GROSSBERG-KARSHON TWISTED CUBES AND HESITANT WALK AVOIDANCE

MEGUMI HARADA AND EUNJEONG LEE

ABSTRACT. Let G be a complex semisimple simply connected linear algebraic group. Let λ be a dominant weight for G and $\mathcal{I} = (i_1, i_2, \ldots, i_n)$ a word decomposition for an element $w = s_{i_1} s_{i_2} \cdots s_{i_n}$ of the Weyl group of G, where the s_i are the simple reflections. In the 1990s, Grossberg and Karshon introduced a virtual lattice polytope associated to λ and $\mathcal{I}$, which they called a **twisted cube**, whose lattice points encode (counted with sign according to a density function) characters of representations of G. In recent work, the first author and Jihyeon Yang prove that the Grossberg-Karshon twisted cube is untwisted (so the support of the density function is a closed convex polytope) precisely when a certain torus-invariant divisor on a toric variety, constructed from the data of λ and $\mathcal{I}$, is basepoint-free. This corresponds to the situation in which the Grossberg-Karshon character formula is a true combinatorial formula, in the sense that there are no terms appearing with a minus sign. In this note, we translate this toric-geometric condition to the combinatorics of $\mathcal{I}$ and λ. More precisely, we introduce the notion of **hesitant λ-walks** and then prove that the associated Grossberg-Karshon twisted cube is untwisted precisely when $\mathcal{I}$ is hesitant-λ-walk-avoiding.

CONTENTS

INTRODUCTION

Let G be a complex semisimple simply connected linear algebraic group. Building combinatorial models for G-representations is a fruitful technique in modern representation theory; a famous example is the theory of crystal bases and string polytopes. In a different direction, given a dominant weight λ and a choice of word expression $\mathcal{I} = (i_1, i_2, \ldots, i_n)$ of an element $w = s_{i_1} s_{i_2} \cdots s_{i_n}$ in the Weyl group, Grossberg and Karshon [3] introduced a combinatorial object called a **twisted cube** $(C(\mathbf{c}, \ell), \rho)$ where $C(\mathbf{c}, \ell)$ is a subset of $\mathbb{R}^n$ and ρ is a support function with support precisely $C(\mathbf{c}, \ell)$. The lattice points of $C(\mathbf{c}, \ell)$ encode (counted with $\pm$ sign according to ρ) the character of the G-representation V_λ [3, Theorems 5 and 6]. Here the parameters $\mathbf{c}$ and ℓ are determined from λ and $\mathcal{I}$. These twisted cubes are combinatorially much simpler than general string polytopes but they are not "true" polytopes in the sense that their faces may have various angles and the intersection of faces may not be a face (cf. [3, §2.5 and Figure 1.1 therein]), and in general they may be neither closed nor convex (see Example 1.2). In particular, the Grossberg-Karshon character formula is not a purely combinatorial 'positive' formula, since it may involve minus signs.

The main result of this note gives necessary and sufficient conditions on a dominant weight λ and a (not necessarily reduced) word expression $\mathcal{I} = (i_1, \ldots, i_n)$ of an element $w \in W$, such that the associated Grossberg-Karshon twisted cube is untwisted (cf. Definition 1.3), i.e., $C(\mathbf{c}, \ell)$ is a closed, convex polytope and ρ is identically equal to 1 on $C(\mathbf{c}, \ell)$. This is precisely the situation in which the Grossberg-Karshon character formula is a 'true' combinatorial formula, in the sense that it is a purely 'positive' formula (with no terms appearing with a minus sign).

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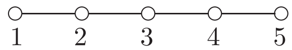
In order to state our result it is useful to introduce some terminology (see Section 2 for details). Roughly, we say that a word $\mathcal{I} = (i_1, \ldots, i_n)$ is a **diagram walk** (or simply **walk**) if successive roots are adjacent in the Dynkin diagram: for instance, in type A_5



the word $\mathcal{I} = (2, 4, 5)$ with corresponding simple roots (s_2, s_4, s_5) is not a walk since s_2 and s_4 are not adjacent, but $\mathcal{I} = (1, 2, 3, 2, 1)$ is a walk. Moreover, given a dominant weight $\lambda = \lambda_1\varpi_1 + \cdots + \lambda_r\varpi_r$ written as a linear combination of the fundamental weights $\{\varpi_1, \ldots, \varpi_r\}$, we say $\mathcal{I} = (i_1, i_2, \ldots, i_n)$ is a λ-walk if it is a walk and if it ends at a root which appears in λ, i.e. $\lambda_{i_n} > 0$. A **hesitant λ-walk** is a word $\mathcal{I} = (i_0, i_1, \ldots, i_n)$ where $i_0 = i_1$, so there is a repetition at the first step, and the subword $(i_1, i_2, \ldots, i_n)$ is a λ-walk. Finally, a word is **hesitant-λ-walk-avoiding** if there is no subword which is a hesitant λ-walk. With this terminology we can state the main result of this paper.

Theorem. *Let $\mathcal{I} = (i_1, i_2, \ldots, i_n)$ be a word decomposition of an element $w = s_{i_1}s_{i_2}\cdots s_{i_n}$ of the Weyl group W and let $\lambda = \lambda_1\varpi_1 + \lambda_2\varpi_2 + \cdots + \lambda_r\varpi_r$ be a dominant weight. Then the corresponding Grossberg-Karshon twisted cube $(C(\mathbf{c}, \ell), \rho)$ is untwisted if and only if $\mathcal{I}$ is hesitant-λ-walk-avoiding.*

We note that "pattern avoidance" is an important notion in the study of Schubert varieties and Schubert calculus, first pioneered by Lakshmibai and Sandhya [9] and further studied by many others (see e.g. [1] and references therein). It would be interesting to explore the relation between our notion of hesitant-λ-walk-avoidance with the other types of pattern avoidance in the theory of flag and Schubert varieties.

We additionally remark that Kiritchenko recently has defined *divided difference operators D_i on polytopes* and, using these D_i inductively together with a fixed choice of reduced word decomposition for the longest element in the Weyl group of G, she constructs (possibly virtual) polytopes whose lattice points encode the character of irreducible G-representations [8, Theorem 3.6]. Kiritchenko's virtual polytopes are generalizations of both Gel'fand-Cetlin polytopes and the Grossberg-Karshon twisted polytopes. It would be interesting to explore whether our methods can be further generalized to study Kiritchenko's virtual polytopes (see Section 5).

This paper is organized as follows. In Section 1 we recall the necessary definitions and background from previous papers. In particular, we recall the results of the first author and Jihyeon Yang [4, Proposition 2.1 and Theorem 2.4] which characterize the untwistedness of the Grossberg-Karshon twisted cube in terms of the Cartier data associated to a certain toric divisor on a toric variety; this is a key tool for our proof. In Section 2 we introduce the notions of diagram walks and hesitant λ-walks and make the statement of our main theorem. We prove the the necessity of hesitant-λ-walk-avoidance in Section 3. The proof of sufficiency, which occupies Section 4, is in part a case-by-case analysis according to Lie type. We briefly record some open questions in Section 5.

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1. BACKGROUND

We begin by recalling the definition of **twisted cubes** given by Grossberg and Karshon [3, §2.5]. We follow the exposition in [4]. Fix a positive integer n. A twisted cube is a pair $(C(\mathbf{c}, \ell), \rho)$ where $C(\mathbf{c}, \ell)$ is a subset of $\mathbb{R}^n$ and $\rho : \mathbb{R}^n \to \mathbb{R}$ is a density function with support precisely equal to $C(\mathbf{c}, \ell)$. Here $\mathbf{c} = \{c_{jk}\}_{1 \le j < k \le n}$ and $\ell = \{\ell_1, \ell_2, \ldots, \ell_n\}$ are fixed integers. (The general definition in [3] only requires the ℓ_i to be real numbers, but since we restrict attention to the cases arising from representation theory, our ℓ_i will always be integers.) In order to simplify the notation in what follows, we define the following functions on $\mathbb{R}^n$:

$$
\begin{aligned}
A_n(x) &= A_n(x_1,\dots,x_n) = \ell_n \\
A_j(x) &= A_j(x_1,\dots,x_n) = \ell_j - \sum_{k>j} c_{jk}x_k \text{ for all } 1 \le j \le n-1.
\end{aligned}
\tag{1.1}
$$

We also define a function $\operatorname{sgn} : \mathbb{R} \to \{\pm 1\}$ by $\operatorname{sgn}(x) = 1$ for $x < 0$ and $\operatorname{sgn}(x) = -1$ for $x \ge 0$.

We now give the precise definition.

Definition 1.1. Let $n, \mathbf{c}, \ell$ and A_j be as above. Let $C(\mathbf{c}, \ell)$ denote the following subset of $\mathbb{R}^n$:

$$C(\mathbf{c},\ell) := \{x = (x_1,\dots,x_n) \in \mathbb{R}^n \mid \text{ for all } 1 \le j \le n, A_j(x) < x_j < 0 \text{ or } 0 \le x_j \le A_j(x)\} \subseteq \mathbb{R}^n. \tag{1.2}$$

Moreover we define a density function $\rho : \mathbb{R}^n \to \mathbb{R}$ by

$$\rho(x) = \begin{cases} (-1)^n \prod_{k=1}^n \operatorname{sgn}(x_k) & \text{if } x \in C(\mathbf{c},\ell) \\ 0 & \text{else.} \end{cases} \tag{1.3}$$

Evidently $\operatorname{supp}(\rho) = C(\mathbf{c}, \ell)$. We call the pair $(C(,\ell), \rho)$ the **twisted cube associated to $\mathbf{c}$ and ℓ.**

A twisted cube may not be a cube in the standard sense. In particular, the set C may be neither convex nor closed, as the following example shows. See also the discussion in [3, §2.5].

Example 1.2. Let $n = 2$ and let $\ell = (\ell_1 = 3, \ell_2 = 5)$ and $\mathbf{c} = \{c_{12} = 1\}$. Then

$$C = \{(x_1,x_2) \in \mathbb{R}^2 \mid 0 \le x_2 \le 5 \text{ and } (3 - x_2 < x_1 < 0 \text{ or } 0 \le x_1 \le 3 - x_2)\}.$$

See Figure 1.1. The value of the density function ρ is recorded within each region.

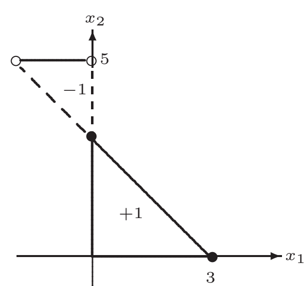


FIGURE 1.1

Note in particular that C does *not* contain the points $\{(0, x_2) \mid 3 < x_2 < 5\}$ and the points $\{(x_1, x_2) \mid 3 < x_2 < 5 \text{ and } x_1 = 3 - x_2\}$, so C is not closed, and it is also not convex.

As mentioned in the introduction, the main goal of this note is to give necessary and sufficient conditions for the *untwistedness* of the twisted cube, stated in terms of the combinatorics of the defining parameters. The following makes the notion precise.

Definition 1.3. (cf. [4, Definition 2.2]) We say that Grossberg-Karshon twisted cube $(C = C(\mathbf{c}, \ell), \rho)$ is **untwisted** if C is a closed convex polytope, and the support for ρ is constant and equal to 1 on C and 0 elsewhere. We say the twisted cube is **twisted** if it is not untwisted.

The main result of [4] characterizes the untwistedness of the Grossberg-Karshon twisted cube in terms of the basepoint-freeness of a certain toric divisor on a toric variety constructed from the data of $\mathbf{c}$ and ℓ, which in turn can be stated in terms of the so-called Cartier data $\{m_\sigma\}$ associated to the divisor. In particular, in this paper we will not require the geometric perspective; instead we work with the integer vectors m_σ, which can be derived directly from the constants $\mathbf{c}$ and ℓ. Before quoting the relevant result from [4] we need some terminology.

Let $\{e_1^+, \dots, e_n^+\}$ be the standard basis of $\mathbb{R}^n$. For $\sigma = (\sigma_1, \dots, \sigma_n) \in \{+,-\}^n$, define $m_\sigma = (m_{\sigma,1}, \dots, m_{\sigma,n}) = \sum m_{\sigma,k} e_k^+ \in \mathbb{Z}^n$ as follows, using the functions $A_k(x)$ defined in (1.1).

$$m_{\sigma,k} = \begin{cases} 0 & \text{if } \sigma_k = + \\ A_k(m_{\sigma,k+1}, \dots, m_{\sigma,n}) & \text{if } \sigma_k = -. \end{cases} \tag{1.4}$$

We will also need a certain polytope P_D as follows:

$$P_D = \{x \in \mathbb{R}^n \mid 0 \leq x_j \leq A_j(x) \text{ for all } 1 \leq j \leq n\} \subseteq \mathbb{R}^n. \tag{1.5}$$

With this notation in place we can quote the following.

Theorem 1.4. *(cf. [4, Proposition 2.1]) Let $n, \mathbf{c}$ and ℓ be as above and let $(C(\mathbf{c}, \ell), \rho)$ denote the corresponding Grossberg-Karshon twisted polytope. Then $(C(\mathbf{c}, \ell), \rho)$ is untwisted if and only if $m_{\sigma,k} \geq 0$ for all $\sigma \in \{+, -\}^n$ and for all k with $1 \leq k \leq n$.*

Recall that the goal of this note is to analyze the case when the defining parameters for the Grossberg-Karshon twisted polytope arise from certain representation-theoretic data. We now briefly describe how to derive the $\mathbf{c}$ and ℓ in this case.

Following [3], let G be a complex semisimple simply-connected linear algebraic group of rank r over an algebraically closed field $\mathbf{k}$. Choose a Cartan subgroup $H \subset G$, and a Borel subgroup Let $\{\alpha_1, \ldots, \alpha_r\}$ denote the simple roots, $\{\alpha_1^\vee, \ldots, \alpha_r^\vee\}$ the coroots, and $\{\varpi_1, \ldots, \varpi_r\}$ the fundamental weights (characterized by the relation $\langle \varpi_i, \alpha_j^\vee \rangle = \delta_{ij}$). Let $s_{\alpha_i} \in W$ denote the simple reflection in the Weyl group corresponding to the root α_i.

Fix a choice $\lambda = \lambda_1\varpi_1 + \cdots + \lambda_r\varpi_r$ in the weight lattice, where $\lambda_i \in \mathbb{Z}$. Let $\mathcal{I} = (i_1, \ldots, i_n)$ be a sequence of elements in $[r] := \{1, 2, \ldots, r\}$; this corresponds to a (not necessarily reduced) decomposition of an element $w = s_{\alpha_{i_1}} s_{\alpha_{i_2}} \cdots s_{\alpha_{i_n}}$ in W. For simplicity, we introduce the notation $\beta_j := \alpha_{i_j}$, so β_j is the j-th simple root appearing in the word decomposition. For such λ and $\mathcal{I}$ we define constants $\mathbf{c}, \ell$ by the formulas (cf. [3, §3.7])

$$c_{jk} = \langle \beta_k, \beta_j^\vee \rangle \tag{1.6}$$

for $1 \leq j < k \leq n$, and

$$\ell_1 = \langle \lambda, \beta_1^\vee \rangle, \ldots, \ell_n = \langle \lambda, \beta_n^\vee \rangle. \tag{1.7}$$

Note that if the j-th simple reflection in the given word decomposition $\mathcal{I}$ is equal to α_i, then $\ell_j = \lambda_i$, and that the constants c_{jk} are matrix entries in the Cartan matrix of G.

The following example illustrates these definitions.

Example 1.5. Consider $G = SL(3, \mathbb{C})$ with positive roots $\{\alpha_1, \alpha_2\}$. Let $\lambda = 2\varpi_1 + \varpi_2$ and $\mathcal{I} = (1, 2, 1)$. Then $(\beta_1, \beta_2, \beta_3) = (\alpha_1, \alpha_2, \alpha_1)$ and we have

$$\begin{aligned} c_{12} &= \langle \alpha_2, \alpha_1^\vee \rangle = -1 \\ c_{13} &= \langle \alpha_1, \alpha_1^\vee \rangle = 2 \\ c_{23} &= \langle \alpha_1, \alpha_2^\vee \rangle = -1 \\ \ell = (\ell_1, \ell_2, \ell_3) &= (\langle \lambda, \alpha_1^\vee \rangle = 2, \langle \lambda, \alpha_2^\vee \rangle = 1, \langle \lambda, \alpha_1^\vee \rangle = 2). \end{aligned} \tag{1.8}$$

As mentioned in the introduction, in the setting above Grossberg and Karshon derive a Demazure-type character formula for the irreducible G-representation corresponding to λ, expressed as a sum over the lattice points $\mathbb{Z}^n \cap C(\mathbf{c}, \ell)$ in the Grossberg-Karshon twisted cube $(C(\mathbf{c}, \ell), \rho)$ [3, Theorem 5 and Theorem 6]. The lattice points appear with a plus or minus sign according the density function ρ. Hence their formula is a *positive* formula if ρ is constant and equal to 1 on all of $C(\mathbf{c}, \ell)$. From the point of view of representation theory it is therefore of interest to determine conditions on the weight λ and the word decomposition $\mathcal{I} = (i_1, i_2, \ldots, i_n)$ for an element $w = s_{i_1} s_{i_2} \ldots s_{i_n}$ such that the associated Grossberg-Karshon twisted cube is in fact untwisted. This is the motivation for this note.

2. Diagram walks, hesitant walk avoidance, and the statement of the main theorem

In order to state our main theorem it is useful to introduce some terminology. In what follows, we fix an ordering on the simple roots as in Table 1; our conventions agree with that in the standard textbook of Humphreys [6]. In particular, given an index i with $1 \leq i \leq r$ where r is the rank of G, we may refer to its corresponding simple reflection $s_i := s_{\alpha_i}$, where the index i refers to the ordering of the roots in Table 1.

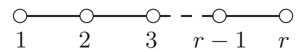
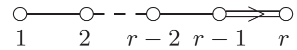
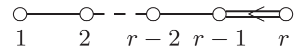
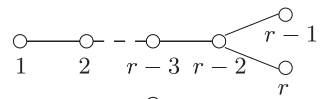
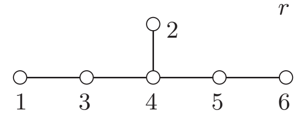
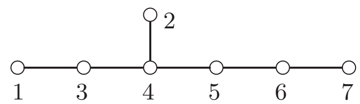
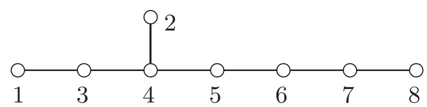
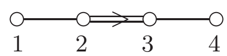
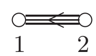
Φ	Dynkin diagram
A_r $(r \geq 1)$	
B_r $(r \geq 2)$	
C_r $(r \geq 3)$	
D_r $(r \geq 4)$	
E_6	
E_7	
E_8	
F_4	
G_2	

TABLE 1. Dynkin diagrams for all Lie types.

Definition 2.1. Let $\mathcal{I} = (i_1, i_2, \ldots, i_n) \in [r]^n$ be a (not necessarily reduced) word decomposition of an element $w = s_{i_1} s_{i_2} \cdots s_{i_n}$ of the Weyl group W. We say that $\mathcal{I}$ is a **diagram walk** (or **walk**) if successive simple roots are adjacent in the corresponding Dynkin diagram, or more precisely, for each $j \in [n-1] = \{1 \leq j \leq n-1\}$, the two successive roots α_{i_j} and $\alpha_{i_{j+1}}$ are distinct, and there is an edge in the corresponding Dynkin diagram connecting α_{i_j} and $\alpha_{i_{j+1}}$. We call i_1 (or α_{i_1}) the **initial root (of the diagram walk $\mathcal{I}$)** and denote it by $IR(\mathcal{I})$. We call i_n (or α_{i_n}) the **final root (of the diagram walk $\mathcal{I}$)** and denote it $FR(\mathcal{I})$.

Example 2.2. (1) In type A, the words $s_2 s_3 s_4 s_5 s_4 s_3$ and $s_1 s_2 s_1 s_2 s_3$ are both diagram walks. Note that the second word is not reduced.
(2) In type B, $s_{r-2} s_{r-1} s_r$ is a diagram walk.
(3) In type E_8, $s_1 s_3 s_4 s_2 s_4 s_5$ is a diagram walk.

In what follows, we also find it useful to consider words which are 'almost' diagram walks, except that the word begins with a repetition (thus disqualifying it from being a walk), i.e. the initial root appears twice.

Definition 2.3. Let $\mathcal{I} = (i_0, i_1, i_2, \ldots, i_n)$ be a (not necessarily reduced) word decomposition of an element $w = s_{i_0} s_{i_1} \cdots s_{i_n}$ of the Weyl group W. We say that $\mathcal{I}$ is a **hesitant (diagram) walk** if

- the length of the word is at least 2, i.e. $n \geq 1$,
- the first two roots are the same, i.e., $i_0 = i_1$, and
- the subword $(i_1, \ldots, i_n)$ is a diagram walk.

In other words, except for the 'hesitation' at the first step, the remainder of the word is a diagram walk. We refer to the subword $(i_1, \ldots, i_n)$ as the **walking component** of the hesitant walk.

A few remarks are in order. First, we emphasize that a hesitant walk, despite the terminology, is not actually a diagram walk; it becomes a diagram walk only after deleting the first entry in the word. Furthermore, it is clear that a hesitant (diagram) walk is never a reduced word decomposition (because of the two repeated roots at the beginning). On the other hand, it is possible for a reduced word decomposition to *contain* a hesitant walk as a subword: for instance, for $G = SL(4, \mathbb{C})$, the reduced word decomposition $s_1s_2s_3s_1s_2s_1$ for the longest element in the Weyl group S_4 contains $s_1s_1s_2$ as a subword, which is a hesitant walk.

Definition 2.4. Let $\mathcal{I} = (i_1, i_2, \ldots, i_n)$ be a word decomposition of an element $w = s_{i_1}s_{i_2}\cdots s_{i_n}$ of the Weyl group W. We say that $\mathcal{I}$ is **hesitant-walk-avoiding** if there is no subword $\mathcal{J} = (i_{j_0}, i_{j_1}, \ldots, i_{j_s})$ of $\mathcal{I}$ which is a hesitant walk.

Example 2.5. Let $G = SL(4, \mathbb{C})$ with Weyl group S_4. The reduced word decomposition $s_1s_2s_3$ is hesitant-walk-avoiding.

In what follows we will also be interested in dominant weights λ in the character lattice $X(T)$ associated to G. As in Section 1, we may express λ as a linear combination of the fundamental weights $\varpi_1, \ldots, \varpi_r$ corresponding to the simple roots $\alpha_1, \ldots, \alpha_r$. Thus we write

$$\lambda = \lambda_1\varpi_1 + \cdots + \lambda_r\varpi_r$$

and since we assume λ is dominant, $\lambda_i \geq 0$ for all $i = 1, \ldots, r$.

Definition 2.6. Let λ be as above. We say that a simple root α_i **appears in** λ if the corresponding coefficient is strictly positive, i.e.

$$\lambda_i = \langle \lambda, \alpha_i^\vee \rangle > 0. \tag{2.1}$$

We now introduce some terminology which relates diagram walks and hesitant walks to the dominant weight λ.

Definition 2.7. Let λ and $\mathcal{I}$ be as above. We will say that $\mathcal{I}$ is a **λ-walk** if

- $\mathcal{I}$ is a diagram walk, and
- the final root $FR(\mathcal{I})$ of the walk $\mathcal{I}$ appears in λ.

Similarly, we say that $\mathcal{I}$ is a **hesitant λ-walk** if it is a hesitant walk, and the final root of its walking component appears in λ. Finally, a word $\mathcal{I}$ is **hesitant-λ-walk-avoiding** if there is no subword $\mathcal{J}$ of $\mathcal{I}$ which is a hesitant λ-walk.

Example 2.8. Let $G = SL(4, \mathbb{C})$ with Weyl group S_4. Consider the reduced word decomposition $\mathcal{I} = (1, 2, 3, 1, 2, 1)$ of the longest element $w_0 = s_1s_2s_3s_1s_2s_1$ of S_4 and $\lambda = 3\varpi_3$. Then $\mathcal{I}$ is hesitant-λ-walk-avoiding.

Given the terminology introduced above we may now state our main theorem.

Theorem 2.9. *Let $\mathcal{I} = (i_1, i_2, \ldots, i_n)$ be a word decomposition of an element $w = s_{i_1}\cdots s_{i_n}$ of W and let $\lambda = \lambda_1\varpi_1 + \lambda_2\varpi_2 + \cdots + \lambda_r\varpi_r$ be a dominant weight. Let $\mathbf{c} = \{c_{jk}\}$ and $\ell = (\ell_1, \ldots, \ell_n)$ be determined from λ and $\mathcal{I}$ as in* (1.6) *and* (1.7)*. Then the corresponding Grossberg-Karshon twisted cube $(C(\mathbf{c}, \ell), \rho)$ is untwisted if and only if $\mathcal{I}$ is hesitant-λ-walk-avoiding.*

The proof of the above theorem occupies Sections 3 and 4.

3. Proof of the main theorem: necessity

We begin the proof of Theorem 2.9 by first proving the "only if" part of the statement, i.e., that hesitant-λ-walk-avoidance implies the untwistedness of the Grossberg-Karshon twisted cube.

We need some preliminary lemmas. Recall that the $m_\sigma = (m_{\sigma,1}, \ldots, m_{\sigma,n})$ are integer vectors defined by (1.4) associated to the defining constants $\mathbf{c}$ and ℓ of the twisted cube.

Lemma 3.1. *Let $\{c_{ij}\}_{1\leq i<j\leq n}$ and $\ell_1, \ldots, \ell_n$ be fixed integers. Assume that $\ell_i \geq 0$ for all i. If there exists an element σ of $\{+, -\}^n$ and $k \in [n]$ such that $m_{\sigma,k} > 0$ and $m_{\sigma,i} \geq 0$ for $i > k$, then there exists an increasing sequence $\mathcal{J}$ of indices $1 \leq j_1 < j_2 < \cdots < j_s \leq n$, with $s \geq 1$, such that*

1. *$j_1 = k$,*
2. *$\ell_{j_s} > 0$, and*
3. *$c_{j_tj_{t+1}} < 0$ for $t = 1, \ldots, s-1$.*

Proof. Let σ and k be as above. We may explicitly construct the subsequence $\mathcal{J}$ as follows. First suppose $\ell_k > 0$. In this case, the subsequence $\mathcal{J} = (j_1 = k)$ satisfies the required three conditions (the third being vacuous), so we are done. If on the other hand $\ell_k = 0$, we set $j_1 = k$ and then define j_2 as follows. By assumption $m_{\sigma,k} > 0$ so we know $\sigma_k = -$, and by definition of the m_σ we know

$$\begin{aligned} m_{\sigma,k} &= \ell_k - \sum_{i>k} c_{ki} m_{\sigma,i} \\ &= -\sum_{i>k} c_{ki} m_{\sigma,i}. \end{aligned} \tag{3.1}$$

Since $m_{\sigma,i} \geq 0$ for $i \geq k$ by assumption, in order for $m_{\sigma,k}$ to be strictly positive there must exist an index $J > k$ with $c_{kJ} < 0$ and $m_{\sigma,J} > 0$. Choose j_2 to be the minimal such index. If $\ell_{j_2} > 0$, then the sequence $\mathcal{J} = (j_1 = k, j_2)$ satisfies the required three conditions and we are done. Otherwise, we may repeat the above argument as many times as necessary (i.e. as long as $\ell_{j_t} = 0$). Since the indices j_t are bounded above by n, this process must stop, i.e. there must exist some $s \geq 1$ such that the sequence $\mathcal{J} = (j_1, \ldots, j_s)$ found in this manner satisfies the requirements.

□

In the case when the constants $\mathbf{c}$ and ℓ are obtained from the data of a weight λ and a word $\mathcal{I}$ we can interpret Lemma 3.1 using the terminology introduced in Section 2.

Corollary 3.2. *Let $\mathcal{I} = (i_1, i_2, \ldots, i_n)$ be a word decomposition of an element $w = s_{i_1} \cdots s_{i_n}$ of W and let $\lambda = \lambda_1\varpi_1 + \lambda_2\varpi_2 + \cdots + \lambda_r\varpi_r$ be a dominant weight, i.e. $\lambda_i \geq 0$ for all i. Let $\mathbf{c}, \ell$ and $\{m_\sigma\}_{\sigma \in \{+,-\}^n}$ be determined from $\mathcal{I}$ and λ as in* (1.6), (1.7) *and* (1.4). *If there exists an element σ of $\{+,-\}^n$ and $k \in [n]$ such that $m_{\sigma,k} > 0$ and $m_{\sigma,i} \geq 0$ for $i > k$, then there exists a subword $\mathcal{J} = (i_{j_1}, i_{j_2}, \ldots, i_{j_s})$ of $\mathcal{I}$, of length at least 1 (i.e. $s \geq 1$), such that $j_1 = k$ and $\mathcal{J}$ is a λ-walk (i.e. it is a diagram walk and the final root $FR(\mathcal{J})$ appears in λ).*

Proof. First observe that by the definition (1.7) of the ℓ_i and by assumption on λ, we have $\ell_i \geq 0$ for all i, and $\ell_i > 0$ exactly when β_i, the i-th simple root in $\mathcal{I}$, appears in λ. Let σ and k be as above. Then by Lemma 3.1 there exists a subword $\mathcal{J} = (i_{j_1} = i_k, i_{j_2}, \ldots, i_{j_s})$ of length at least 1 such that the $j_1 = k$ and $FR(\mathcal{J})$ appears in λ. It remains to check that $\mathcal{J}$ is a diagram walk. Recall that by definition $c_{j\ell} = \langle \beta_\ell, \beta_j^\vee \rangle$. Hence $c_{j\ell} < 0$ if and only if there is an edge in the corresponding Dynkin diagram connecting the roots α_{i_j} and α_{i_ℓ}, so by the conditions on $\mathcal{J}$ in Lemma 3.1 we see that $\mathcal{J}$ is a diagram walk, as desired. □

The next result is the main technical fact we need.

Lemma 3.3. *Let $\{c_{ij}\}_{1 \leq i < j \leq n}$ and $\ell_1, \ldots, \ell_n$ be fixed integers and let $(C(\mathbf{c}, \ell), \rho)$ be the corresponding Grossberg-Karshon twisted cube. Assume that $\ell_i \geq 0$ for all i. If $(C(\mathbf{c}, \ell), \rho)$ is twisted, then there exists an increasing subsequence $\mathcal{J} = (j_0 < j_1 < \cdots < j_s)$ of indices of length at least 2 (i.e. $s \geq 1$) such that*

1. *$\ell_{j_s} > 0$,*
2. *$c_{j_0 j_1} > 0$, and*
3. *$c_{j_t j_{t+1}} < 0$ for all $t = 1, \ldots, s-1$.*

Proof. By Theorem 1.4, there exist an element σ of $\{+,-\}^n$ and an index k such that $m_{\sigma,k} < 0$. For such a choice of σ we may assume without loss of generality that k is chosen to be the maximal such index, i.e. that $m_{\sigma,k} < 0$ and $m_{\sigma,s} \geq 0$ for $s > k$. Recall that by definition

$$m_{\sigma,k} = \ell_k - \sum_{s>k} c_{ks} m_{\sigma,s}.$$

By assumption $m_{\sigma,k} < 0$ so we have $\sum_{s>k} c_{ks} m_{\sigma,s} > \ell_k \geq 0$. Since also $m_{\sigma,s} \geq 0$ for $s > k$, this implies that there exists some $p > k$ with $c_{kp} > 0$ and $m_{\sigma,p} > 0$. Applying Lemma 3.1 we obtain an increasing sequence $(j_1 = p, j_2, \ldots, j_s)$ of indices with $s \geq 1$ such that $\ell_{j_s} > 0$ and $c_{j_t j_{t+1}} < 0$ for all $t = 1, \ldots, s-1$. Then by choosing $j_0 = k < j_1 = p$ and since $c_{j_0 j_1} = c_{kp} > 0$ by construction of p, we obtain a sequence $\mathcal{J} = (j_0 = k, j_1 = p, \ldots, j_s)$ satisfying the required conditions.

□

The proof of the "only if" part of Theorem 2.9 is a straightforward consequence of the above lemma.

Proof of the "only if" part of Theorem 2.9. Suppose the Grossberg-Karshon twisted cube $(C(\mathbf{c}, \ell), \rho)$ is twisted. By the dominance assumption on λ and by definition of the ℓ_i, we know $\ell_i \geq 0$ for all i. Thus we may apply Lemma 3.3. Note also $\ell_{j_s} > 0$ precisely when the root β_{j_s} appears in λ. Moreover, by definition, we know that $c_{j_0 j_1} := \langle \beta_{j_1}, \beta_{j_0}^\vee \rangle > 0$ if and only if $\beta_{j_0} = \beta_{j_1}$ (equivalently $i_{j_0} = i_{j_1}$) and $c_{j_t j_{t+1}} < 0$ if and only if there is an edge in the corresponding Dynkin diagram connecting the roots β_{j_t} and $\beta_{j_{t+1}}$. Thus the subword $(i_{j_0}, i_{j_1}, \ldots, i_{j_s})$ of $\mathcal{I}$ corresponding to the subsequence $(j_0, j_1, \ldots, j_s)$ of indices obtained from Lemma 3.3 is a hesitant λ-walk, as desired. $\square$

4. Proof of the main theorem: sufficiency

We now proceed to prove the "if" part of the main theorem, i.e., that untwistedness implies hesitant-λ-walk-avoidance. Part of the proof will be a case-by-case analysis of the possible Lie types of G.

For convenience we recall the Cartan matrices for all Lie types (see for example [6, p.58-59]).

$$A_r : \begin{bmatrix} 2 & -1 & 0 & & \cdots & & & 0 \\ -1 & 2 & -1 & 0 & \cdots & & & 0 \\ 0 & -1 & 2 & -1 & 0 & \cdots & & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & & \cdots & -1 & 2 & -1 \\ 0 & 0 & 0 & & \cdots & 0 & -1 & 2 \end{bmatrix} \qquad E_6 : \begin{bmatrix} 2 & 0 & -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}$$

$$B_r : \begin{bmatrix} 2 & -1 & 0 & & \cdots & & & 0 \\ -1 & 2 & -1 & 0 & \cdots & & & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & & \cdots & -1 & 2 & -2 \\ 0 & 0 & 0 & & \cdots & 0 & -1 & 2 \end{bmatrix} \qquad E_7 : \begin{bmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}$$

$$C_r : \begin{bmatrix} 2 & -1 & 0 & & \cdots & & & 0 \\ -1 & 2 & -1 & 0 & \cdots & & & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & & \cdots & -1 & 2 & -1 \\ 0 & 0 & 0 & & \cdots & 0 & -2 & 2 \end{bmatrix} \qquad E_8 : \begin{bmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}$$

$$D_r : \begin{bmatrix} 2 & -1 & 0 & & \cdots & & & & 0 \\ -1 & 2 & -1 & & \cdots & & & & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & & \cdot\,\cdot & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & & \cdot\,\cdot & & -1 & 2 & -1 & -1 \\ 0 & 0 & & \cdot\,\cdot & & 0 & -1 & 2 & 0 \\ 0 & 0 & & \cdot\,\cdot & & 0 & -1 & 0 & 2 \end{bmatrix} \qquad F_4 : \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \qquad G_2 : \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$$

TABLE 2. Cartan matrices for all Lie types.

In the discussion below it will be useful to restrict attention to hesitant λ-walks which are minimal in an appropriate sense. We make this precise in the definition below.

Definition 4.1. Let λ be a dominant weight and let $\mathcal{I} = (i_0, \ldots, i_n)$ be a hesitant λ-walk. We say that $\mathcal{I}$ is **minimal** if

(1) $\{i_1, \ldots, i_n\}$ are all distinct, i.e. the walking component of $\mathcal{I}$ visits any given vertex of the Dynkin diagram at most once, and
(2) if $n \geq 2$, then $\beta_0, \ldots, \beta_{n-1}$ do not appear in λ.

Example 4.2. Let $G = SL(6, \mathbb{C})$.

- Let $\lambda = \varpi_2$. The hesitant λ-walk $\mathcal{J} = (5, 5, 4, 3, 4, 3, 2)$ is not minimal since the walking component revisits some vertices multiple times, but the subword $\mathcal{J}' = (5, 5, 4, 3, 2)$ is minimal.
- Let $\lambda = \varpi_2 + \varpi_5$. In this case the above hesitant λ-walk $(5, 5, 4, 3, 2)$ is not minimal since $\beta_0 = \beta_1 = \alpha_5$ already appears in λ. The subword $(5, 5)$ is minimal.

It is clear from the definition that for any dominant $\lambda \neq 0$ and a hesitant λ-walk $\mathcal{J}$, there exists a subword $\mathcal{J}'$ of $\mathcal{J}$ which is minimal in the sense of Definition 4.1.

Lemma 4.3. *Let $\lambda \neq 0$ be a dominant weight and $\mathcal{J} = (i_{j_0}, i_{j_1}, \ldots, i_{j_s})$ a hesitant λ-walk. Let $\mathbf{c}, \ell$ be the constants associated to $\mathcal{J}$ and λ as defined in* (1.6) *and* (1.7)*. If $\mathcal{J}$ is minimal, then*

(1) *$c_{j_p j_q} = 0$ if $|p-q| \geq 2$ and $1 \leq p, q \leq s$, and*
(2) *$\ell_{j_p} = 0$ for $0 \leq p \leq s-1$ if $s \geq 2$.*

Proof. By the minimality assumption, and since Dynkin diagrams have no loops, we know that if $|p-q| \geq 2$ and $1 \leq p, q \leq s$ (so j_p and j_q are in the walking component of $\mathcal{J}$) then the roots β_{j_p} are neither adjacent nor equal. This implies the corresponding entry in the Cartan matrix is 0, as desired. The second statement is immediate from the minimality assumption since $\ell_{j_p} > 0$ exactly when β_{j_p} appears in λ. □

We begin with a lemma.

Lemma 4.4. *Let $\{c_{ij}\}_{1 \leq i < j \leq n}$ and $\ell_1, \ldots, \ell_n$ be fixed integers and let $(C(\mathbf{c}, \ell), \rho)$ be the corresponding Grossberg-Karshon twisted cube. Assume that $\ell_i \geq 0$ for all i. If there exists two distinct indices i and j, $1 \leq i < j \leq n$, with $c_{ij} > 1$ and $\ell_i = \ell_j > 0$, then $(C(\mathbf{c}, \ell), \rho)$ is twisted.*

Proof. By Theorem 1.4, it suffices to show that there exists an element σ of $\{+,-\}^n$ and some k with $1 \leq k \leq n$ such that $m_{\sigma,k} < 0$. Let $\sigma = (\sigma_1, \ldots, \sigma_n) \in \{+,-\}^n$ be the element defined by

$$\sigma_k = \begin{cases} - & \text{if } k = i \text{ or } j \\ + & \text{otherwise} \end{cases}$$

and consider the associated $m_\sigma = (m_{\sigma,1}, \ldots, m_{\sigma,n})$. Then by definition of σ and m_σ we have

$$m_{\sigma,j} = \ell_j - \sum_{s>j} c_{js} m_{\sigma,s}$$

$$m_{\sigma,i} = \ell_i - \left(c_{ij} m_{\sigma,j} - \sum_{\substack{s>i \\ s \neq j}} c_{is} m_{\sigma,s} \right).$$

Since $\sigma_k = +$ for $k \neq i, j$, we have that $m_{\sigma,k} = 0$ for $k \neq i, j$. Hence the above equations can be simplified to

$$m_{\sigma,j} = \ell_j$$
$$m_{\sigma,i} = \ell_i - c_{ij} m_{\sigma,j} = \ell_i - c_{ij} \ell_j$$

By assumption $\ell_i = \ell_j$ so

$$m_{\sigma,i} = \ell_i (1 - c_{ij}).$$

Since $c_{ij} > 1$ and $\ell_i > 0$, we obtain $m_{\sigma,i} < 0$, as desired. □

As in the previous section, the above lemma can be interpreted in terms of hesitant λ-walks.

Corollary 4.5. *Let $\mathcal{I} = (i_1, i_2, \ldots, i_n)$ be a word decomposition of an element $w = s_{i_1} \cdots s_{i_n}$ of W and let $\lambda = \lambda_1 \varpi_1 + \lambda_2 \varpi_2 + \cdots + \lambda_r \varpi_r$ be a dominant weight, i.e. $\lambda_i \geq 0$ for all i. Let $\mathbf{c} = \{c_{jk}\}, \ell = (\ell_1, \ldots, \ell_n)$ and $\{m_\sigma\}_{\sigma \in \{+,-\}^n}$ be determined from $\mathcal{I}$ and λ as in* (1.6), (1.7) *and* (1.4) *and let $(C(\mathbf{c}, \ell), \rho)$ denote the corresponding Grossberg-Karshon twisted cube. If $\mathcal{I}$ contains a subword $\mathcal{J} = (j_0, j_1)$ of length 2 which is a hesitant λ-walk, then $(C(\mathbf{c}, \ell), \rho)$ is twisted.*

Proof. By definition of a hesitant λ-walk, if $\mathcal{J} = (j_0, j_1)$ is a hesitant λ-walk then $i_{j_0} = i_{j_1}$ (equivalently $\beta_{j_0} = \beta_{j_1}$) and $\beta_{j_0} = \beta_{j_1}$ appears in λ. This implies $c_{j_0 j_1} = 2 > 1$ and $\ell_{j_0} = \ell_{j_1} > 0$. The result now follows from Lemma 4.4. □

Proof of "if" part of Theorem 2.9. Suppose $\mathcal{J} = \{i_{j_0}, i_{j_1}, \cdots, i_{j_s}\}$ is a subword of $\mathcal{I}$ which is a hesitant λ-walk. We may without loss of generality assume that $\mathcal{J}$ is minimal in the sense of Definition 4.1. We then wish to show that $(C(\mathbf{c}, \ell), \rho)$ is twisted. If the length of $\mathcal{J}$ is 2, i.e. $s = 1$, then this follows from Corollary 4.5. Thus we may now assume that the length is at least 3, i.e. $s \geq 2$. To prove that $(C(\mathbf{c}, \ell), \rho)$ is twisted, by Theorem

1.4 it is enough to find an element σ of $\{+,-\}^n$ and a $k \in [n]$ such that $m_{\sigma,k} < 0$. To achieve this, consider the element $\sigma = (\sigma_1, \ldots, \sigma_n) \in \{+,-\}^n$ defined by

$$\sigma_p = \begin{cases} - & \text{if } p \in \{j_0, j_1, \ldots, j_s\} \\ + & \text{otherwise.} \end{cases}$$

By definition of m_σ, we then have

$$\begin{aligned} m_{\sigma,j_s} &= \ell_{j_s} - \sum_{p>j_s} c_{j_s p} m_{\sigma,p} \\ m_{\sigma,j_t} &= \ell_{j_t} - \left(c_{j_t j_{t+1}} m_{\sigma,j_{t+1}} + \sum_{\substack{p>j_t \\ p\neq j_{t+1}}} c_{j_t p} m_{\sigma,p} \right) \text{ for } 1 \le t \le s-1 \\ m_{\sigma,j_0} &= \ell_{j_0} - \left(c_{j_0 j_1} m_{\sigma,j_1} + c_{j_0 j_2} m_{\sigma,j_2} + \sum_{\substack{p>j_0 \\ p\neq j_1,j_2}} c_{j_0 p} m_{\sigma,p} \right). \end{aligned} \tag{4.1}$$

Since $\mathcal{J}$ is a hesitant λ-walk, we know $\ell_{j_s} > 0$. On the other hand, by the minimality assumption on $\mathcal{J}$ and Lemma 4.3, we know $\ell_{j_t} = 0$ for all t with $0 \le t \le s-1$. Moreover, again by minimality and Lemma 4.3 we know that $c_{j_t j_r} = 0$ for $j_r > j_t$ and $j_r \neq j_{t+1}$. Also by construction of the σ, for $p \notin \mathcal{J} = \{j_0, j_1, \ldots, j_s\}$ we have $\sigma_p = +$ and hence $m_{\sigma,p} = 0$. Finally, since $\mathcal{J}$ is a hesitant λ-walk, we have $\beta_{j_0} = \beta_{j_1}$ and hence $c_{j_0 j_1} = \langle \beta_{j_0}, \beta_{j_1}^\vee \rangle = 2$. From these considerations we can simplify (4.1) to

$$\begin{aligned} m_{\sigma,j_s} &= \ell_{j_s} > 0 \\ m_{\sigma,j_t} &= -c_{j_t j_{t+1}} m_{\sigma,j_{t+1}} \text{ for } 1 \le t \le s-1 \\ m_{\sigma,j_0} &= -\left(2m_{\sigma,j_1} + c_{j_0 j_2} m_{\sigma,j_2}\right). \end{aligned} \tag{4.2}$$

We now claim that $m_{\sigma,j_0} < 0$; as already noted, this suffices to prove the theorem. In order to prove this claim we need to know that values of the constants $c_{j_t j_{t+1}}$ and $c_{j_0 j_2}$ appearing in (4.2). By the assumption that $\mathcal{J}$ is a hesitant λ-walk, these constants are equal to the corresponding entry of the Cartan matrices for simple roots which are adjacent in the Dynkin diagram. For the case-by-case analysis below we refer to the list of Dynkin diagrams and Cartan matrices in Tables 1 and 2. Suppose first that the hesitant λ-walk only crosses edges of the form ○—○ or, if it crosses a double edge ○═○ or triple edge ○≡○ then it does so only by going in the direction *agreeing with* the arrow drawn on the edge in the Dynkin diagram (e.g. in type B, if $i_{j_t} = r-1$ and $i_{j_{t+1}} = r$ and in type G, if $i_{j_t} = 2$ and $i_{j_{t+1}} = 1$). In this situation, the corresponding constants $c_{j_t j_{t+1}}$ and $c_{j_0 j_2}$ are all equal to -1. So we consider this case first. In this setting we have

$$\begin{aligned} m_{\sigma,j_s} &= \ell_{j_s} > 0 \\ m_{\sigma,j_t} &= m_{\sigma,j_{t+1}} \text{ for } 1 \le t \le s-1 \\ m_{\sigma,j_0} &= -(2m_{\sigma,j_1} - m_{\sigma,j_2}) \end{aligned} \tag{4.3}$$

so $m_{\sigma,j_1} = m_{\sigma,j_2} = \cdots = m_{\sigma,j_s} = \ell_{j_s}$ and $m_{\sigma,j_0} = -\ell_{j_s} < 0$, as desired.

Next we consider the possibility that the hesitant λ-walk crosses a "double" edge in a direction *against* the direction of the arrow on the edge. Since we assume the hesitant λ-walk is minimal, it can only cross such an edge once. In particular, in type B (respectively type C) this implies that the hesitant λ-walk must be of the form $i_{j_0} = i_{j_1} = r$ and $i_{j_2} = r-1, i_{j_3} = r-2, \ldots, i_{j_s} = r-s+1$ (respectively $i_{j_0} = i_{j_1} = r-s+1, i_{j_2} = r-s+2, \ldots, i_{j_{s-1}} = r-1$ and $i_{j_s} = r$) for some $s \ge 2$. We consider these cases next.

In type B consider the hesitant λ-walk of the form $i_{j_0} = i_{j_1} = r$ and $i_{j_2} = r-1, i_{j_3} = r-2, \ldots, i_{j_s} = r-s+1$ for some $s \ge 2$. In this case the equations (4.2) become

$$\begin{aligned} m_{\sigma,j_s} &= \ell_{j_s} > 0 \\ m_{\sigma,j_{s-1}} &= \cdots = m_{\sigma,j_2} = \ell_{j_s} \\ m_{\sigma,j_1} &= 2m_{\sigma,j_2} = 2\ell_{j_s} \\ m_{\sigma,j_0} &= -(2m_{\sigma,j_1} + (-2)m_{\sigma,j_2}) = -2\ell_{j_s} < 0 \end{aligned}$$

so we obtain $m_{\sigma,j_0} < 0$ as desired. In type C, consider the hesitant λ-walk $i_{j_0} = i_{j_1} = r - s + 1, i_{j_2} = r - s + 2, \ldots, i_{j_{s-1}} = r - 1$ and $i_{j_s} = r$ for $s \geq 2$. Note that the case $s = 2$ is already covered in the argument for type B above so we may assume $s \geq 3$. It is straightforward to see that here we obtain from (4.2) that $m_{\sigma,j_s} = \ell_{j_s} > 0, m_{\sigma,j_{s-1}} = \cdots = m_{\sigma,j_1} = 2\ell_{j_s}$, and $m_{\sigma,j_0} = -2\ell_{j_s} < 0$. Thus $m_{\sigma,j_0} < 0$ as desired.

The only remaining cases are in the exceptional Lie types F and G, but many cases of hesitant λ-walks in type F are already handled by the considerations in type B and C above. Thus the only remaining cases are: $(4, 4, 3, 2, 1)$ in type F and $(1, 1, 2)$ in type G. Both are straightforward and left to the reader.

□

5. Open questions

The study of Grossberg-Karshon twisted cubes is related to representation theory and to the recent theory of Newton-Okounkov bodies and divided-difference operators on polytopes. Moreover, in this manuscript we have introduced the notion of hesitant λ-walks as well as hesitant-λ-walk-avoidance. Below, we briefly mention some possible avenues for further exploration.

(1) The Grossberg-Karshon twisted cubes are a special case of the virtual polytopes produced by Kiritchenko's divided-difference operators [8]. We may ask whether our methods generalize to Kiritchenko's setting to provide combinatorial conditions on a dominant weight λ and choice of word decomposition $\mathcal{I}$ which guarantee that the corresponding virtual polytope from Kiritchenko's construction is a "true" polytope. (See also Kiritchenko's discussion in [8, §3.3].)
(2) In the cases when the Grossberg-Karshon twisted polytope is untwisted (i.e. it is a "true" polytope), it would be of interest to study the relationship between the Grossberg-Karshon polytope and other polytopes appearing in representation theory and Schubert calculus, such as Gel'fand-Cetlin polytopes, or (more generally) string polytopes, or (even more generally) Newton-Okounkov bodies of Bott-Samelson varieties (see [2, 5, 7]).
(3) "Pattern avoidance" is a recurring and important theme in the study of Schubert varieties. We may ask whether, and how, hesitant-λ-walk-avoidance relates to the known results in this direction [1].

References

[1] H. Abe and S. Billey, *Consequences of the Lakshmibai-Sandhya Theorem: the ubiquity of permutation patterns in Schubert calculus and related geometry*, ArXiv:1403.4345.
[2] D. Anderson, *Okounkov bodies and toric degenerations*, Math. Ann. **356** (2013), no. 3, 1183-1202.
[3] M. Grossberg and Y. Karshon, *Bott towers, completeintegrability, and the extended character of representations*, Duke Math. J., 76(1):23-58, 1994.
[4] M. Harada and J. Yang, *Grossberg-Karshon twisted cubes and basepoint-free divisors*, ArXiv:1407.4147.
[5] M. Harada and J. Yang, *Newton-Okounkov bodies of Bott-Samelson varieties*, in preparation.
[6] J. E. Humphreys, *Introduction to Lie Algebras and Representation Theory*, Springer-Verlag, New York, 1972.
[7] K. Kaveh, *Crystal bases and Newton-Okounkov bodies*, Arxiv:1101.1687.
[8] V. Kiritchenko, *Divided difference operators on polytopes*, Arxiv:1307.7234.
[9] V. Lakshmibai and B. Sandhya, *Criterion for smoothness of Schubert varieties in $SL(n)/B$*. Proc. Indian Acad. Sci. (Math Sci.) 100(1): 4552, 1990.

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