Cohomological non-rigidity of eight-dimensional complex projective towers

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Abstract. A complex projective tower or simply a $\mathbb{C}P$-tower is an iterated complex projective fibrations starting from a point. In this paper, we classify certain class of 8-dimensional $\mathbb{C}P$-towers up to diffeomorphism. As a consequence, we show that cohomological rigidity is not satisfied by the collection of 8-dimensional $\mathbb{C}P$-towers, i.e., there is a two distinct 8-dimensional $\mathbb{C}P$-towers which have the same cohomology rings.

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1. Introduction

Let $\mathcal{M}$ be a collection of diffeomorphism classes of smooth manifolds and $H^* \mathcal{M}$ be the isomorphism classes of cohomology rings of manifolds in $\mathcal{M}$. Let $H^* : \mathcal{M} \to H^* \mathcal{M}$ be the map defined by $M \in \mathcal{M} \mapsto H^*(M; \mathbb{Z})$. In general, $H^*$ is not bijective. However, if we restrict the class of manifolds then this map sometimes becomes a bijection; e.g., if $\mathcal{M}$ is a collection of oriented 2-dimensional manifolds then it is well-known that the map $H^*$ is bijective. We say such collection $\mathcal{M}$ is cohomologically rigid or $\mathcal{M}$ satisfies cohomological rigidity. The problem asking whether the map $H^* : \mathcal{M} \to H^* \mathcal{M}$ is bijective or not is called a cohomological rigidity problem. In this paper, we study the cohomological rigidity problem for complex projective towers (or simply a $\mathbb{C}P$-tower) introduced in [KuSu].

A $\mathbb{C}P$-tower of height $m$ is a sequence of complex projective fibrations

$$
C_m \xrightarrow{\pi_m} C_{m-1} \xrightarrow{\pi_{m-1}} \cdots \xrightarrow{\pi_2} C_1 \xrightarrow{\pi_1} C_0 = \{ \text{a point} \}
$$

where $C_i = P(\xi_{i-1})$ is the projectivization of a complex vector bundle $\xi_{i-1}$ over $C_{i-1}$. We call each $C_i$ the $i$th stage of the tower. If we forget the tower structure, then we call $C_i$ an $(i$-stage) $\mathbb{C}P$-manifolds. In [KuSu], we show that the diffeomorphism types of 6-dimensional $\mathbb{C}P$-manifolds are

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There exists a bijective map $\phi: \mathcal{VECT}_2(\mathbb{C}P^3) \to \mathbb{Z}_2 \oplus \mathbb{Z} \oplus \mathbb{Z}$ such that $\phi(\xi) = (\alpha(\xi), c_1(\xi), c_2(\xi))$, where $c_1(\xi)$ and $c_2(\xi)$ are the first and the second Chern classes of $\xi$, and $\alpha(\xi)$ is a mod 2 element which is 0 when $c_1(\xi)$ is odd.

By Theorem 1.1, any element in $\mathcal{VECT}_2(\mathbb{C}P^3)$ can be denoted by $\eta(\alpha, c_1, c_2)$, where $(\alpha, c_1, c_2) \in \mathbb{Z}_2 \oplus \mathbb{Z} \oplus \mathbb{Z}$ such that $\alpha \equiv 0 \pmod{2}$ when $c_1 \equiv 1 \pmod{2}$. On the other hand, it can be seen easily that $P(\eta(\alpha, c_1, c_2))$ is diffeomorphic to $P(\eta(0, 0, c_2 - c_1^2/4))$ if $c_1 \equiv 1 \pmod{2}$, and is diffeomorphic to $P(\eta(\alpha, 0, c_2 - c_1^2/4))$ if $c_1 \equiv 0 \pmod{2}$, see Lemma 3.2.

Let $N(u) := P(\eta(0, 1, u))$, and let $\mathcal{N} := \{N(u) \mid u \in \mathbb{Z}\}$. Similarly, let $M_\alpha(u) := P(\eta(\alpha, 0, u))$, and let $\mathcal{M} := \{M_\alpha(u) \mid \alpha \in \{0, 1\}, u \in \mathbb{Z}\}$. We now state the main result of the paper (see Theorem 4.2 for (1) and see Theorem 5.2 for more precise statement of (2)).

**Theorem 1.2.** For the classes $\mathcal{M}$ and $\mathcal{N}$, we have the following.

1. The class $\mathcal{N}$ is cohomologically rigid. In fact, the following are equivalent:
   a. $N(u)$ is diffeomorphic to $N(u')$;
   b. $u = u'$;
   c. $H^*(N(u); \mathbb{Z}) \cong H^*(N(u'); \mathbb{Z})$ as graded rings.
2. The class $\mathcal{M}$ is not cohomologically rigid. In fact, $H^*(M_0(u); \mathbb{Z}) \not\cong H^*(M_1(u); \mathbb{Z})$ as graded rings for all $u$, but if $\frac{u(u+1)}{12} \in \mathbb{Z}$ then $M_0(u)$ is not diffeomorphic, actually not homotopic, to $M_1(u)$.

The second part of the theorem is proved in Proposition 5.4 by showing that $\pi_6(M_0(u)) \not\cong \pi_6(M_1(u))$ when $\frac{u(u+1)}{12} \in \mathbb{Z}$.

The organization of this paper is as follows. In Section 2, as examples of $\mathbb{C}P$-towers, we explain when flag manifolds admit the structure of $\mathbb{C}P$-tower. In Section 3, we recall some basic facts from [KuSu]. In Section 4, we show that $\mathcal{N}$ satisfies the cohomological rigidity. In Section 5, we compute the 6-dimensional homotopy group of the elements in some class of $\mathcal{M}$ and show that $\mathcal{M}$ does not satisfies the cohomological rigidity.

## 2. Flag manifolds of type $A$ and $C$

The $\mathbb{C}P$-towers contain many interesting classes of manifolds. In the previous paper [KuSu], we introduce that generalized Bott manifolds or the Milnor surface admits the structure of $\mathbb{C}P$-towers. We first introduce the other two examples of $\mathbb{C}P$-towers. Let $\mathcal{CP}M^6_n$ be the collection of 2n-dimensional $m$-stage $\mathbb{C}P$-manifolds up to diffeomorphism.

**Example 2.1.** The flag manifold $\mathcal{FL}(\mathbb{C}^{n+1}) = \{\{0\} \subset V_1 \subset \cdots \subset V_n \subset \mathbb{C}^{n+1}\}$, called type $A$, is well-known to be diffeomorphic to the homogeneous space $U(n+1)/T^{n+1}(\cong SU(n+1)/T^n)$. We will show that the flag manifold $U(n+1)/T^{n+1}$ is a $\mathbb{C}P$-tower with height $n$. Recall that if $M$ is a smooth manifold with free $K$ action and $H$ is a subgroup of $K$, then we have a diffeomorphism $M/H \cong M \times_K (K/H)$. Also recall that $\mathcal{CP}^n \cong U(n+1)/(T^1 \times U(n))$. By using these facts, it is
The flag manifold of type C is defined by the homogeneous space

\[ \frac{U(n)}{T} \]

where the \( U(n) \) action on \( \mathbb{C}P^n \) in each stage is induced from the usual \( U(k) \) action on \( \mathbb{C}^k \). Hence, the flag manifold \( U(n+1)/T^{n+1} \) of type A is an element of \( \mathcal{CP}M_{n^2+n} \).

**Example 2.2.** The flag manifold of type C is defined by the homogeneous space \( Sp(n)/T^n \). We claim that \( Sp(n)/T^n \) is a \( \mathbb{C}P \)-tower with height \( n \). It is well known that \( Sp(n)/(T^1 \times Sp(n-1)) \cong S^{4n-1}/T^1 \cong \mathbb{C}P^{2n-1} \), because \( Sp(n)/Sp(n-1) \cong S^{4n-1} \). By using this fact and the method similar to that demonstrated in Example 2.1, it is easy to check that there is the following \( \mathbb{C}P \)-tower structure of height \( n \) in \( Sp(n)/T^n \):

\[
\begin{align*}
Sp(n) \times (T^1 \times Sp(n-1)) & \rightarrow (Sp(n-1) \times (T^1 \times Sp(n-2))) \rightarrow \cdots \rightarrow (Sp(2) \times (T^1 \times Sp(1))) \rightarrow \mathbb{C}P^1 \\
\vdots & \quad \vdots & \quad \vdots \\
Sp(n) \times (T^1 \times Sp(n-1)) & \rightarrow (Sp(n-1) \times (T^1 \times Sp(n-2))) \rightarrow \mathbb{C}P^{2n-5} \\
& \quad \vdots \\
& \quad \vdots \\
& \quad \vdots \\
& \rightarrow \mathbb{C}P^{2n-1},
\end{align*}
\]

where the \( Sp(k) \)-action on \( \mathbb{C}P^{2k-1} \) in each stage is induced from the \( Sp(k) \)-action on \( \mathbb{C}^{2k}(\cong \mathbb{H}^k) \) induced by the following representation to \( U(2k) \):

\[
A + Bj \rightarrow \begin{pmatrix} A & -B \\ B & A \end{pmatrix}.
\]

Here \( A, B \in M(k; \mathbb{C}) \) satisfy \( A\overline{A} + B\overline{B} = I_k \) and \( BA - AB = O \). Hence, the flag manifold \( Sp(n)/T^n \) of type C is an element of \( \mathcal{CP}M_{n^2+n} \).

**Remark 2.3.** As is well-known, both of the flag manifolds \( U(n+1)/T^{n+1} \) and \( Sp(n)/T^n \) with \( n \geq 2 \) do not admit the structure of a toric manifold (see e.g. [BuPa]). On the other hand, \( U(2)/T^2 \cong Sp(1)/T^1 \cong \mathbb{C}P^1 \) is a toric manifold.

Moreover, by computing the generators of flag manifolds of other types (\( B_n (n \geq 3), D_n (n \geq 4), G_2, F_4, E_6, E_7, E_8 \)), they do not admit the structure of \( \mathbb{C}P \)-towers, see [Bo] (or [FIM] for classical types). Namely, we have the following proposition:

**Proposition 2.4.** Let \( M \) be a flag manifold denoted by \( G/T \), where \( G \) is a compact simple Lie group and \( T \) is its maximal torus. If \( M \) admits the structure of a \( \mathbb{C}P \)-tower, then \( G \) must be a compact Lie group of type \( A \) or \( C \).

The following problem also naturally arises (also see Remark 5.5).

**Problem 2.5.** Let \( H^* : \mathcal{CP}M \rightarrow H^* \mathcal{CP}M \) be the map defined by taking the cohomology rings. Classify diffeomorphism types of all manifolds in the class \( (H^*)^{-1}(H^*(U(n+1)/T^{n+1})) \) and \( (H^*)^{-1}(H^*(Sp(n)/T^n)) \).
3. Some preliminaries

In this section, we recall some basic facts.

3.1. Preliminaries from [KuSu]. We first recall some basic facts from [KuSu, Section 2]. Let \( \xi \) be an \( n \)-dimensional complex vector bundle over a topological space \( X \), and let \( P(\xi) \) denote its projectivization. Then, the following formula holds (see [KuSu]):

\[
H^*(P(\xi); \mathbb{Z}) \cong H^*(X; \mathbb{Z})[x]/(x^{n+1} + \sum_{i=1}^{n} (-1)^i c_i(\pi^*\xi)x^{n+1-i})
\]

where \( \pi^*\xi \) is the pull-back of \( \xi \) along \( \pi : P(\xi) \to X \) and \( c_i(\pi^*\xi) \) is the \( i \)-th Chern class of \( \pi^*\xi \). Here \( c \) can be viewed as the first Chern class of the canonical line bundle over \( P(\xi) \), i.e., the complex 1-dimensional sub-bundle \( \gamma_1 \) in \( \pi^*\xi \to P(\xi) \) such that the restriction \( \gamma_1|_{\gamma^1(\xi)} \) is the canonical line bundle over \( \pi^{-1}(a) \cong \mathbb{C}P^{n-1} \) for all \( a \in X \). Therefore, \( \deg x = 2 \). Since it is well-known that the induced homomorphism \( \pi^* : H^*(X; \mathbb{Z}) \to H^*(P(\xi); \mathbb{Z}) \) is injective, we often abuse the notation \( c_i(\pi^*\xi) \) by \( c_i(\xi) \). The formula (3.1) is called the Borel-Hirzebruch formula.

In order to prove the main theorem, we often use the following two lemmas.

**Lemma 3.1.** Let \( \gamma \) be any line bundle over \( M \), and let \( P(\xi) \) be the projectivization of a complex vector bundle \( \xi \) over \( M \). Then, \( P(\xi) \) is diffeomorphic to \( P(\xi \otimes \gamma) \).

**Lemma 3.2.** Let \( \gamma \) be a complex line bundle, and let \( \xi \) be a 2-dimensional complex vector bundle over a manifold \( M \). Then the Chern classes of the tensor product \( \xi \otimes \gamma \) are as follows.

\[
c_1(\xi \otimes \gamma) = c_1(\xi) + 2c_1(\gamma);
\]

\[
c_2(\xi \otimes \gamma) = c_1(\gamma)^2 + c_1(\gamma)c_1(\xi) + c_2(\xi).
\]

3.2. Atiyah-Rees’s theorem. By Theorem 1.1, all of the complex 2-plane bundles over \( \mathbb{C}P^3 \) can be denoted by \( \eta(\alpha) \), for some \( (\alpha, c_1, c_2) \in \mathbb{Z}^2 \times \mathbb{Z} \). Using Lemma 3.1, its projectivization \( P(\eta(\alpha) \otimes \gamma) \) is diffeomorphic to \( P(\eta(\alpha) \otimes \gamma) \) for any line bundle \( \gamma \) over \( \mathbb{C}P^3 \). Moreover, by Lemma 3.2 and the proof of the main theorem in [AtRe], we also have

\[
\eta(\alpha, c_1, c_2) \otimes \gamma \equiv \eta(\alpha, c_1 + 2c_1(\gamma), c_1(\gamma)^2 + c_1(\gamma)c_1 + c_2).
\]

Therefore, we may assume \( c_1 \in \{0, 1\} \). Consequently, in order to classify all \( P(\eta(\alpha, c_1, c_2)) \) up to diffeomorphisms, it is enough to classify the following:

\[
M_0(u) = P(\eta(0,0, u));
\]

\[
M_1(u) = P(\eta(1,0, u));
\]

\[
N(u) = P(\eta(0,1, u)),
\]

where \( u \in \mathbb{Z} \). We denote the class of \( M_0(u) \), \( M_1(u) \) up to diffeomorphism by \( \mathcal{M} \) and that of \( N(u) \) by \( \mathcal{N} \). Then, both classes \( \mathcal{M} \) and \( \mathcal{N} \) are the subclasses of \( \mathbb{C}P^3 \times \mathbb{C}P^3 \) consisting of 8-dimensional 2-stage \( \mathbb{CP} \)-manifolds.

3.3. Intersection of two classes \( \mathcal{M} \) and \( \mathcal{N} \) are empty. Finally, in this section, we prove \( \mathcal{M} \cap \mathcal{N} = \emptyset \) by comparing their cohomology rings. Namely, we prove the following lemma:

**Lemma 3.3.** Two cohomology rings \( H^*(M_0(u)) \) and \( H^*(N(u')) \) are not isomorphic for any \( u, u' \in \mathbb{Z} \).

**Proof.** By the Borel-Hirzebruch formula (3.1), we have ring isomorphisms

\[
H^*(M_0(u)) \cong \mathbb{Z}[X,Y]/(X^4, uX^2 + Y^2), \quad \text{and}
\]

\[
H^*(N(u')) \cong \mathbb{Z}[x,y]/(x^4, u'x^2 + xy + y^2).
\]

Assume that there is an isomorphism map \( f : H^*(M_0(u)) \to H^*(N(u')) \). Then we may put

\[
f(X) = ax + by, \quad \text{and}
\]

\[
f(Y) = cx + dy,
\]

where \( a, b, c, d \in \mathbb{Z} \). Since \( \deg x = 2 \) and \( \deg y = 2 \), we have

\[
a = b = c = d = 0,
\]

\[
X = Y = 0.
\]
for some $a$, $b$, $c$, $d \in \mathbb{Z}$ such that $ad - bc = \epsilon = \pm 1$. By taking the inverse of $f$, we also have

$$f^{-1}(x) = dx + ay, \quad \text{and} \quad f^{-1}(y) = -cx + a\epsilon y.$$  

From the ring structures of $H^*(M_\alpha(u))$ and $H^*(N(u'))$, we have $f(uX^2 + Y^2) = 0$ and $f^{-1}(y^2 + xy + u'x^2) = 0$. Therefore we have the following equations:

(3.2) \quad u(a^2 - u'b^2) + (c^2 - u'd^2) = 0;

(3.3) \quad u(2ab - b^2) + (2cd - d^2) = 0;

(3.4) \quad c^2 - a^2u - cd + abu + u'd^2 - b^2uu' = 0;

(3.5) \quad -2ac + cb + ad - 2bdu' = 0.

Because $f^{-1}(x^4) = (dX - bY)^4 = 0$, we also have

$$bd(d^2 - ub^2) = 0.$$  

Therefore $bd = 0$, or otherwise $d^2 = ub^2$. We first assume $bd = 0$. Then, there are two cases: $b = 0$ and $d = 0$. If $b = 0$, then $|a| = |d| = 1$. However, by using (3.3), we have $2cd = 1$. This gives a contradiction. If $d = 0$, then $|b| = |c| = 1$. By using (3.5), we have $c(-2a + b) = 0$, i.e., $b = 2a$ by $|c| = 1$. However, this contradicts to $|b| = 1$. Hence, $bd \neq 0$ and $d^2 = ub^2$, i.e., $|d| = \sqrt{|u||b|}$. In this case, because $ad - bc = \epsilon = \pm 1$, we have $|b| = 1$ and $d^2 = u$. Let $b = e' = \pm 1$ and $d = \sqrt{\epsilon}e''$, where $e'' = \pm 1$. Then, it follows from $ad - bc = \epsilon$ that $c = -\epsilon e' + a\sqrt{\epsilon}e''$. Therefore, by using (3.2), we have the following equation:

$$u(a^2 - u'b^2) + (c^2 - u'd^2) = u(a^2 - u') + (-\epsilon e' + a\sqrt{\epsilon}e'')^2 - uu' = 2uu' - 2u' + 1 - 2a\sqrt{\epsilon}e'' = 0.$$  

However, this gives the equation $1 = 2(-uu' + uu' + a\sqrt{\epsilon}e'')$, which is a contradiction. Hence, $H^*(M_\alpha(u)) \neq H^*(N(u'))$ for all $u$, $u' \in \mathbb{Z}$.  

Hence, we have the following corollary:

**Corollary 3.4.** There are no intersections between two classes $M$ and $N$.

## 4. Cohomological rigidity of $N$

In this section, we shall prove the cohomological rigidity of the class $N$. To show that, it is enough to prove the following lemma.

**Lemma 4.1.** The following two statements are equivalent.

1. $H^*(N(u)) \cong H^*(N(u'))$.
2. $u = u' \in \mathbb{Z}$.

**Proof.** Because (2) $\Rightarrow$ (1) is trivial, it is enough to show (1) $\Rightarrow$ (2). Assume there is an isomorphism $f : H^*(N(u)) \cong H^*(N(u'))$ where

$$H^*(N(u)) \cong \mathbb{Z}[X, Y]/(X^4, uX^2 + xy + Y^2);$$  

$$H^*(N(u')) \cong \mathbb{Z}[x, y]/(x^4, u'x^2 + xy + y^2).$$  

Again, we use the same representation for $f$ as in the proof of Lemma 3.3. Because $f(Y^2 + XY + uX^2) = 0$ and $f^{-1}(y^2 + xy + u'x^2) = 0$, we have that

(4.1) \quad c^2 - d^2u' = -ua^2 + b^2u' - ac + bdu';

(4.2) \quad 2cd - d^2 = -2abu + b^2u - ad - bc + bd;

(4.3) \quad c^2 - a^2u = -u'd^2 + b^2u' + cd - bau;

(4.4) \quad -2ac - a^2 = 2bd - b^2u' + a^2u - ad - bc - ab.

Because $f(X^4) = 0$ and $f^{-1}(x^4) = 0$, there are the following two cases:

1. $b = 0$;
(2) $b \neq 0$ and $4a^3 - 6a^2b + 4ab^2(1-u') + b^3(2u'-1) = -4d^3 - 6d^2b - 4db^2(1-u) + b^3(2u-1) = 0$.

If $b = 0$, then $|a| = |d| = 1$. Therefore, by (4.2), $2c = d - a$, i.e., $c = 0$ if $d = a$ or $c = -a$ if $d = -a$. Because $c^2 - u' = -u - ac$ by (4.1), we have that $u = u'$.

Assume $b \neq 0$. By the equation $4a^3 - 6a^2b + 4ab^2(1-u') + b^3(2u'-1) = 0$, we have $b$ is even. Substituting $a = A + \frac{b}{2}$ for some $A \in \mathbb{Z}$ to this equation (i.e., Tschirnhaus’s transformation), we have the following equation:

$$\begin{align*}
4(A + \frac{b}{2})^3 - 6(A + \frac{b}{2})^2b + 4(A + \frac{b}{2})b^2(1-u') + b^3(2u'-1) \\
= 4(A^3 + 3A^2\frac{b}{2} + 6A\frac{b^2}{4} + \frac{b^3}{8}) - 6(A^2 + Ab + \frac{b^2}{4})b + 4(2\frac{b^3}{2})(1-u') + b^3(2u'-1) \\
= 4A^3 + 6A^2b + 3Ab^2 + \frac{b^3}{2} - 6A^2b - 6Ab^2 - \frac{3b^3}{2} + 4Ab^2 - 4Ab^2u' - 2b^3u' + 2b^3u' - b^3 \\
= 4A^3 + Ab^2 - 4Ab^2u' \\
= A(4A^2 + b^2 - 2u') = 0
\end{align*}$$

Therefore, there are the two cases: $A = 0$ or $A \neq 0$. We first assume $A \neq 0$. Then, by using the equation $4A^2 + b^2 - 2u' = 0$, we have $u' \geq 1$. Now, there is the following commutative diagram:

$$\begin{array}{ccc}
H^2(N(u)) &=& ZX \oplus ZY \\
\downarrow f \\
H^2(N(u')) &=& Zx \oplus Zy \\
&\xrightarrow{ax+by} & Zx^2 \oplus Zx \\
&\downarrow f & \\
&= & H^4(N(u'))
\end{array}$$

Because $X$ and $f$ are isomorphisms, so is $ax + by$ in the diagram. Using the indicated generators as bases, the determinant of the map $f \circ X : H^2(N(u)) \to H^4(N(u'))$ is equal to the determinant of the map $(ax + by) \circ f : H^2(N(u)) \to H^4(N(u'))$, which is equal to

$$(4.5) \quad a^2 - ab + b^2u' = \epsilon_1 = \pm 1.$$ 

Because $a \in \mathbb{Z}$, the discriminant of this equation satisfies

$$(b^2 - 4(b^2u' - \epsilon_1)) = b^2(1 - 4u') + 4\epsilon_1 \geq 0$$

Because $u' \geq 1$, we have that

$$0 < b^2 \leq \frac{4\epsilon_1}{4u'-1} < 1.$$ 

This gives a contradiction to $b \in \mathbb{Z}$. Therefore, we have $A = 0$, i.e., $a = A = \frac{b}{2}$. Because $ad - bc = \epsilon(= \pm 1)$, we also have that $a = \epsilon' = \pm 1$, $b = 2\epsilon'$ and $d = 2c = \epsilon'c$. Hence, by (4.5), we have $-1 + 4u' = \epsilon_1$, i.e., $u' = 0$ and $\epsilon_1 = -1$. By applying a similar method to the one used to derive (4.5) for $f^{-1}(x)$, we have

$$(4.6) \quad d^2 + db + b^2u = \epsilon_2 = \pm 1.$$ 

Substituting (4.5) and (4.6) to (4.3) and (4.4), we have

$$\begin{align*}
c^2 &= u\epsilon_1 - u'd^2 + cd = -u + cd; \\
-2ac &= \epsilon_1 + 2bud' - ad - bc = -1 - (d + 2c)\epsilon'.
\end{align*}$$

By using the second equation above, we also have $d = -\epsilon'$; therefore, by $d - 2c = \epsilon'c$, we have $c = \frac{-\epsilon' - \epsilon}{2} = 0$ or $-\epsilon'$. If $c = 0$, then $u = 0$ by the first equation above; if $c = -\epsilon'$ then we also have $u = 0$ by $d = -\epsilon'$ and the first equation above. This implies that $u = u = 0$ for the case $b \neq 0$.

This establishes the statement. □

Therefore, by Theorem 1.1 and Lemma 4.1, we have the following theorem.

**Theorem 4.2.** The following three statements are equivalent.
(1) Two spaces \( N(u) \) and \( N(u') \) are diffeomorphic.
(2) Two cohomology rings \( H^*(N(u)) \) and \( H^*(N(u')) \) are isomorphic.
(3) \( u = u' \in \mathbb{Z} \).

In particular, the class \( N \) is cohomologically rigid.

This establishes Theorem 1.2 (1).

5. Cohomological non-rigidity of \( CPM^2 \)

In this section, we prove that \( M \) is not cohomologically rigid. We first show the following fact about the cohomology rings of elements in \( M \).

**Lemma 5.1.** The following two statements are equivalent.
(1) \( H^*(M_\alpha(u)) \cong H^*(M_\alpha'(u')) \) where \( \alpha, \alpha' \in \{0, 1\} \).
(2) \( u = u' \in \mathbb{Z} \).

**Proof.** Because (2) \( \Rightarrow \) (1) is trivial, it is enough to show (1) \( \Rightarrow \) (2). Assume there is an isomorphism \( f : H^*(M_\alpha(u)) \cong H^*(M_\alpha'(u')) \) where
\[
H^*(M_\alpha(u)) \cong \mathbb{Z}[X, Y]/(X^4, uX^2 + Y^2);
H^*(M_\alpha'(u')) \cong \mathbb{Z}[x, y]/(x^4, u'x^2 + y^2).
\]
We may use the same representation for \( f \) as in the proof of Lemma 3.3. Note that \( f(ux^2 + y^2) = 0 \) and \( f^{-1}(u'x^2 + y^2) = 0 \). By using the representation of \( f \), we have the following equations:
\[
\begin{align*}
(5.1) & \quad ua^2 - uu'b^2 + c^2 - u'd^2 = 0; \\
(5.2) & \quad uab + cd = 0; \\
(5.3) & \quad u'd^2 - uu'b^2 + c^2 - a^2u = 0; \\
(5.4) & \quad u'bd + ac = 0.
\end{align*}
\]
By (5.1) and (5.3), we have
\[
\begin{align*}
(5.5) & \quad c^2 = b^2u'; \\
(5.6) & \quad ua^2 = u'd^2.
\end{align*}
\]
Because \( X^4 = 0 \), we also have that
\[
ab(a^2 - b^2u') = 0.
\]
We first assume \( ab \neq 0 \). Then
\[
a^2 = b^2u'
\]
by this equation. Together with (5.5) and (5.6), we have that
\[
c^2b^2 = b^2uu' = b^2a^2u = b^2d^2u' = a^2d^2.
\]
This implies that
\[
(ad - bc)(ad + bc) = \epsilon(ad + bc) = 0.
\]
Hence, \( ad = -bc \). However this gives a contradiction because \( ad - bc = 2ad = \epsilon = \pm 1 \). Consequently, we have \( ab = 0 \). Since \( ad - bc = \epsilon \), if \( a = 0 \) then \( |b| = |c| = 1 \); therefore, we have \( u = u' = \pm 1 \) by (5.5); if \( b = 0 \) then \( |a| = |d| = 1 \); therefore, we have \( u = u' \) by (5.6). This establishes the statement. \( \square \)

Lemma 5.1 says that cohomology rings of \( M \) are not affected by \( \alpha \in \mathbb{Z}_2 \). On the other hand, the goal of this section is to prove the following theorem, i.e., some topological types of \( M \) are affected by \( \alpha \in \mathbb{Z}_2 \).

**Theorem 5.2.** Assume \( u(u + 1)/12 \in \mathbb{Z} \). The following three statements are equivalent.
(1) Two spaces \( M_\alpha(u) \) and \( M_\beta(u') \) are diffeomorphic.
(2) \( (\alpha, u) = (\beta, u') \in \mathbb{Z}_2 \times \mathbb{Z} \).
(3) Two spaces \( M_\alpha(u) \) and \( M_\beta(u') \) are homotopy equivalent.
In order to prove Theorem 5.2, we first compute the 6-dimensional homotopy group of $M_\alpha(u)$ in Proposition 5.4. Now $M_\alpha(u)$ can be defined by the following pull-back diagram:

\[
\begin{array}{c}
M_\alpha(u) \\
\downarrow \\
\mathbb{C}P^3 \\
\downarrow \\
BU(2)
\end{array}
\xrightarrow{\mu_{\alpha,u}}
\begin{array}{c}
EU(2) \times_{U(2)} \mathbb{C}P^1 \\
\downarrow \\
\mathbb{C}P^3 \\
\downarrow \\
BU(2)
\end{array}
\]

Let $p : S^7 \to \mathbb{C}P^3$ be the canonical $S^1$-fibration and $P(\xi_{\alpha,u})$ be the pull-back of $M_\alpha(u)$ along $p$. Namely, we have the following diagram:

\[
(5.7) \quad \begin{array}{c}
P(\xi_{\alpha,u}) \\
\downarrow \\
S^7 \\
\downarrow p \\
\mathbb{C}P^3 \\
\downarrow \\
BU(2)
\end{array}
\xrightarrow{\mu_{\alpha,u}}
\begin{array}{c}
M_\alpha(u) \\
\downarrow \\
EU(2) \times_{U(2)} \mathbb{C}P^1 \\
\downarrow \\
\mathbb{C}P^3 \\
\downarrow \\
BU(2)
\end{array}
\]

Then, we have the following lemma.

**Lemma 5.3.** For $* \geq 3$, $\pi_*(P(\xi_{\alpha,u})) \cong \pi_*(M_\alpha(u))$.

**Proof.** Because $P(\xi_{\alpha,u})$ is the pull-back of $M_\alpha(u)$, the homotopy exact sequences of $P(\xi_{\alpha,u})$ and $M_\alpha(u)$ satisfy the following commutative diagram:

\[
\begin{array}{ccccccccc}
\pi_{*+1}(S^7) & \longrightarrow & \pi_*(\mathbb{C}P^1) & \longrightarrow & \pi_*(P(\xi_{\alpha,u})) & \longrightarrow & \pi_*(S^7) & \longrightarrow & \pi_{*+1}(\mathbb{C}P^1) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\pi_{*+1}(\mathbb{C}P^3) & \longrightarrow & \pi_*(\mathbb{C}P^1) & \longrightarrow & \pi_*(M_\alpha(u)) & \longrightarrow & \pi_*(\mathbb{C}P^3) & \longrightarrow & \pi_{*+1}(\mathbb{C}P^1)
\end{array}
\]

From the homotopy exact sequence of the fibration $S^1 \to S^7 \to \mathbb{C}P^3$, we have $\pi_*(S^7) \cong \pi_*(\mathbb{C}P^3)$ for $* \geq 3$. Therefore, by using the 5 lemma, we have the statement. \qed

Now we may prove the following proposition.

**Proposition 5.4.** Assume $u(u+1)/12 \in \mathbb{Z}$. The following two isomorphisms hold.

1. $\pi_6(P(\xi_{\alpha,u})) \cong \pi_6(M_\alpha(u)) \cong \mathbb{Z}_{12}$ if $\alpha \equiv u(u+1)/12 \pmod{2}$

2. $\pi_6(P(\xi_{\beta,u})) \cong \pi_6(M_\beta(u)) \cong \mathbb{Z}_6$ if $\beta \not\equiv u(u+1)/12 \pmod{2}$

**Proof.** We first claim the 1st statement. If $u(u+1)/12 \in \mathbb{Z}$ and $\alpha \equiv u(u+1)/12 \pmod{2}$, then it follows from [AtRe] that $\xi_{\alpha,u}$ is induced from the rank 2 complex vector bundle over $\mathbb{C}P^4$. Namely, there is the following commutative diagram:

\[
(5.8) \quad \begin{array}{c}
\xi_{\alpha,u} \\
\downarrow \\
S^7 \\
\downarrow p \\
\mathbb{C}P^3 \\
\downarrow \\
\mathbb{C}P^4 \\
\downarrow \\
BU(2)
\end{array}
\xrightarrow{\mu_{\alpha,u}}
\begin{array}{c}
\eta_{(\alpha,0,u)} \\
\downarrow \\
EU(2) \times_{U(2)} \mathbb{C}^2 \\
\downarrow \\
\mathbb{C}P^3 \\
\downarrow \\
BU(2)
\end{array}
\]

On the other hand, we have that $\pi_7(\mathbb{C}P^4) \cong \pi_7(S^9) = \{0\}$, by using the homotopy exact sequence for the fibration $S^1 \to S^9 \to \mathbb{C}P^4$. This implies that $\xi_{\alpha,u}$ is the trivial $\mathbb{C}^2$-bundle over $S^7$. Therefore,

$$P(\xi_{\alpha,u}) = S^7 \times \mathbb{C}P^1$$

when $u(u+1)/12 \in \mathbb{Z}$ and $\alpha \equiv u(u+1)/12 \pmod{2}$. Hence, we also have that

$$\pi_6(M_\alpha(u)) \cong \pi_6(S^7 \times \mathbb{C}P^1) \cong \pi_6(\mathbb{C}P^1) \cong \mathbb{Z}_{12}.$$ 

Next we claim the 2nd statement. Let $\mu_{\alpha,u} : \mathbb{C}P^3 \to BU(2)$ be a continuous map which induces the above $\eta_{(\alpha,0,u)}$, and $\beta$ be the element in $\mathbb{Z}_2$ which is not equal to $\alpha$. Let $x \in \mathbb{C}P^3$ and $s = \mu_{\alpha,u}(x) \in BU(2)$ be base points. Take a disk neighborhood around $x \in \mathbb{C}P^3$ and pinch
its boundary to a point, i.e., the boundary of $D^6 \subset CP^3$ pinches to a point, then we obtain the surjective map

$$\rho: CP^3 \to CP^3 \vee S^6,$$

where $CP^3 \vee S^6$ may be regarded as the wedge sum with respect to the base points $x \in CP^3$ and $y \in S^6$. Due to theorem of Atiyah-Rees [AtRe], we have $\eta_{(\beta,0,u)} \neq \eta_{(0,0,u)}$. This implies that the vector bundle $\eta_{(\beta,0,u)}$ is induced from the following continuous map:

$$\mu_{\beta,u}: CP^3 \xrightarrow{\rho} CP^3 \vee S^6 \xrightarrow{\nu_a} BU(2)$$

where $\nu_a = \mu_{\alpha,u} \vee \kappa$ for the generator $\kappa \in \pi_6(BU(2),s) \cong \mathbb{Z}_2$. Hence, we have the following commutative diagram.

From the $CP^1$-fibrations $CP^1 \to P(\xi_{\beta,u}) \to S^7$ and $CP^1 \to EU(2) \times_{U(2)} CP^1 \cong BT^2 \to BU(2)$ in the above diagram (5.10), there is the following commutative diagram.

$$\pi_7(S^7) \cong \mathbb{Z} \xrightarrow{\pi_6(CP^1)} \pi_6(P(\xi_{\beta,u})) \xrightarrow{\pi_6(S^7)} \{0\} \cong \mathbb{Z}_{12}$$

This shows that the following exact sequence:

$$Z \cong \pi_7(S^7) \to \pi_7(BU(2))\cong \mathbb{Z}_{12} \to \pi_6(P(\xi_{\beta,u})) \to \{0\}.$$
Remark 5.5. For example, the relation \( u(u + 1)/12 \in \mathbb{Z} \) is true for the case when \( u = 0 \) and \( u = 3 \). In these cases, by using Proposition 5.4, we have

\[
\pi_6(M_\alpha(0)) \cong \begin{cases} 
\mathbb{Z}_{12} & \text{for } \alpha \equiv 0 \\
\mathbb{Z}_6 & \text{for } \alpha \equiv 1
\end{cases}
\]

and

\[
\pi_6(M_\alpha(3)) \cong \begin{cases} 
\mathbb{Z}_6 & \text{for } \alpha \equiv 0 \\
\mathbb{Z}_{12} & \text{for } \alpha \equiv 1
\end{cases}
\]

On the other hand, the case when \( u = 1 \) does not satisfy the relation \( u(u + 1)/12 \in \mathbb{Z} \). It follows from the cohomology ring of the flag manifold of type \( C \) (see e.g. [Bo] or [FIM]) that the flag manifold \( Sp(2)/T^2 \) is one of this case, i.e., \( M_0(1) \) or \( M_1(1) \). However, by using the homotopy exact sequence for the fibration \( T^2 \to Sp(2) \to Sp(2)/T^2 \) and the computation in [MiTo], we have that

\[
\pi_6(Sp(2)/T^2) \cong \pi_6(Sp(2)) = 0.
\]

Therefore, Proposition 5.4 is not true for the case when \( u(u + 1)/12 \not\in \mathbb{Z} \).

Let us prove Theorem 5.2.

Proof of Theorem 5.2. By using Theorem 1.1, (2) \( \Rightarrow \) (1) is trivial. The statement (1) \( \Rightarrow \) (3) is also trivial. We claim (3) \( \Rightarrow \) (2). Assume \( M_\alpha(u) \) and \( M_\beta(u') \) are homotopy equivalent. Then, \( H^*(M_\alpha(u)) \cong H^*(M_\beta(u')) \). Therefore, it follows from Lemma 5.1 that \( u = u' \). Moreover, in this case, \( \pi_6(M_\alpha(u)) \cong \pi_6(M_\beta(u)) \). If \( \alpha \equiv \beta \mod 2 \), then this gives a contradiction to Proposition 5.2. Hence, \( \alpha \equiv \beta \mod 2 \). We have (3) \( \Rightarrow \) (2). This establishes Theorem 5.2. \( \square \)

In summary, by Lemma 5.1 and Theorem 5.2, we have the following corollary:

Corollary 5.6. The set of 8-dimensional \( \mathbb{C}P \)-manifolds does not satisfy the cohomological rigidity.

This establishes Theorem 1.2 (2).

Note that if we restrict the class of 8-dimensional \( \mathbb{C}P \)-manifolds to the 8-dimensional generalized Bott manifolds with height 2, then cohomological rigidity holds by [CMS10]. On the other hand, the following question seems to be natural to ask for the class of \( \mathbb{C}P \)-manifolds \( \mathbb{C}PM \) instead of the cohomological rigidity problem.

Problem 5.7. Is the class of \( \mathbb{C}P \)-manifolds \( \mathbb{C}PM \) (up to diffeomorphism) determined by their homotopy types? More precisely, are \( M_1, M_2 \in \mathbb{C}PM \) diffeomorphic if they have the same homotopy types?

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