpha \to +0 \), \( \varepsilon \to +0 \), we obtain
\[
\limsup_{p \to \infty} \|u_p\|_{L^\infty(\partial \Omega)} \leq \sqrt{e}
\]
as desired. \( \square \)

By Theorem 1 and Hölder’s inequality, we also obtain
Corollary 8  There exists $C_1, C_2 > 0$ such that

$$C_1 \leq p \int_{\partial \Omega} u^p ds_x \leq C_2$$

holds.

4. Proof of Theorem 2.

In this section, we prove Theorem 2. First, we recall an $L^1$ estimate from [6], which is a variant of the one by Brezis and Merle [2].

Lemma 9  Let $u$ be a solution to

$$\begin{cases}
-\Delta u + u = 0 & \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} = h & \text{on } \partial \Omega
\end{cases}$$

with $h \in L^1(\partial \Omega)$, where $\Omega$ is a smooth bounded domain in $\mathbb{R}^2$. For any $\varepsilon \in (0, \pi)$, there exists a constant $C > 0$ depending only on $\varepsilon$ and $\Omega$, independent of $u$ and $h$, such that

$$\int_{\partial \Omega} \exp \left( \frac{(\pi - \varepsilon) |u(x)|}{\|h\|_{L^1(\partial \Omega)}} \right) ds_x \leq C$$

(4.1)

holds true.

Also we need an elliptic $L^1$ estimate by Brezis and Strauss [3] for weak solutions with the $L^1$ Neumann data.

Lemma 10  Let $u$ be a weak solution of

$$\begin{cases}
-\Delta u + u = f & \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} = g & \text{on } \partial \Omega
\end{cases}$$

with $f \in L^1(\Omega)$ and $g \in L^1(\partial \Omega)$, where $\Omega$ is a smooth bounded domain in $\mathbb{R}^N$, $N \geq 2$. Then we have $u \in W^{1,q}(\Omega)$ for all $1 \leq q < \frac{N}{N-1}$ and

$$\|u\|_{W^{1,q}(\Omega)} \leq C_q \left( \|f\|_{L^1(\Omega)} + \|g\|_{L^1(\partial \Omega)} \right)$$

holds.
For the proof, see [3]:Lemma 23.

Now, following [10], [11], we define the notion of \( \delta \)-regular points. Put
\[ u_n = u_{p_n} \]
for any subsequence of \( u_p \). Since \( u_n \) satisfies
\[ \int_{\partial \Omega} \frac{u_{p_n}^n}{u_{p_n}^n} ds_x = 1, \]
we can select a subsequence \( p_n \to \infty \) (without changing the notation) and a
Radon measure \( \mu \geq 0 \) on \( \partial \Omega \) such that
\[ f_n := \frac{u_{p_n}^n}{\int_{\partial \Omega} u_{p_n}^n ds_x} \rightharpoonup \mu \]
weakly in the sense of Radon measures on \( \partial \Omega \), i.e.,
\[ \int_{\partial \Omega} f_n \varphi \ ds_x \to \int_{\partial \Omega} \varphi \ d\mu \]
for all \( \varphi \in C(\partial \Omega) \). As in [11], we define
\[ L_0 = \frac{1}{2\sqrt{e}} \lim sup_{p \to \infty} \left( p \int_{\partial \Omega} u_p^p ds_x \right). \quad (4.2) \]
By Corollary 5 and Hölder’s inequality, we have
\[ L_0 \leq \pi \sqrt{e}. \]
For some \( \delta > 0 \) fixed, we call a point \( x_0 \in \partial \Omega \) a \( \delta \)-regular point if there is a
function \( \varphi \in C(\partial \Omega) \), \( 0 \leq \varphi \leq 1 \) with \( \varphi = 1 \) in a neighborhood of \( x_0 \) such that
\[ \int_{\partial \Omega} \varphi \ d\mu < \frac{\pi}{L_0 + 2\delta} \]
holds. Define \( S = \{ x_0 \in \partial \Omega \mid x_0 \text{ is not a } \delta \text{-regular point for any } \delta > 0. \} \). Then,
\[ \mu(\{x_0\}) \geq \frac{\pi}{L_0 + 2\delta} \quad (4.3) \]
for all \( x_0 \in S \) and for any \( \delta > 0. \)
Here, following the argument in [11], we prove a key lemma in the proof of Theorem 2.
Lemma 11 Let \( x_0 \in \partial \Omega \) be a \( \delta \)-regular point for some \( \delta > 0 \). Then \( v_n = \frac{u_n}{\| u_n \|_{L^\infty(\partial \Omega)}} \) is bounded in \( L^\infty(B_{R_0}(x_0) \cap \Omega) \) for some \( R_0 > 0 \).

Proof. Let \( x_0 \in \partial \Omega \) be a \( \delta \)-regular point. Then by definition, there exists \( R > 0 \) such that
\[
\int_{\partial \Omega \cap B_R(x_0)} f_n \, ds < \frac{\pi}{L_0 + \delta}
\]
holds for all \( n \) large. Put \( a_n = \chi_{B_R(x_0)} f_n \) and \( b_n = (1 - \chi_{B_R(x_0)}) f_n \) where \( \chi_{B_R(x_0)} \) denotes the characteristic function of \( B_R(x_0) \). Split \( v_n = v_{1n} + v_{2n} \), where \( v_{1n}, v_{2n} \) is a solution to
\[
\begin{aligned}
-\Delta v_{1n} + v_{1n} &= 0 \quad \text{in } \Omega, \\
\frac{\partial v_{1n}}{\partial \nu} &= a_n \quad \text{on } \partial \Omega,
\end{aligned}
\]
\[
\begin{aligned}
-\Delta v_{2n} + v_{2n} &= 0 \quad \text{in } \Omega, \\
\frac{\partial v_{2n}}{\partial \nu} &= b_n \quad \text{on } \partial \Omega
\end{aligned}
\]
respectively. By the maximum principle, we have \( v_{1n}, v_{2n} > 0 \). Since \( b_n = 0 \) on \( B_R(x_0) \), elliptic estimates imply that
\[
\| v_{2n} \|_{L^\infty(B_{R/2}(x_0) \cap \Omega)} \leq C \| v_{2n} \|_{L^1(B_R(x_0) \cap \Omega)} \leq C,
\]
where we used the fact \( \| v_{2n} \|_{L^1(\Omega)} = \| \Delta v_{2n} \|_{L^1(\Omega)} = \| b_n \|_{L^1(\partial \Omega)} \leq C \) for the last inequality. Thus we have to consider \( v_{1n} \) only.

Claim: For any \( x \in \partial \Omega \), we have
\[
f_n(x) \leq \exp \left( (L_0 + \delta/2) v_n(x) \right) \tag{4.4}
\]
for \( n \) large.

Indeed, put
\[
\alpha_n = \frac{\| u_n \|_{L^\infty(\partial \Omega)}}{(\int_{\partial \Omega} u_n^m \, ds_x)^{1/p_n}}.
\]
Then by Lemma 7 and Corollary 8, we have
\[
\limsup_{n \to \infty} \alpha_n \leq \sqrt{e}.
\]
Since the function \( s \mapsto \frac{\log s}{s} \) is monotone increasing if \( 0 < s < e \), and
\[
\frac{u_n(x)}{(\int_{\partial \Omega} u_n^m \, ds_x)^{1/p_n}} \leq \alpha_n \quad \text{for any } x \in \partial \Omega,
\]
we observe that for fixed \( \varepsilon > 0 \),
\[
\frac{\log \frac{u_n(x)}{(\int_{\partial \Omega} u_n^m \, ds_x)^{1/p_n}}}{\alpha_n} \leq \frac{\log \alpha_n}{\alpha_n} \leq \frac{1}{2\sqrt{e}} + \varepsilon
\]
\]
holds for large $n$. Thus

$$f_n(x) = \exp \left( p_n \log \frac{u_n(x)}{\int_{\Omega} u_n^p \, ds_x}^{1/p_n} \right) \leq \exp \left( \frac{p_n u_n(x)}{\int_{\partial \Omega} u_n^p \, ds_x}^{1/p_n} \left( \frac{1}{2\sqrt{e}} + \varepsilon \right) \right)$$

$$= \exp \left( p_n v_n(x) \left( \int_{\partial \Omega} u_n^p \, ds_x \right)^{1-1/p_n} \left( \frac{1}{2\sqrt{e}} + \varepsilon \right) \right)$$

$$\leq \exp \left( \limsup_{n \to \infty} p_n \left( \int_{\partial \Omega} u_n^p \, ds_x \right) v_n(x) \left( \frac{1}{2\sqrt{e}} + 2\varepsilon \right) \right)$$

$$= \exp \left( \left( \frac{1}{2\sqrt{e}} + 2\varepsilon \right) 2\sqrt{e} L_0 v_n(x) \right) = \exp \left( (L_0 + 4\varepsilon \sqrt{e} L_0) v_n(x) \right).$$

Thus if we choose $\varepsilon > 0$ so small, we have the claim (4.4).

By this claim and the fact that $v_{2n}$ is uniformly bounded in $B_{R/2}(x_0)$, for sufficiently small $\delta_0 > 0$ so that $(1 + \delta_0) L_0 + \delta/2 < 1$, we have

$$\int_{B_{R/2}(x_0) \cap \partial \Omega} f_n^{1+\delta_0} \, ds_x \leq \int_{B_{R/2}(x_0) \cap \partial \Omega} \exp \left( (1+\delta_0)(L_0 + \delta/2)v_n(x) \right) \, ds_x$$

$$\leq C \int_{B_{R/2}(x_0) \cap \partial \Omega} \exp \left( (1+\delta_0)(L_0 + \delta/2)v_{1n}(x) \right) \, ds_x$$

$$\leq C \int_{B_{R/2}(x_0) \cap \partial \Omega} \exp \left( \pi (1+\delta_0) \frac{L_0 + \delta/2}{L_0 + \delta} v_{1n}(x) \right) \, ds_x$$

$$= C \int_{B_{R/2}(x_0) \cap \partial \Omega} \exp \left( \pi (1-\varepsilon_0) v_{1n}(x) \right) \, ds_x,$$

where $1 - \varepsilon_0 = (1 + \delta_0) L_0 + \delta/2$. Thus by Lemma 9, we have

$$\int_{B_{R/2}(x_0) \cap \partial \Omega} f_n^{1+\delta_0} \, ds_x \leq C$$

for some $C > 0$ independent of $n$. This fact and elliptic estimates imply that

$$\limsup_{n \to \infty} \|v_n\|_{L^\infty(\Omega \cap B_{R/4}(x_0))} \leq C,$$

which proves Lemma.

Now, we estimate the cardinality of the set $S$. By Theorem 1, we have

$$v_n(x_n) = \frac{\|u_n\|_{L^\infty(\partial \Omega)}}{\int_{\partial \Omega} u_n^p \, ds_x} \geq \frac{C_1}{\int_{\partial \Omega} u_n^p \, ds_x} \to \infty$$
for a sequence $x_n \in \partial \Omega$ such that $u_n(x_n) = \|u_n\|_{L^\infty(\partial\Omega)}$. Thus by Lemma 11, we see $x_0 = \lim_{n \to \infty} x_n \in S$ and $\sharp S \geq 1$. On the other hand, by (4.3) we have

$$1 = \lim_{n \to \infty} \|f_n\|_{L^1(\partial\Omega)} \geq \mu(\partial\Omega) \geq \frac{\pi}{L_0} \sharp S,$$

which leads to

$$1 \leq \sharp S \leq \frac{L_0 + 2\delta}{\pi} \leq \sqrt{\epsilon} + \frac{2\delta}{\pi} \approx 1.64 \cdots + \frac{2\delta}{\pi}.$$

Thus we have $\sharp S = 1$ if $\delta > 0$ is chosen small.

Let $S = \{x_0\}$ for some point $x_0 \in \partial\Omega$. By Lemma 11, we can conclude easily that $f_n \rightharpoonup \delta_{x_0}$ in the sense of Radon measures on $\partial\Omega$:

$$\int_{\partial\Omega} f_n \varphi ds_x \to \varphi(x_0), \quad \text{as } n \to \infty$$

for any $\varphi \in C(\partial\Omega)$, since $v_n$ is locally uniformly bounded on $\partial\Omega \setminus \{x_0\}$ and $f_n \to 0$ uniformly on any compact sets of $\partial\Omega \setminus \{x_0\}$.

Now, by the $L^1$ estimate in Lemma 10, we have $v_n$ is uniformly bounded in $W^{1,q}(\Omega)$ for any $1 \leq q < 2$. Thus, by choosing a subsequence, we have a function $\bar{G}$ such that $v_n \rightharpoonup \bar{G}$ weakly in $W^{1,q}(\Omega)$ for any $1 \leq q < 2$, $v_n \to \bar{G}$ strongly in $L^t(\Omega)$ and $L^t(\partial\Omega)$ respectively for any $1 \leq t < \infty$. The last convergence follows by the compact embedding $W^{1,q}(\Omega) \hookrightarrow L^t(\Omega)$ for any $1 \leq t < \frac{q}{2-q}$. Thus by taking the limit in the equation

$$\int_\Omega (-\Delta \varphi + \varphi)v_n dx = \int_{\partial\Omega} f_n \varphi ds_x - \int_{\partial\Omega} \frac{\partial \varphi}{\partial \nu} v_n ds_x$$

for any $\varphi \in C^1(\Omega)$, we obtain

$$\int_\Omega (-\Delta \varphi + \varphi)\bar{G} dx + \int_{\partial\Omega} \frac{\partial \varphi}{\partial \nu} \bar{G} ds_x = \varphi(x_0),$$

which implies $\bar{G}$ is the solution of (1.4) with $y = x_0$.

Finally, we prove the statement (3) of Theorem 2. We borrow the idea of [6] and derive Pohozaev-type identities in balls around the peak point. We may assume $x_0 = 0$ without loss of generality. As in [6], we use a conformal diffeomorphism $\Psi : H \cap B_{R_0} \to \Omega \cap B_r$ which flattens the boundary $\partial\Omega$, where $H = \{(y_1, y_2) \mid y_2 > 0\}$ denotes the upper half space and $R_0 > 0$ is a
radius sufficiently small. We may choose $\Psi$ is at least $C^3$, up to $\partial H \cap B_{R_0}$, $\Psi(0) = 0$ and $D\Psi(0) = Id$. Set $\tilde{u}_n(y) = u_n(\Psi(y))$ for $y = (y_1, y_2) \in H \cap B_{R_0}$. Then by the conformality of $\Psi$, $\tilde{u}_n$ satisfies

$$\begin{cases}
  -\Delta \tilde{u}_n + b(y)\tilde{u}_n = 0 & \text{in } H \cap B_{R_0}, \\
  \frac{\partial \tilde{u}_n}{\partial \nu} = h(y)\tilde{u}_n^n & \text{on } \partial H \cap B_{R_0},
\end{cases} \quad (4.5)$$

where $\tilde{\nu}$ is the unit outer normal vector to $\partial(H \cap B_{R_0})$, $b$ and $h$ are defined

$$b(y) = |\det D\Psi(y)|, \quad h(y) = |D\Psi(y)e|$$

with $e = (0, -1)$. Note that $\tilde{\nu}(y) = \nu(\Psi(y))$ for $y \in \partial H \cap B_{R_0}$. Note also that, by using a clever idea of [6], we can modify $\Psi$ to prescribe the number

$$\alpha = \left. \frac{\partial h}{\partial y_1} \right|_{y=0} = \left. \frac{\partial h}{\partial y_1} \right|_{y(0)}.$$

Let $D \subset \mathbb{R}^N$ be a bounded domain and recall the Pohozaev identity for the equation $-\Delta u = f(y, u), y \in D$:

$$\begin{align*}
  N \int_D F(y, u) dy - \left( \frac{N-2}{2} \right) \int_D |\nabla u|^2 dy + \int_D (y - y_0, \nabla_y F(y, u)) dy \\
  = \int_{\partial D} (y - y_0, \nu) F(y, u) ds_y + \int_{\partial D} (y - y_0, \nabla u) \left( \frac{\partial u}{\partial \nu} \right) ds_y \\
  - \frac{1}{2} \int_{\partial D} (y - y_0, \nu) |\nabla u|^2 ds_y
\end{align*}$$

for any $y_0 \in \mathbb{R}^N$, where $u$ is a smooth solution. Applying this to (4.5) for $N = 2$, $D = H \cap B_R$ for $0 < R < R_0$, $f(y, \tilde{u}_n) = -b(y)\tilde{u}_n$ and $F(y, \tilde{u}_n) = -\frac{b(y)}{2} \tilde{u}_n^2$, we obtain

$$\begin{align*}
  &\int_{H \cap B_R} b(y)\tilde{u}_n^2(y) dy + \int_{H \cap B_R} (y - y_0, \nabla b(y)) \frac{1}{2} \tilde{u}_n^2(y) dy \\
  = &\int_{\partial(H \cap B_R)} (y - y_0, \nu) \frac{1}{2} b(y)\tilde{u}_n^2(y) ds_y - \int_{\partial(H \cap B_R)} (y - y_0, \nabla \tilde{u}_n(y)) \left( \frac{\partial \tilde{u}_n}{\partial \tilde{\nu}} \right) ds_y \\
  + &\frac{1}{2} \int_{\partial(H \cap B_R)} (y - y_0, \tilde{\nu}) |\nabla \tilde{u}_n|^2 ds_y.
\end{align*}$$
where and from now on, \( \tilde{\nu} \) will be used again to denote the unit normal to \( \partial (H \cap B_R) \). Differentiating with respect to \( y_0 \), we have, in turn,

\[
\int_{\partial (H \cap B_R)} \nabla \tilde{u}_n(y) \left( \frac{\partial \tilde{u}_n}{\partial \tilde{\nu}} \right) ds_y = \frac{1}{2} \int_{\partial (H \cap B_R)} \left( |\nabla \tilde{u}_n|^2 + b(y) \tilde{u}_n^2 \right) \tilde{\nu} ds_y - \frac{1}{2} \int_{H \cap B_R} \nabla b(y) \tilde{u}_n^2 dy.
\]

Since \( \tilde{\nu} = (\tilde{\nu}_1, \tilde{\nu}_2) = (0, -1) \) on \( \partial H \cap B_R \), the first component of the above vector equation reads

\[
\int_{\partial H \cap B_R} (\tilde{u}_n)_{y_1} h(y) \tilde{u}_n^{p_n}(y) ds_y + \int_{H \cap \partial B_R} (\tilde{u}_n)_{y_1}(y) \left( \frac{\partial \tilde{u}_n}{\partial \tilde{\nu}} \right) ds_y 
= \frac{1}{2} \int_{H \cap \partial B_R} \left( |\nabla \tilde{u}_n|^2 + b(y) \tilde{u}_n^2 \right) \tilde{\nu}_1 ds_y - \frac{1}{2} \int_{H \cap B_R} b_{y_1}(y) \tilde{u}_n^2 dy,
\]

where \( (\cdot)_{y_1} \) denotes the derivative with respect to \( y_1 \). Let \( \gamma_n = \int_{\partial B_R} u^{n_{y_1}} ds_x \).

From the fact that \( \tilde{f}_n(y) = \frac{\tilde{u}_n^{p_n}}{\gamma_n} \ast \delta_0 \) in the sense of Radon measures on \( \partial H \cap B_R \), Corollary 8 and \( \|\tilde{u}_n\|_{L^\infty(\partial H \cap B_R)} = O(1) \) uniformly in \( n \), we see

\[
\tilde{g}_n(y) = \frac{1}{\gamma_n^2} \frac{\tilde{u}_n^{p_n+1}(y)}{p_n + 1} + \frac{1}{(p_n + 1)\gamma_n} \tilde{f}_n(y) \tilde{u}_n(y)
\]
satisfies that \( \text{supp}(\tilde{g}_n) \to \{0\} \) and \( \int_{\partial H \cap B_R} \tilde{g}_n ds_y = O(1) \) as \( n \to \infty \). Thus, by choosing a subsequence, we have the convergence

\[
\tilde{g}_n(y) = \frac{1}{\gamma_n^2} \frac{\tilde{u}_n^{p_n+1}(y)}{p_n + 1} \ast C_0 \delta_0
\]
in the sense of Radon measures on \( \partial H \cap B_R \), where \( C_0 = \lim_{n \to \infty} \int_{\partial H \cap B_R} \tilde{g}_n ds_y \) (up to a subsequence). By using this fact, we have

\[
\frac{1}{\gamma_n^2} \int_{\partial H \cap B_R} (\tilde{u}_n)_{y_1} h(y) \tilde{u}_n^{p_n}(y) ds_y
= \left[ \frac{h(y) \tilde{u}_n^{p_n+1}}{\gamma_n^2} \right]_{y_1 = -R}^{y_1 = R} - \int_{\partial H \cap B_R} h_{y_1}(y) \tilde{u}_n^{p_n+1}(y) \gamma_n^2 ds_y
\to 0 - C_0 h_{y_1}(0) = -C_0 \alpha
\]
as \( n \to \infty \). Thus after dividing (4.6) by \( \gamma_n^2 \) and then letting \( n \to \infty \), we obtain

\[
-C_0\alpha + \int_{H \cap \partial B_R} \tilde{G}_{y_1}(y) \left( \frac{\partial \tilde{G}}{\partial \nu} \right) ds_y \tag{4.7}
\]

\[
= \frac{1}{2} \int_{H \cap \partial B_R} \left( |\nabla \tilde{G}|^2 + b(y)\tilde{G}^2 \right) \tilde{\nu}_1 ds_y - \frac{1}{2} \int_{H \cap \partial B_R} b_{y_1}(y)\tilde{G}^2(y) dy,
\]

where \( \tilde{G}(y) = G(\Psi(y), 0) \) is a limit function of \( \tilde{v}_n(y) = v_n(\Psi(y)) = \tilde{u}_n(y) \). At this point, we have the same formula as the equation (117) in [6], thus we obtain the result. Indeed, decompose \( G(x, 0) = s(x) + w(x) \) where

\[
s(x) = \frac{1}{\pi} \log |x|^{-1}, \quad w(x) = H(x, 0),
\]

and put \( \tilde{s}(y) = s(\Psi(y)) \), \( \tilde{w}(y) = H(\Psi(y), 0) \) so that \( \tilde{G} = \tilde{s} + \tilde{w} \). Then after some computation using the fact that \( \tilde{w} \) satisfies

\[
-\Delta \tilde{w} + b(y)\tilde{w} = -b(y)\tilde{s}(y) \quad \text{in} \ H \cap B_R,
\]

we have from (4.7) that

\[
-C_0\alpha + \int_{H \cap \partial B_R} (\tilde{s}_\nu \tilde{s}_{y_1} + \tilde{s}_\nu \tilde{w}_{y_1} + \tilde{s}_{y_1} \tilde{w}_\nu) ds_y
\]

\[
= \int_{H \cap \partial B_R} \left( \frac{1}{2} |\nabla \tilde{s}|^2 + \nabla \tilde{s} \cdot \nabla \tilde{w} \right) \tilde{\nu}_1 ds_y + \int_{H \cap \partial B_R} \left( \frac{1}{2} \tilde{s}_y^2 + \tilde{s}\tilde{w} \right) b(y)\tilde{\nu}_1 ds_y
\]

\[
- \int_{\partial H \cap \partial B_R} b_{y_1}(y) \left( \frac{1}{2} \tilde{s}_y^2 + \tilde{s}\tilde{w} \right) ds_y + \int_{\partial H \cap \partial B_R} \tilde{w}_\nu \tilde{w}_{y_1} ds_y
\]

\[
- \int_{H \cap B_R} b(y)\tilde{s}(y)\tilde{w}_{y_1} dy. \tag{4.8}
\]

By Lemma 9.3 in [6], we know estimates

\[
\lim_{R \to 0} \int_{H \cap \partial B_R} \tilde{s}_\nu \tilde{s}_{y_1} ds_x = \frac{3\alpha}{4\pi}, \quad \lim_{R \to 0} \int_{H \cap \partial B_R} \tilde{s}_\nu \tilde{w}_{y_1} ds_x = -\tilde{w}_{y_1}(0),
\]

\[
\lim_{R \to 0} \frac{1}{2} \int_{H \cap \partial B_R} |\nabla \tilde{s}|^2 \tilde{\nu}_1 ds_x = \frac{\alpha}{4\pi}, \quad \lim_{R \to 0} \int_{H \cap \partial B_R} \nabla \tilde{s} \cdot \nabla \tilde{w} \tilde{\nu}_1 ds_x = -\frac{1}{2} \tilde{w}_{y_1}(0)
\]

18
and other terms in (4.8) go to 0 as $R \to 0$. Thus we take the limit in (4.8) as $R \to 0$ to obtain the relation

$$-C_0 \alpha + \frac{3\alpha}{4\pi} - \tilde{w}_{y_1}(0) = \frac{\alpha}{4\pi} - \frac{1}{2} \tilde{w}_{y_1}(0),$$

which leads to

$$\alpha \left( \frac{1}{2\pi} - C_0 \right) = \frac{1}{2} \tilde{w}_{y_1}(0).$$

Since $\alpha \in \mathbb{R}$ can be chosen arbitrary, we conclude that $C_0 = \frac{1}{2\pi}$ and $\tilde{w}_{y_1}(0) = 0$. This last equation means the desired conclusion of Theorem 2 (3).

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**References**


