# Spectral Analysis of Mean-Field Hamiltonian of Nonrelativistic Many-Body Systems

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**Abstract**

We conduct the spectral analysis of a time-dependent Schrödinger operator that can be considered as the quantized Hamiltonian of a nonrelativistic many-body system in the mean-field regime. The basic properties of these Hamiltonians in general two-particle interactions satisfying suitable decay conditions are shown. Specifically, we determine the location of the essential spectrum and the finiteness or infiniteness of the discrete spectrum. The Hartree equation analysis plays an essential role in our study. [1]

*Key words*: Mean-field Hamiltonian, Hartree equation.

## 1 Introduction

In this paper we conduct a spectral analysis of a time-dependent Schrödinger operator $H_t$ on $L^2(\mathbb{R}^d_x)$ defined by

$$H_t = -\Delta + V * |\varphi_t|^2.$$

Here, $\Delta$ is the generalized Laplacian on $L^2(\mathbb{R}^d)$, $V$ is a real-valued function on $\mathbb{R}^d$, and $\varphi_t : \mathbb{R}^d \to \mathbb{C}$ is a time-global solution of the Hartree equation

$$i\partial_t \varphi_t = -\Delta\varphi_t + (V * |\varphi_t|^2)\varphi_t \tag{1.1}$$

where $*$ denotes the convolution in $\mathbb{R}^d$. Therefore, the term $V * |\varphi_t|^2$ can be formally expressed as

$$V * |\varphi_t|^2(x) = \int_{\mathbb{R}^d} V(x-y)|\varphi_t|^2(y)dy, \quad x \in \mathbb{R}^d.$$

The operator $H_t$ is the quantum counterpart of the classical mean-field Hamiltonian, which describes a particle influenced by the collective force of many other particles with pairwise potential $V$ and macroscopic density $\rho_t(x) = |\varphi_t(x)|^2$ (see e.q., [G]). The state of this system, described by the density operator $D(t) = \langle \varphi_t, \cdot \rangle \varphi_t$, is governed by the von Neumann equation

$$i\partial_t D(t) = [H_t, D(t)],$$

[1] *2010 Mathematics Subject Classification*: 35J10, 35Q40, 35Q41

which is equivalent to the Hartree equation (1.1). The operator $H_t$ has recently been applied for the mean-field analysis of many-boson systems (see e.q., [AB] and references therein). We are interested in the relationship between the mean-field Hamiltonian $H_t$ and the Hartree solution $\varphi_t$. The Hartree equation has been thoroughly investigated in partial differential equation theory, but the spectral property of $H_t$ is less well-known. In this paper, we investigate the basic properties of $H_t$ using several results from the Hartree theory.

The present paper is organized as follows: In Section 2, we first investigate the properties of the potential $V * |\varphi_t|^2$ from a suitable two-particle interaction $V : \mathbb{R}^d \to \mathbb{R}$ and any initial value satisfying $\varphi \in H^2(\mathbb{R}^d)$ of equation (1.1). Next, we prove some spectral properties of $H_t$ in a generalized spatial dimension $d$. In particular, we prove the stability theorem for the essential spectrum (Theorem 2.5) and the existence of a ground state for $H_t$ (Corollary 2.6). Using the general perturbation theory, recently developed by Arai [Ar], we prove that discrete eigenvalues of $H_t$ are stable (Theorem 2.9, Corollary 2.10). Finally, in Section 3, we show the finiteness or infiniteness of discrete eigenvalues of $H_t$ for $d = 3$ (Theorems 3.1–3.2).

# 2 Mean-Field Hamiltonian in General Dimensions

In the following, we denote $L^p(\mathbb{R}^d)$ (resp. $H^s(\mathbb{R}^d)$) as $L^p$ (resp. $H^s$) for $0 < p \leq \infty$ and $s \in \mathbb{R}$. Unless otherwise stated, we assume that $V : \mathbb{R}^d \to \mathbb{R}$ satisfies the following two conditions:

(A) $V$ is even; i.e., $V(x) = V(-x)$ for any $x \in \mathbb{R}^d$.

(B) There exist some $V_1 \in L^p$ and $V_2 \in L^\infty$ such that $V = V_1 + V_2$, where $p = 2$ if $d = 1, 2, 3$ and $p > d/2$ if $d \geq 4$.

The following proposition is well-known:

**Proposition 2.1.** For any $\varphi_0 \in H^2$, there exists a time-global solution $\varphi_t$ of (1.1) uniquely defined on a suitable open interval $I_{\varphi_0}$ of $\mathbb{R}$ containing the origin such that $\varphi_t \in C(I_{\varphi_0}, H^2) \cap C^1(I_{\varphi_0}, L^2)$. Furthermore, mass and energy are conserved:

$$\begin{aligned} \|\varphi_t\|_{L^2} &= \|\varphi\|_{L^2}, \quad t \in I_{\varphi_0}, \\ E(\varphi_t) :&= \|\nabla\varphi_t\|_{L^2}^2 + \frac{1}{2}\int_{\mathbb{R}^d} V * |\varphi_t|^2(x)|\varphi_t|^2(x)dx \\ &= E(\varphi_0), \quad t \in I_{\varphi_0}. \end{aligned} \tag{2.1}$$

The proof is shown in Chapter 4 of [Caz].

In the following, we set

$$V_t = V * |\varphi_t|^2, \quad t \in \mathbb{R}$$

and

$$\langle x \rangle^\theta = (1 + |x|^2)^{\theta/2}, \quad x \in \mathbb{R}^d,\ \theta \in \mathbb{R}.$$

Before proving the self-adjointness of $H_t$, we need the following two lemmas. Since the first lemma is well-known, its proof is omitted (see e.q., [Is]).

**Lemma 2.2.** Let $V : \mathbb{R}^d \to \mathbb{R}$ satisfy (B). Then we have

$$\|Vf\|_{L^2} \leq t^{-\frac{d}{2p}} \|\langle x \rangle^{-2}\|_{L^p} \|V\|_{L^p} \|1 - t\Delta f\|_{L^2} \tag{2.2}$$

for any $t > 0$ and $f \in D(-\Delta)$.

**Lemma 2.3.** Let $q > 1$ and $V \in L^q$. Then $V_t \in L^q$ and

$$\|V_t\|_{L^q} \leq \|V\|_{L^q} \|\varphi_0\|_{L^2}^2 \tag{2.3}$$

holds for any $t \in I_{\varphi_0}$. Moreover, we have

$$\|V_t - V_s\|_{L^q} \leq 2\|V\|_{L^q} \|\varphi_0\|_{L^2} \|\varphi_t - \varphi_s\|_{L^2},\ t, s \in I_{\varphi_0}. \tag{2.4}$$

*Proof.* Since $|\varphi_t|^2 \in L^1$ and $\||\varphi_t|^2\|_{L^1} = \|\varphi_0\|_{L^2}^2$, the Hölder-Young inequality gives

$$\begin{aligned} \|V_t\|_{L^q} &\leq \|V\|_{L^q} \||\varphi_t|^2\|_{L^1} \\ &= \|V\|_{L^q} \|\varphi_0\|_{L^2}^2, \end{aligned}$$

which implies (2.3). To prove (2.4), we first calculate (2.5) by the Minkowski inequality as follows:

$$\begin{aligned} \|V_t - V_s\|_{L^q} &= \|V * (|\varphi_t|^2 - |\varphi_s|^2)\|_{L^q} \\ &= \|V * \{(|\varphi_t| - |\varphi_s|)(|\varphi_t| + |\varphi_s|)\}\|_{L^q} \\ &\leq \|V * \{(|\varphi_t| - |\varphi_s|)|\varphi_t|\}\|_{L^q} + \|V * \{(|\varphi_t| - |\varphi_s|)|\varphi_s|\}\|_{L^q} \\ &\leq \|V\|_{L^q} \|(|\varphi_t| - |\varphi_s|)|\varphi_t|\|_{L^1} + \|V\|_{L^q} \|(|\varphi_t| - |\varphi_s|)|\varphi_s|\|_{L^1}. \end{aligned} \tag{2.5}$$

The final inequality of (2.5) again uses the Hölder-Young inequality. Applying the Schwarz inequality and the conservation of mass, the final right-hand side of the above inequality becomes

$$\begin{aligned} &\|V\|_{L^q} \||\varphi_t| - |\varphi_s|\|_{L^2} \|\varphi_t\|_{L^2} + \|V\|_{L^q} \||\varphi_t| - |\varphi_s|\|_{L^2} \|\varphi_s\|_{L^2} \\ &\leq \|V\|_{L^q} \|\varphi_t - \varphi_s\|_{L^2} \|\varphi_t\|_{L^2} + \|V\|_{L^q} \|\varphi_t - \varphi_s\|_{L^2} \|\varphi_s\|_{L^2} \\ &= 2\|V\|_{L^p} \|\varphi_t - \varphi_s\|_{L^2} \|\varphi_0\|_{L^2}. \end{aligned} \tag{2.6}$$

Combining (2.5) and (2.6), we obtain (2.4). $\square$

**Proposition 2.4.** For any $V : \mathbb{R}^d \to \mathbb{R}$ satisfying assumptions (A) and (B), the following properties hold:
(i) $H_t$ is a self-adjoint operator on $L^2$ with $D(H_t) = D(-\Delta)$ for any $t \in I_{\varphi_0}$.
(ii) For any fixed $t_0 \in I_{\varphi_0}$, $H_t$ converges to $H_{t_0}$ in the norm resolvent sense as $t \to t_0$.

*Proof.* Property (i) follows directly from Lemmas 2.2 and 2.3 and the Kato-Rellich theorem. To prove (ii), we calculate

$$(H_t - z)^{-1} - (H_s - z)^{-1} = (H_t - z)^{-1}(V_s - V_t)(H_s - z)^{-1}$$

for $z \in \mathbb{C} \setminus \mathbb{R}$. Hence by Lemma 2.7 below, we have

$$\begin{aligned}\|(H_t - z)^{-1} - (H_s - z)^{-1}\| &\leq \|(H_t - z)^{-1}(V_s - V_t)(H_s - z)^{-1}\| \\ &\leq \frac{\alpha C(s,z)}{|\mathrm{Im} z|}\|\varphi_t - \varphi_s\|_{L^2}.\end{aligned}$$

Since $\varphi_t \in C^1(I_{\varphi_0}, L^2)$ by Proposition 2.1, the proof is complete. $\square$

Imposing stronger assumptions on $V$, we can identify the essential spectrum $\sigma_{\mathrm{ess}}(H_t)$. For this purpose, we specify a class of potentials $V$.

We say that $V \in L^p + L^{\infty,\epsilon}$ if for any $\epsilon > 0$ there exist $V_1 \in L^p$ and $V_2 \in L^\infty$ such that

$$V = V_1 + V_2, \ \|V_2\|_{L^\infty} < \epsilon.$$

**Theorem 2.5.** Assume that $V : \mathbb{R}^d \to \mathbb{R}$ satisfies (A) and belongs to the class of $L^p(\mathbb{R}^d) + L^{\infty,\epsilon}(\mathbb{R}^d)$ for $p = 2$ $(d = 3)$, $p > d/2$ $(d \geq 4)$. Then we have

$$\sigma_{\mathrm{ess}}(H_t) = [0, \infty). \tag{2.7}$$

*Proof.* As is well-known (see e.q., [ReSi IV]), it is sufficient to show that $V_t \in L^p(\mathbb{R}^d) + L^{\infty,\epsilon}(\mathbb{R}^d)$. By the Hölder inequality and the conservation of mass, we have

$$\begin{aligned}\|V_2 * |\varphi_t|^2\|_{L^\infty} &\leq \|V_2\|_{L^\infty}\|\varphi_t\|_{L^2}^2 \\ &\leq \epsilon\|\varphi_0\|_{L^2}^2.\end{aligned} \tag{2.8}$$

Combining this inequality with Lemma 2.3 completes the proof. $\square$

As a corollary, we note the existence of a ground state of $H_t$.

**Corollary 2.6.** Let $V$ be given as in Theorem 2.5. Also, assume that $E(\varphi_0) < 0$. Then, $H_t$ has a ground state for any $t \in I_{\varphi_0}$.

*Proof.* By the min-max principle (see e.q., [ReSi, IV]), finding a unit vector $\psi \in D(H_t) \subset H^2$ such that $\langle \psi, H_t \psi \rangle < 0$ is sufficient, since we know that $\inf \sigma_{\mathrm{ess}}(H_t)$=0 by Theorem 2.5. However, since energy is conserved, we have

$$\begin{aligned}\langle \varphi_t, H_t \varphi_t \rangle &= \langle \varphi_t, -\Delta \varphi_t \rangle + \langle \varphi_t, V_t \varphi_t \rangle \\ &\leq 2\langle \varphi_t, -\Delta \varphi_t \rangle + \langle \varphi_t, V_t \varphi_t \rangle \\ &= 2E(\varphi_t) = 2E(\varphi_0) < 0.\end{aligned}$$

Taking $\psi = \varphi_t / \|\varphi_t\|$, the proof is completed. □

Next, we develop a stability theorem for a discrete eigenvalue of $H_t$. For this purpose, we adopt the general perturbation theory established in [Ar].

Let us denote the resolvent set of a self-adjoint operator $T$ by $\rho(T)$. For $z \in \rho(T)$, we set

$$R_T(z) = (T - z)^{-1}.$$

We first prove the following lemma:

**Lemma 2.7.** There exists a positive constant $\alpha$ dependent only on $V$ and $\varphi_0$ such that

$$\|(V_t - V_s) R_{H_s}(z)\| \leq \alpha C(s, z) \|\varphi_t - \varphi_s\|_{L^2}$$

holds for all $s, t \in I_{\varphi_0}$ and $z \in \rho(H)$. Here we set

$$C(s, z) = \|R_{H_s}(z)\| + \|H_s R_{H_s}(z)\|.$$

**Remark 2.8.** Since $\sup_{z \in K} C(s, z) < \infty$ holds for all compact subsets $K \subset \rho(H)$, it follows that

$$\lim_{t \to s} \|(V_t - V_s) R_{H_s}(z)\| = 0 \tag{2.9}$$

uniformly in $z$ on each compact subset $K \subset \rho(H)$.

*Proof.* Since

$$\begin{aligned}\|(V_t - V_s) R_{H_s}(z)\| &\leq \|(V_t - V_s)(-\Delta + 1)^{-1}\| \|(-\Delta + 1) R_{H_s}(z)\| \\ &=: B_1 B_2,\end{aligned} \tag{2.10}$$

we only need to estimate $B_1$ and $B_2$. We first estimate $B_1$. Setting $t = 1$ in (2.2) together with (2.4) we see that

$$\begin{aligned}B_1 &\leq \|\langle x \rangle^{-2}\|_{L^p} \|V_t - V_s\|_{L^p} \\ &\leq 2\|\langle x \rangle^{-2}\|_{L^p} \|V\|_{L^p} \|\varphi_0\|_{L^2} \|\varphi_t - \varphi_s\|_{L^2}.\end{aligned} \tag{2.11}$$

Next we estimate $B_2$. First we note that, by (2.2) and (2.3),

$$\begin{aligned}\|V_s f\|_{L^2} &\le \|\langle x\rangle^{-2}\|_{L^p}\|V_s\|_{L^p}\{t^{1-\frac{d}{2p}}\|-\Delta f\| + t^{-\frac{d}{2}}\|f\|\} \\ &\le \|\langle x\rangle^{-2}\|_{L^p}\|V\|_{L^p}\|\varphi_0\|_{L^2}^2\{t^{1-\frac{d}{2p}}\|-\Delta f\|_{L^2} + t^{-\frac{d}{2}}\|f\|_{L^2}\}\end{aligned}$$

holds for any $f \in D(-\Delta)$. Since $1 - d/(2p) > 0$ by assumption (B), we obtain

$$\|V_s f\|_{L^2} \le \frac{1}{2}\|-\Delta f\|_{L^2} + C\|f\|_{L^2}$$

for any $f \in D(-\Delta)$, where $C > 0$ depends only on $V$. Hence we see that

$$\begin{aligned}\|-\Delta R_{H_s}(z)\| &= \|(H_s - V_s)R_{H_s}(z)\| \\ &\le \|H_s R_{H_s}(z)\| + \|V_s R_{H_s}(z)\| \\ &\le \|H_s R_{H_s}(z)\| + \frac{1}{2}\|-\Delta R_{H_s}(z)\| + C\|R_{H_s}(z)\|,\end{aligned}$$

which implies the inequality

$$B_2 \le 2\|H_s R_{H_s}(z)\| + (2C+1)\|R_{H_s}(z)\|. \tag{2.12}$$

Combining (2.10), (2.11), and (2.12), we complete the proof. □

In stating the following result, we require some notations defined in [Ar]. Suppose that $t_0 \in I_{\varphi_0}$ is arbitrary and fixed. Suppose also that $H_{t_0}$ has an isolated eigenvalue $E_0 \in \mathbb{R}$. Let $r_0$ be a constant satisfying

$$0 < r_0 < \min_{E\in\sigma(H_{t_0})\setminus\{E_0\}} |E - E_0|.$$

Then, we define

$$C_{r_0}(E_0) = \{z \in \mathbb{C} \,|\, |z - E_0| < r_0\}.$$

Note that $C_{r_0}(E_0) \subset \mathbb{R} \setminus \sigma(H_{t_0})$. By virtue of Remark 2.8, there exists $\delta > 0$ such that

$$\sup_{z\in C_{r_0}(E_0)} \|(V_t - V_{t_0})R_{H_{t_0}}(z)\| < \left(1 + r\sup_{z\in C_{r_0}(E_0)}\|R_{H_{t_0}}(z)\|\right)^{-1}$$

for any $t \in (t_0 - \delta, t_0 + \delta)$.

We now state the following:

**Theorem 2.9.** Suppose that the multiplicity $m(E_0)$ of $E_0$ is finite. Then, for all $t \in (t_0 - \delta, t_0 + \delta)$, $H_t$ has exactly $m(E_0)$ eigenvalues in the interval $(E_0 - r_0, E_0 + r_0)$, including multiplicities, and $\sigma(H_t) \cap (E_0 - r_0, E_0 + r_0)$ consists of these eigenvalues alone.

**Corollary 2.10.** Suppose in addition that $m(E_0) = 1$ and let $\Omega_0$ be a normalized eigenvector of $H_{t_0}$ with an eigenvalue $E_0$, i.e., $H_{t_0}\Omega_0 = E_0\Omega_0$ and $\|\Omega_0\| = 1$. Then, for all $t \in (t_0 - \delta, t_0 + \delta)$, $H_t$ has a simple eigenvalue $E_t$ in the interval $(E_0 - r_0, E_0 + r_0)$ given by

$$E_t = E_0 + \frac{\langle \Omega_0, V_t - V_{t_0}\Omega_0 \rangle + \sum_{n=1}^{\infty} c_n(t)}{1 + \sum_{n=1}^{\infty} d_n(t)},$$

where

$$\begin{aligned} c_n(t) :&= \frac{(-1)^{n+1}}{2\pi i} \int_{C_{r_0}(E_0)} dz \langle \Omega_0, [(V_t - V_{t_0})R_{H_t}(z)]^{n+1}\Omega_0 \rangle, \\ d_n(t) :&= \frac{(-1)^{n+1}}{2\pi i} \int_{C_{r_0}(E_0)} dz \frac{\langle \Omega_0, [(V_t - V_{t_0})R_{H_t}(z)]^{n}\Omega_0 \rangle}{E_0 - z} \end{aligned}$$

and $\sigma(H_t) \cap (E_0 - r_0, E_0 + r_0) = \{E_t\}$. Moreover, a normalized eigenvector of $H_t$ is given as

$$\Omega_t = \frac{\Omega_0 + \sum_{n=1}^{\infty} \Omega_{t,n}}{\sqrt{1 + \sum_{n=1}^{\infty} d_n(t)}},$$

where

$$\Omega_{t,n} := \frac{(-1)^{n+1}}{2\pi i} \int_{C_{r_0}(E_0)} dz (H_{t_0} - z)^{-1} [(V_t - V_{t_0})R_{H_t}(z)]^{n}\Omega_0.$$

*Proof.* Both of the above statements are direct consequences of Theorem A.3 and Corollary A.4 in [Ar]. In that reference, equation (2.9) is given as Hypothesis (A). □

# 3 Mean-Field Hamiltonian in Three Spatial Dimensions

In this section we consider the case $d = 3$, and show the finiteness or infiniteness of the discrete spectrum $\sigma_{\mathrm{disc}}(H_t)$ of $H_t$.

First we treat the finiteness. We assume that

$$\text{(C)}\ V \le 0,\ V \in L^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3).$$

Clearly, condition (C) implies condition (B) in Section 2.

**Theorem 3.1.** Assume that $V$ satisfies (A) and (C). Then $\sigma_{\mathrm{disc}}(H_t) \subset (-\infty, 0)$. Moreover, the number of discrete eigenvalues $N(H_t)$ can be estimated by $a(\|V\|_{L^1}\|\varphi_t\|_{L^3})^{2/3}$, where $a > 0$ is some universal constant.

*Proof.* The first statement immediately follows from Theorem 2.5. The second follows from the Lieb-Cwickel-Rosenbljum bound (see e.g., [Si2])

$$N(H_t) \le a\|V_t\|_{L^{\frac{3}{2}}}^{\frac{2}{3}}$$

and the estimate

$$\|V_t\|_{L^{\frac{3}{2}}} \le \|V\|_{L^1}\|\varphi_t\|_{L^3}$$

given by the Hölder-Young inequality. $\square$

Finally, we consider the infiniteness of $\sigma_{\rm disc}(H_t)$ with

$$V(x) := -\frac{a}{|x|^\gamma},\ a > 0,\ \gamma \in (0, 2) \tag{3.1}$$

and $\varphi_0$ in $\mathcal{S}(\mathbb{R}^3)$, denoting the space of rapidly decreasing $C^\infty$ functions on $\mathbb{R}^3$. For such $V$ and $\varphi_0$, the following proposition holds:

**Proposition 3.2.** There exists a unique solution $\varphi_t \in C^1(\mathbb{R}, \mathcal{S}(\mathbb{R}^3))$ of (1.1). The proof is provided in Proposition 3.1 of [HaOz].

We also need the following lemma:

**Lemma 3.3.** For any pair $(\alpha, \beta)$ satisfying $0 < \alpha < d < \beta$, we have

$$\int_{\mathbb{R}^d} |x-y|^{-\alpha}\langle y\rangle^{-\beta} \le C\langle x\rangle^{-\alpha}.$$

For the proof, the reader is referred to Appendix 2, Lemma 1(c) in [AS] and Lemma 17.2 in [Is].

We can now state the following:

**Theorem 3.2.** Let $V : \mathbb{R}^3 \to \mathbb{R}$ be of the form (3.1). Then, $H_t$ is a self-adjoint operator on $L^2$ and $\sigma_{\rm ess}(H_t) = [0, \infty)$. Moreover, $\sigma_{\rm disc}(H_t) \subset (-\infty, 0)$ is infinite.

*Proof.* Let us consider the first part of the statement. By the proof of Lemma 2.5 in [HaOz], we have

$$\|V_t\|_{L^\infty} \le C\|\nabla\varphi_t\|^\gamma\|\varphi_t\|^{2-\gamma}.$$

Moreover, by the virtue of Lemma 3.3, we see that

$$\begin{aligned}\left|\int_{\mathbb{R}^3} |x-y|^{-\gamma}|\varphi_t|^2(y)dy\right| &\le \|\langle y\rangle^4|\varphi_t|^2\|_{L^\infty}\int_{\mathbb{R}^3} |x-y|^{-\gamma}\langle y\rangle^{-4}dy\\ &\le C\langle y\rangle^{-\gamma},\end{aligned}$$

since $\varphi_t \in \mathcal{S}(\mathbb{R}^3)$ is true by Proposition 3.2. Therefore, we know that $V_t \in L^\infty$ and $V_t$ tends to 0 as $|x| \to \infty$, implying that $V_t \in L^2 + L^{\infty,\epsilon}$. Hence, as when proving Theorem 2.5, we have completed the proof of the first part.

It remains to prove the infiniteness of $\sigma_{\rm disc}(H_t)$. By a well-known theorem (see e.q., [Ar, Si2]), it is sufficient to show that there exists $C, R > 0$ such that

$$V_t(x) \le -\frac{C}{|x|^\gamma},\ |x| \ge R.$$

Following the proof of Proposition 5 in [Si1], we need only to show that

$$\int_{|y|\ge r_0} |x-y|^\gamma |\varphi_t(x-y)|^2 |y|^{-\gamma} dy \to 0 \quad \text{as} \quad |x| \to \infty. \tag{3.2}$$

Since

$$|y|^\gamma |\varphi_t(y)|^2 \le \| \langle y \rangle^{4+\gamma} |\varphi_t(y)|^2 \|_{L^\infty} \langle y \rangle^{-4},$$

the above integral can be estimated by

$$\begin{aligned}\int_{\mathbb{R}^3} |x-y|^\gamma |\varphi_t(x-y)|^2 |y|^{-\gamma} dy &= \int_{\mathbb{R}^3} |y|^\gamma |\varphi_t(y)|^2 |x-y|^{-\gamma} dy \\ &\le C \int_{\mathbb{R}^3} \langle y \rangle^{-4} |x-y|^{-\gamma} \\ &\le C \langle x \rangle^{-\gamma},\end{aligned}$$

which demonstrates the truth of (3.2). □

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