

ON INTERSECTIONS OF THE GAUSS IMAGES OF ISOPARAMETRIC HYPERSURFACES

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ABSTRACT. We know that the *Gauss image*, i.e. the image of the Gauss map of an isoparametric hypersurface in the unit standard hypersphere $S^{n+1}(1) \subset \mathbb{R}^{n+2}$ provides a compact minimal Lagrangian submanifold embedded in the complex hyperquadric $Q_n(\mathbb{C})$. In this paper we shall discuss some properties and related problems on the intersection of the Gauss images of two isoparametric hypersurfaces. This work is in progress as a joint work with Hui Ma (Tsinghua University, Beijing).

INTRODUCTION

Let (M, ω) be a $2n$ -dimensional symplectic manifold with a symplectic form ω . A smooth immersion (resp. embedding) $\varphi : L \rightarrow M$ of a smooth manifold L into a symplectic manifold M is called a *Lagrangian immersion* (resp. *Lagrangian embedding*) if $\varphi^*\omega = 0$ and $\dim L = n$.

It is well-known that *any smooth manifold N^n is a Lagrangian submanifold!* The cotangent vector bundle T^*N has the standard symplectic structure $\omega_{std} = d\theta_{can}$, where θ_{can} is a canonical 1-form on T^*N defined by $(\theta_{can})_{(x,\alpha)}(X) := \alpha(\pi_*X)$ for each $X \in T_{(x,\alpha)}(T^*N)$. Then the smooth manifold N^n can be identified with the zero section $\{0\}$ of T^*N , which is a Lagrangian submanifold embedded in T^*N relative to ω_{std} . For each 1-form ϕ on N , $\phi(N)$ considered as a submanifold embedded in T^*N is a Lagrangian submanifold of T^*N relative to ω_{std} if and only if ϕ is a closed 1-form on N . Moreover, a Lagrangian submanifold $\phi(N)$ embedded in (T^*N, ω_{std}) is said to be a *Hamiltonian deformation* of N when ϕ is an exact 1-form on N , that is, $\phi = df$ for some $f \in C^\infty(N)$. Now we observe that $(x, 0) \in N \cap df(N)$ if and only if $(df)_x = 0$. A critical point of f on N is nothing but an intersection point of Lagrangian submanifolds N and $df(N)$! Moreover, we can observe that Lagrangian submanifolds N and $df(N)$ intersect transversally at $(x, 0)$ if and only if x is a non-degenerate critical point of f . Hence we have that Lagrangian submanifolds N and $df(N)$ intersect transversally if and only if f is a Morse function on N . Suppose that N is compact. Then by Morse inequality we obtain an

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inequality on intersection numbers of Lagrangian submanifolds

$$\sharp(N \cap \phi(N)) \geq SB(N; \mathbf{Z}_2)$$

for each Hamiltonian deformation $\phi(N)$ of N with transversal $N \cap \phi(N)$. Here $SB(N; \mathbf{Z}_2) = \dim_{\mathbf{Z}_2} H^*(N; \mathbf{Z}_2)$ denotes the sum of Betti numbers of N over \mathbf{Z}_2 . By the equivalence theorem for Lagrangian submanifolds ([20]) it is well-known that any closed Lagrangian submanifold L embedded in M has a tubular neighborhood symplectic diffeomorphic to a neighborhood of the zero section of the cotangent vector bundle T^*L of L . Therefore any closed Lagrangian submanifold L embedded in a symplectic manifold M satisfies the inequality

$$\sharp(L \cap \phi(L)) \geq SB(L; \mathbf{Z}_2)$$

for each *sufficiently small* Hamiltonian deformation $\phi(L)$ of L with transversal $L \cap \phi(L)$. The famous *Arnold conjecture* is asking to study whether the global version of this inequality holds or not, that is,

$$\sharp(L \cap \phi(L)) \geq SB(L; \mathbf{Z}_2)$$

for each $\phi \in \text{Hamil}(M, \omega)$ with transversal $L \cap \phi(L)$. Here $\text{Hamil}(M, \omega)$ denotes the set of all time-dependent Hamiltonian diffeomorphisms of M , which is a group. It is well-known that Andreas Floer provides the breakthrough to the Arnold conjecture.

In this paper we shall give attention to a very special and nice class of compact Lagrangian submanifolds embedded in complex hyperquadrics $Q_n(\mathbb{C})$, which is a rank two Hermitian symmetric space of compact type. We know that the *Gauss image*, i.e. the image of the Gauss map of an isoparametric hypersurface in the unit standard hypersphere $S^{n+1}(1) \subset \mathbb{R}^{n+2}$ provides a compact minimal Lagrangian submanifold embedded in the complex hyperquadric $Q_n(\mathbb{C})$. In this paper we shall discuss some properties and related problems on the intersection of the Gauss images of two isoparametric hypersurfaces in the unit standard hypersphere.

The intersection point of the Gauss images of two isoparametric hypersurfaces will be characterized in terms of critical points of an isoparametric function restricted to another isoparametric hypersurface. Moreover, the transversal property at an intersection point of two Gauss images will be shown to be equivalent to the non-degeneracy of the critical point. Based on such fundamental observations and using the structure theory of isoparametric hypersurfaces, we shall discuss some results on the non-empty property of the intersection and the intersection number of two Gauss images.

Further interesting problems are to determine the intersection numbers $\sharp(L_0 \cap L_1)$ and to compute the Lagrangian intersection Floer cohomology $I^*(L_0, L_1 : Q_n(\mathbb{C}))$ for the Gauss images of isoparametric hypersurfaces.

This work is in progress as a joint work with Associate Professor Hui Ma (Tsinghua University, Beijing).

1. THE GAUSS IMAGES OF ISOPARAMETRIC HYPERSURFACES IN THE UNIT STANDARD HYPERSPHERE

Let $Q_n(\mathbb{C})$ be a complex hyperquadrics of $\mathbb{C}P^{n+1}$ defined by the homogeneous quadratic equation $z_0^2 + z_1^2 + \cdots + z_{n+1}^2 = 0$. Let $\widetilde{Gr}_2(\mathbb{R}^{n+2})$ be the real Grassmann manifold of all oriented 2-dimensional vector subspaces of \mathbb{R}^{n+2} and $Gr_2(\mathbb{R}^{n+2})$ the real Grassmann manifold of all 2-dimensional vector subspaces of \mathbb{R}^{n+2} . We denote by $[W]$ a 2-dimensional oriented vector subspace W of \mathbb{R}^{n+2} and by $-[W]$ the same vector subspace W of \mathbb{R}^{n+2} equipped with an orientation opposite to $[W]$. The map $\widetilde{Gr}_2(\mathbb{R}^{n+2}) \ni [W] \mapsto W \in Gr_2(\mathbb{R}^{n+2})$ is the universal covering with the deck transformation group \mathbb{Z}_2 generated by an involutive isometry

$$\widetilde{Gr}_2(\mathbb{R}^{n+2}) \ni [W] \mapsto -[W] \in \widetilde{Gr}_2(\mathbb{R}^{n+2}).$$

Then we have the identification

$$Q_n(\mathbb{C}) \ni [\mathbf{a} + \sqrt{-1}\mathbf{b}] \longleftrightarrow [W] = \mathbf{a} \wedge \mathbf{b} \in \widetilde{Gr}_2(\mathbb{R}^{n+2}),$$

where $\{\mathbf{a}, \mathbf{b}\}$ denotes an orthonormal basis of W compatible with the orientation of $[W]$.

Let N^n be an oriented hypersurface of the unit standard hypersphere $S^{n+1}(1) \subset \mathbb{R}^{n+2}$. Denote by $\mathbf{x}(p)$ the position vector of a point $p \in N^n$ and by $\mathbf{n}(p)$ the unit normal vector at a point $p \in N^n$ to N^n in $S^{n+1}(1)$ compatible to the orientation. The *Gauss map* $\mathcal{G} : N^n \rightarrow Q_n(\mathbb{C})$ of N^n is defined by

$$\mathcal{G} : N^n \ni p \longrightarrow [\mathbf{x}(p) + \sqrt{-1}\mathbf{n}(p)] = \mathbf{x}(p) \wedge \mathbf{n}(p) \in Q_n(\mathbb{C}) \cong \widetilde{Gr}_2(\mathbb{R}^{n+2}).$$

Then it is known

Proposition 1.1. *The Gauss map $\mathcal{G} : N^n \rightarrow Q_n(\mathbb{C})$ is always a Lagrangian immersion.*

Note that the hypersurfaces parallel to N^n has the same image under the Gauss map. and a deformation of N^n in $S^{n+1}(1)$ corresponds to a Hamiltonian deformation of the Gauss map.

Palmer ([16]) gave the formula describing the mean curvature form of the Gauss map \mathcal{G} in terms of the principal curvatures of the original oriented hypersurface N^n .

Suppose that N^n is an oriented hypersurface with constant principal curvatures in $S^{n+1}(1)$, the so-called *isoparametric hypersurface*. Denote by g the number of distinct principal curvatures. The hypersurfaces parallel to N^n form a family of oriented hypersurfaces with constant principal curvatures in $S^{n+1}(1)$, the so-called *isoparametric family*. The structure theory of isoparametric hypersurfaces and isoparametric families was established by E. Cartan and H. F. Münzner ([10]) Let $f : S^{n+1}(1) \rightarrow [-1, 1]$ be an *isoparametric function* on $S^{n+1}(1)$ defining the isoparametric family of N^n , which is extended to the *Cartan-Münzner* polynomial $F : \mathbb{R}^{n+2} \rightarrow \mathbb{R}$ of degree g . The family of level subsets $\{f^{-1}(c) \mid c \in [-1, 1]\}$ coincides with the family of hypersurfaces

parallel to N^n ($c \in (-1, 1)$) and its focal manifolds ($c = -1, 1$). Note that $\mathbf{n}(p) = \frac{1}{\|(\text{grad}f)_p\|}(\text{grad}f)_p$ for each $p \in N^n$.

We know that their multiplicities satisfy $m_1 = m_3 = \cdots = m_{2i-1} = \cdots$ and $m_2 = m_4 = \cdots = m_{2i} = \cdots$. Thus $\frac{2n}{g}$ must be an integer given as

$$\frac{2n}{g} = \begin{cases} m_1 + m_2 & \text{if } g \text{ is even,} \\ 2m_1 & \text{if } g \text{ is odd.} \end{cases}$$

The famous result of Münzner ([11]) is that g must be 1, 2, 3, 4 or 6.

From the Palmer's formula ([1]) we see

Proposition 1.2. *The Gauss map $\mathcal{G} : N^n \rightarrow Q_n(\mathbb{C})$ is a minimal Lagrangian immersion.*

The Lagrangian immersion \mathcal{G} and the Gauss image $\mathcal{G}(N^n)$ of an isoparametric hypersurface have the following properties. It follows from [5], [15], [7] that

- Proposition 1.3.**
- (1) *The Gauss image $\mathcal{G}(N^n)$ is a compact smooth minimal Lagrangian submanifold embedded in $Q_n(\mathbb{C})$.*
 - (2) *The Gauss map \mathcal{G} gives a covering map $\mathcal{G} : N^n \rightarrow \mathcal{G}(N^n)$ over the Gauss image with the deck transformation group \mathbb{Z}_g . Note that the \mathbb{Z}_g -action does not preserve the induced metric on N^n from $S^{n+1}(1)$ if $g \geq 3$.*
 - (3) *$\mathcal{G}(N^n)$ is invariant under the deck transformation group \mathbb{Z}_2 of the universal covering $Q_n(\mathbb{C}) = \widetilde{Gr}_2(\mathbb{R}^{n+2}) \rightarrow Gr_2(\mathbb{R}^{n+2})$.*
 - (4) *$\frac{2n}{g}$ is even (resp. odd) if and only if $\mathcal{G}(N^n)$ is orientable (resp. non-orientable).*
 - (5) *$\mathcal{G}(N^n)$ is a monotone and cyclic Lagrangian submanifold in $Q_n(\mathbb{C})$ with minimal Maslov number equal to $\frac{2n}{g}$.*

Hence we see that the Lagrangian intersection Floer cohomology for the Gauss images of isoparametric hypersurfaces is well-defined by Y. G. Oh's works ([12], [13], [14]).

A submanifold of a Riemannian manifold is said to be *homogeneous* if it is obtained as an orbit of a connected Lie subgroup of its isometry group. In the classification theory of isoparametric hypersurfaces, it is well-known that any homogeneous isoparametric hypersurface in the standard sphere is obtained as a principal orbit of the isotropy representation of a Riemannian symmetric pair (U, K) of rank 2 (Hsiang-Lawson [4], Takagi-Takahashi [17]). By Elie Cartan, Dorfmeister-Nehr and R. Miyaoka ([9]), it is known that for $g = 1, 2, 3, 6$ isoparametric hypersurfaces are homogeneous. *Non-homogeneous* isoparametric hypersurfaces appear only in the case of $g = 4$. Non-homogeneous isoparametric hypersurfaces was discovered first by Ozeki-Takeuchi and generalized

by Ferus-Karcher-Münzner. Isoparametric hypersurfaces with $g = 4$ were classified except for the case $(m_1, m_2) = (7, 8)$ by Cecil-Q. S. Chi-Jensen [1], [2], [3].

Note that $g = 1$ or 2 if and only if $\mathcal{G}(N^n)$ is a totally geodesic Lagrangian submanifold of $Q_n(\mathbb{C})$, that is, a real form (real hyperquadric) of a complex hyperquadric.

In the joint works of the author and Hui Ma, we have done

- (1) Classification of all compact homogeneous Lagrangian submanifolds in complex hyperquadrics ([5]).
- (2) Determination of Hamiltonian stability, Hamiltonian rigidity and strict Hamiltonian stability for the Gauss images of all homogeneous isoparametric hypersurfaces:
 - (a) $g = 1, 2, 3$ ([5]).
 - (b) $g = 4$, (U, K) is of classical type ([7]).
 - (c) $g = 6$ and $g = 4$, (U, K) is of exceptional type ([8]).

2. INTERSECTIONS OF THE GAUSS IMAGES OF ISOPARAMETRIC HYPERSURFACES

Let N_0 and N_1 be two compact isoparametric hypersurfaces embedded in the unit standard sphere $S^{n+1}(1)$ with g_0 and g_1 distinct constant principal curvatures and their multiplicities (m_1^0, m_2^0) and (m_1^1, m_2^1) , respectively. Let f_0 and f_1 be the isoparametric functions on $S^{n+1}(1)$ corresponding to N_0 and N_1 , and let F_0 and F_1 denote the Cartan-Münzner polynomials on \mathbb{R}^{n+2} corresponding to $f_0 = F_0|_{S^{n+1}(1)}$ and $f_1 = F_1|_{S^{n+1}(1)}$, respectively.

Let \mathbf{x}_0 and \mathbf{x}_1 denote the position vectors of points of N_0 and N_1 , and let \mathbf{n}_0 and \mathbf{n}_1 denote unit normal vector fields to N_0 and N_1 in $S^{n+1}(1)$, respectively. Let $\mathcal{G}_0 : N_0 \rightarrow Q_n(\mathbb{C})$ and $\mathcal{G}_1 : N_1 \rightarrow Q_n(\mathbb{C})$ be their Gauss maps. Denote by $L_0 = \mathcal{G}_0(N_0)$ and $L_1 = \mathcal{G}_1(N_1)$ their Gauss images.

Lemma 2.1. *Let $[W] \in L_0 \cap L_1$ be an intersection point of L_0 and L_1 . If $p_0 \in N_0$, $p_1 \in N_1$ and $[W] = \mathcal{G}_0(p_0) = \mathcal{G}_1(p_1) \in L_0 \cap L_1$, then $p_0 \in N_0$ is a critical point of the function $f_1|_{N_0}$ and $p_1 \in N_1$ is a critical point of the function $f_0|_{N_1}$.*

Proof. First by the definition of the Gauss map we see that the assumption $[W] = \mathcal{G}_0(p_0) = \mathcal{G}_1(p_1) \in L_0 \cap L_1$ is equivalent to the condition that $[W] = \mathbf{x}_0(p_0) \wedge \mathbf{n}_0(p_0) = \mathbf{x}_1(p_1) \wedge \mathbf{n}_1(p_1)$. Thus using the properties of isoparametric families, we have $(\text{grad} f_1)_{p_0} \perp T_{p_0}N_0$ and $(\text{grad} f_0)_{p_1} \perp T_{p_1}N_1$, where note that $(\text{grad} f_1)_{p_0} = 0$ or $(\text{grad} f_0)_{p_1} = 0$ can happen. It implies that $p_0 \in N_0$ is a critical point of the function $f_1|_{N_0}$ and $p_1 \in N_1$ is a critical point of the function $f_0|_{N_1}$. \square

Remark 1. By changing a choice of N_0 (resp. N_1) in its isoparametric family, we may assume that $(\text{grad} f_1)_{p_0} \neq 0$ (resp. $(\text{grad} f_0)_{p_1} \neq 0$).

Conversely, we have

Lemma 2.2. *If $p_0 \in N_0$ is a critical point of the function $f_1|_{N_0}$ and $(\text{grad} f_1)_{p_0} \neq 0$, then $\mathcal{G}_0(p_0) \in L_0 \cap L_1$. Similarly, if $p_1 \in N_1$ is a critical point of the function $f_0|_{N_1}$ and $(\text{grad} f_0)_{p_1} \neq 0$, then $\mathcal{G}_1(p_1) \in L_0 \cap L_1$.*

Proof. Since $(\text{grad} f_1)_{p_0} \neq 0$, we can choose an isoparametric hypersurface $N'_1 = f_1^{-1}(f_1(p_0))$ through p_0 in the isoparametric family of N_1 . Let $\mathbf{x}'_1(p)$ and $\mathbf{n}'_1(p)$ be the position vector of $p \in N'_1$ and the unit normal vector to N'_1 in $S^{n+1}(1)$ at $p \in N'_1$, respectively. Denote by $\mathcal{G}'_1 : N'_1 \rightarrow Q_n(\mathbb{C})$ the Gauss map of N'_1 . Then we have $L_1 = \mathcal{G}_1(N_1) = \mathcal{G}'_1(N'_1)$. Thus $\mathcal{G}_0(p_0) = \mathbf{x}_0(p_0) \wedge \mathbf{n}_0(p_0) = \mathbf{x}'_1(p_0) \wedge \mathbf{n}'_1(p_0) = \mathcal{G}'_1(p_0) \in \mathcal{G}_0(N_0) \cap \mathcal{G}'_1(N'_1) = L_0 \cap L_1$. \square

Therefore we obtain a characterization of an intersection point of L_0 and L_1 as follows:

Theorem . *The following three conditions are equivalent each other:*

- (1) $[W] \in L_0 \cap L_1$.
- (2) *There are N_0 in its isoparametric family and $p_0 \in N_0$ such that $L_0 = \mathcal{G}_0(N_0)$, $[W] = \mathcal{G}_0(p_0)$ and p_0 is a critical point of $f_1|_{N_0}$ with $(\text{grad} f_1)_{p_0} \neq 0$.*
- (3) *There are N_1 in its isoparametric family and $p_1 \in N_1$ such that $L_1 = \mathcal{G}_1(N_1)$, $[W] = \mathcal{G}_1(p_1)$ and p_1 is a critical point of $f_0|_{N_1}$ with $(\text{grad} f_0)_{p_1} \neq 0$.*

Moreover we observe that

$$\mathcal{G}_0^{-1}(L_0 \cap L_1) \subset \{p \in N_0 \mid p \text{ is a critical point of } f_1|_{N_0}\} \subset N_0$$

and

$$\mathcal{G}_1^{-1}(L_0 \cap L_1) \subset \{p \in N_1 \mid p \text{ is a critical point of } f_0|_{N_1}\} \subset N_1$$

have the following symmetry.

Lemma 2.3. *The subset $\mathcal{G}_0^{-1}(L_0 \cap L_1)$ of N_0 is invariant under the group action of \mathbb{Z}_{g_0} on N_0 . Similarly, the subset $\mathcal{G}_1^{-1}(L_0 \cap L_1)$ of N_1 is invariant under the group action of \mathbb{Z}_{g_1} .*

Proof. We observe that

$$\mathcal{G}_0^{-1}(L_0 \cap L_1) = \bigcup_{[W] \in L_0 \cap L_1} W \cap N_0 \subset N_0.$$

Since $\sharp(W \cap N_0) = 2g_0$ and \mathbb{Z}_{g_0} acts freely and transitively on $W \cap N_0$, we obtain this lemma. \square

Proposition 2.1. *$L_0^n \cap L_1^n$ is invariant under the Deck transformation group \mathbb{Z}_2 of $\widetilde{Gr}_2(\mathbb{R}^{n+2}) \rightarrow Gr_2(\mathbb{R}^{n+2})$. In other words, the Deck transformation group \mathbb{Z}_2 of $\widetilde{Gr}_2(\mathbb{R}^{n+2}) \rightarrow Gr_2(\mathbb{R}^{n+2})$ acts freely on $L_0^n \cap L_1^n$.*

Proof. Since each $L_i = \mathcal{G}_i(N_i)$ ($i = 0, 1$) is invariant under the free group action of \mathbb{Z}_2 , $L_0 \cap L_1 = \mathcal{G}_0(N_0) \cap \mathcal{G}_1(N_1)$ is also so. \square

Therefore from these lemmas we obtain

Theorem . $\sharp(L_0 \cap L_1)$ is even and

$$\sharp(L_0 \cap L_1) \geq 2$$

if it is finite.

Proof. We have only to prove that $L_0 \cap L_1 \neq \emptyset$. Set $N_0^t := f_0^{-1}(t)$ for each $t \in (-1, 1)$ and $M_0^\pm := f_0^{-1}(\pm 1)$, which are the isoparametric family and its focal manifolds for f_0 . Set $N_1^t := f_1^{-1}(t)$ for each $t \in (-1, 1)$ and $M_1^\pm := f_1^{-1}(\pm 1)$, which are the isoparametric family and its focal manifolds for f_1 . Assume that $L_0 \cap L_1 = \emptyset$. By the assumption note that, for each $p \in S^{n+1}(1) \setminus (M_0^+ \cup M_0^- \cup M_1^+ \cup M_1^-)$, $(\text{grad} f_0)_p \neq 0$, $(\text{grad} f_1)_p \neq 0$ and they are linearly independent. Because if not, then $\mathcal{G}_0(p) = \mathcal{G}_1(p) \in L_0 \cap L_1$, which is a contradiction. For each $t \in (-1, 1)$, $N_0^t \cap M_1^+ \neq \emptyset$, $N_0^t \cap M_1^- \neq \emptyset$ and the critical point set of $f_1|_{N_0^t}$ coincides with $(N_0^t \cap M_1^+) \cup (N_0^t \cap M_1^-)$. Similarly, for each $t \in (-1, 1)$, $N_1^t \cap M_0^+ \neq \emptyset$, $N_1^t \cap M_0^- \neq \emptyset$ and the critical point set of $f_0|_{N_1^t}$ coincides with $(N_1^t \cap M_0^+) \cup (N_1^t \cap M_0^-)$. As $t \rightarrow \pm 1$, the compactness of N_0^t , N_1^t , M_0^\pm and M_1^\pm implies that $M_0^+ \cap M_1^+ \neq \emptyset$, $M_0^+ \cap M_1^- \neq \emptyset$, $M_0^- \cap M_1^+ \neq \emptyset$ and $M_0^- \cap M_1^- \neq \emptyset$.

If $f_0|_{M_1^+}$ has a critical point $p \in M_1^+$, then $p \in M_0^+ \cup M_0^-$. Because since $\text{grad}(f_0|_{M_1^+})_p = 0$ we have $(\text{grad} f_0)_p \perp T_p M_1^+$. If $(\text{grad} f_0)_p \neq 0$, then $\mathcal{G}_0(p) \in L_0 \cap L_1$, which is a contradiction. Thus $(\text{grad} f_0)_p = 0$ and hence $f_0(p) = \pm 1$, that is, $p \in (M_1^+ \cap M_0^+) \cup (M_1^+ \cap M_0^-)$. Therefore the critical point subset of $f_0|_{M_1^+}$ coincides with $(M_1^+ \cap M_0^+) \cup (M_1^+ \cap M_0^-)$. In particular the level subsets $f_0|_{M_1^+}$ ($t \in (-1, 1)$) form a family of smooth embedded hypersurfaces of M_1^+ . The similar property for each $f_0|_{M_1^-}$, $f_1|_{M_0^+}$ and $f_1|_{M_0^-}$ also holds.

Let $p \in M_1^- \cap M_0^-$. Let U be a small neighborhood of p in $S^{n+1}(1)$. Choose N_0^a in the isoparametric family of f_0 for some $a \in (-1, 1)$ such that N_0^a is the boundary of a tubular neighborhood $\mathcal{N}_0^a = (\cup_{-1 < t < a} N_0^t) \cup M_0^-$ of M_0^- in $S^{n+1}(1)$. Then for some $c \in (-1, 1)$ close to -1 , $(N_1^c \cup \mathcal{N}_1^c) \cap U$ is contained in $\mathcal{N}_0^a \cap U$, where $\mathcal{N}_1^c = (\cup_{-1 < s < c} N_1^s) \cup M_1^-$ denotes a tubular neighborhood of M_1^- in $S^{n+1}(1)$. Since N_0^t collapses to M_0^- as $t \rightarrow -1$, there is N_0^b for some $-1 < b < a < 1$ such that $N_0^b \cap U$ is contained in $(N_1^c \cup \mathcal{N}_1^c) \cap U$. Then there is some t_0 with $-1 < b < t_0 < a$ such that $N_0^{t_0}$ is tangent to N_1^c at a point q . Hence $\mathcal{G}_0(q) = \mathcal{G}_1(q) \in \mathcal{G}_0(N_0^{t_0}) \cap \mathcal{G}_1(N_1^c) = L_0 \cap L_1 \neq \emptyset$. This is a contradiction. We conclude that $L_0 \cap L_1 \neq \emptyset$. \square

We recall that the differential

$$(d\mathcal{G})_p : T_p N \rightarrow T_{\mathcal{G}(p)} \widetilde{Gr}_2(\mathbb{R}^{n+2}) \cong \text{Hom}(\mathbf{x}(p) \wedge \mathbf{n}(p), T_p N)$$

of the Gauss map $\mathcal{G} : N \rightarrow Q_n(\mathbb{C}) \cong \widetilde{Gr}_2(\mathbb{R}^{n+2})$ at $p \in N$ is given by

$$(2.1) \quad \begin{cases} [(d\mathcal{G})_p(X)] \mathbf{x}(p) = X, \\ [(d\mathcal{G})_p(X)] \mathbf{n}(p) = -A_{\mathbf{n}}(X) \end{cases}$$

for each $X \in T_p N$.

Let $[W] \in L_0 \cap L_1$. We may assume that $[W] = \mathcal{G}_0(p)$, $p \in N_0$ and $p \in N_1$, and thus we have $T_p N_0 = T_p N_1$. We know that $p \in N_0$ is a critical point of $f_1|_{N_0}$ and $p \in N_1$ is a critical point of $f_0|_{N_1}$.

Lemma 2.4. *Let $X \in T_p N_0 = T_p N_1$. Then the following three conditions are equivalent each other:*

- (1) $X \in \text{Ker}(\text{Hess}_x(f_1|_{N_0}))$.
- (2) $A_{\mathbf{n}(p)}^0(X) = A_{\mathbf{n}(p)}^1(X)$.
- (3) $(d\mathcal{G}_0)_p(X) = (d\mathcal{G}_1)_p(X) \in T_{[W]}L_0^n \cap T_{[W]}L_1^n$.

Proof. First the equivalence of (2) and (3) follows from (2.1). Next we compute the Hessian of $f_1|_{N_0}$ at p : For each $X \in C^\infty(TN_0)$,

$$\begin{aligned}
\text{Hess}_{p_0}(f_1|_{N_0})(X, X) &= X(X f_1|_{N_0}) \\
&= X(df_1|_{N_0}(X)) \\
&= X(g(\text{grad}(f_1|_{N_0}), X)) \\
&= X(g(\text{grad}(f_1) - \langle \text{grad}(f_1), \mathbf{n}_0 \rangle \mathbf{n}_0, X)) \\
&= X(g(\text{grad}(f_1), X)) \\
&= X((df_1)(X)) \\
&= \text{Hess}_{p_0}(f_1)(X, X) + (df_1)_{p_0}(\tilde{\nabla}_X X) \\
&= \text{Hess}_{p_0}(f_1)(X, X) + (df_1)_{p_0}(\nabla_X X + B^0(X, X)) \\
&= \text{Hess}_{p_0}(f_1)(X, X) + (df_1)_{p_0}(B^0(X, X)) \\
&= \text{Hess}_{p_0}(f_1)(X, X) + g(\text{grad}(f_1), B^0(X, X)) \\
&= \text{Hess}_{p_0}(f_1)(X, X) + g(A_{\text{grad}(f_1)}^0 X, X) \\
&= -\|\text{grad} f_1\| g(A_{\mathbf{n}}^1 X, X) + g(A_{\text{grad}(f_1)}^0 X, X) \\
&= -\|\text{grad} f_1\| g(A_{\mathbf{n}}^1 X, X) + \|\text{grad} f_1\| g(A_{\mathbf{n}}^0 X, X) \\
&= \|\text{grad} f_1\| g((A_{\mathbf{n}}^0 - A_{\mathbf{n}}^1)X, X),
\end{aligned}$$

where $\tilde{\nabla}$, ∇ and B^0 denote the Levi-Civita connections of $S^{n+1}(1)$, N_0 and the second fundamental form of N_0 in $S^{n+1}(1)$. Here we use that if $f : M \rightarrow \mathbf{R}$ is a smooth function on a Riemannian manifold M and $(\text{grad} f)_x \neq 0$ for $x \in M$, then for each smooth vector field X on a neighborhood of x in $f^{-1}(f(x))$ we have the formula

$$\begin{aligned}
(\text{Hess} f)_x(X, X) &= (\nabla_X^M df)_x(X) = X(X f)(x) - (df)_x(\nabla_X^M X) \\
&= -g(A_{\text{grad} f} X, X) = -\|\text{grad} f\| g(A_{\mathbf{n}} X, X).
\end{aligned}$$

Hence we obtain the formula

$$(2.2) \quad \text{Hess}_{p_0}(f_1|_{N_0})(X, Y) = \|\text{grad} f_1\| g((A_{\mathbf{n}}^0 - A_{\mathbf{n}}^1)X, Y)$$

for each $X, Y \in T_p N_0 = T_p N_1$. By (2.2) we have the equivalence of (1) and (2). \square

Therefore we obtain

Lemma 2.5. *Let $[W] \in L_0 \cap L_1$. Suppose that $[W] = \mathcal{G}_0(p_0)$ for $p_0 \in N_0$ and thus p_0 is a critical point of $f_1|_{N_0}$. Then L_0 and L_1 intersect transversally at $[W] \in L_0 \cap L_1$ if and only if the critical point p_0 of $f_1|_{N_0}$ is non-degenerate.*

Moreover, using those results we can show

Theorem . *There are N_0 and N_1 in their isoparametric families such that $L_0 = \mathcal{G}_0(N_0)$, $L_1 = \mathcal{G}_1(N_1)$ and the function $f_1|_{N_0^n}$ (resp. $f_0|_{N_1^n}$) is a Morse function on N_0^n (resp. N_1^n) if and only if L_0^n and L_1^n intersect transversally each other.*

By the Morse inequalities we obtain

Corollary 2.1. *Suppose that L_0 and L_1 intersect transversally each other. Then*

$$(2.3) \quad \begin{aligned} \sharp(L_0 \cap L_1) &= \frac{1}{g_0} \sharp\{p \in N_0 \mid x \text{ is a critical point of } f_1|_{N_0}\} \\ &= \frac{1}{g_1} \sharp\{p \in N_1 \mid x \text{ is a critical point of } f_0|_{N_1}\} \\ &\geq 2 \end{aligned}$$

Proof. By the Morse inequalities for $f_1|_{N_0}$ and $f_0|_{N_1}$ we have

$$(2.4) \quad \sharp\{p \in N_0 \mid x \text{ is a critical point of } f_1|_{N_0}\} \geq SB(N_0, \mathbb{Z}_2)$$

and

$$(2.5) \quad \sharp\{p \in N_1 \mid x \text{ is a critical point of } f_0|_{N_1}\} \geq SB(N_1, \mathbb{Z}_2).$$

Here $SB(L, \mathbb{Z}_2)$ denotes the sum of Betti numbers of L over \mathbb{Z}_2 . Since $SB(N_0, \mathbb{Z}_2) = 2g_0$ and $SB(N_1, \mathbb{Z}_2) = 2g_1$ by Münzner's result ([11]), they become

$$\frac{1}{g_0} \sharp\{p \in N_0 \mid x \text{ is a critical point of } f_1|_{N_0}\} \geq \frac{2g_0}{g_0} = 2$$

and

$$\frac{1}{g_1} \sharp\{p \in N_1 \mid x \text{ is a critical point of } f_0|_{N_1}\} \geq \frac{2g_1}{g_1} = 2.$$

□

In the case when $N_1^n = S^n$ ($g = 1$), the corresponding Cartan-Münzner polynomial F_1 is given by a linear function $F_1(\mathbf{x}) = \langle \mathbf{x}, \mathbf{a} \rangle$ ($\forall \mathbf{x} \in \mathbb{R}^{n+2}$) for some $\mathbf{a} \in \mathbb{R}^{n+2}$.

Theorem . *Assume that $g_1 = 1$. Suppose that L_0 and L_1 intersect transversally each other. Then*

$$(2.6) \quad \sharp(L_0 \cap L_1) = 2 (= SB(L_1, \mathbb{Z}_2)).$$

Proof. Since f_1 is a perfect Morse function on N_0 , we have

$$\sharp\{p \in N_0 \mid x \text{ is a critical point of } f_1|_{N_0}\} = SB(N_0, \mathbb{Z}_2).$$

By using $SB(N_0, \mathbb{Z}_2) = 2g_0$ (Münzner [10], [11]), it becomes

$$\#\{p \in N_0 \mid x \text{ is a critical point of } f_1|_{N_0}\} = 2g_0.$$

Hence we obtain

$$\frac{1}{g_0} \#\{p \in N_0 \mid x \text{ is a critical point of } f_1|_{N_0}\} = 2 \frac{g_0}{g_0} = 2.$$

□

Remark 2. In the case when $g_0 = 1$ or 2 and $g_1 = 1$, it coincides with the results of H. Tasaki and M. S. Tanaka ([19], [18]). From their results we see that if $g_0 = g_1 = 2$, then $\#(L_0 \cap L_1) = 2 \text{Min}\{m_1^0, m_2^0, m_1^1, m_2^1\} + 2$ for transverse $L_0 \cap L_1$.

Problem . Determine the intersection numbers $\#(L_0 \cap L_1)$ for the Gauss images of isoparametric hypersurfaces.

Problem . Compute the Lagrangian intersection Floer cohomology $I^*(L_0, L_1 : Q_n(\mathbb{C}))$ for the Gauss images of isoparametric hypersurfaces.

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