LOWER BOUND OF THE UNKNOTTING NUMBER OF PRIME KNOTS WITH ELEVEN OR TWELVE CROSSINGS

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Abstract. The oriented Gordian distance between two oriented links is the minimal number of crossing changes needed to deform one into the other. We compile a table of oriented Gordian distances between 2-component non-splittable links with up to six crossings. In particular, we give a criterion of oriented Gordian distance two using a special value of the Jones polynomial, which allows us to prove that the unlinking number of the 2-component link 9^2_3 is 3. This is one of the 5 links for which Kohn could not compute the unlinking number.

1. Introduction

The unknotting number of a knot K is the minimal number of crossing changes required to convert a knot into the trivial knot, which we denote by $\mathrm{u}(K)$. The signature is the most useful tool for giving the lower bound of the unknotting number. Also, the special values of the Jones, Q, and HOMFLYPT polynomials provide practical criterion for the lower bound of the unknotting number. We calculate these values for prime knots with up to 12 crossings, and then compare the unknotting numbers compiled in the table of Cha and Livingstone [1]. The first criterion for giving the lower bound of the unknotting number is due to Wendt [?], which uses the first homology group of the cyclic covering space of the 3-sphere S^3 branched over a knot (Eq. (2)). We may consider the criterion using the polynomial invariants are refinements of Wendt's formulaBy . We also calculate the first homology groups of the 3-fold cyclic covering spaces of the 3-sphere S^3 branched over these knots using the "Knot Theory Calculators" in [1]. By these calculations we can improve the table of the unknotting numbers in [1]. Furthermore, for every knot whose determinant is square of prime number ≥ 7 we calculate the 2-fold branched covering space of S^3 to apply Wendt's formula, and for a very few cases we may find the unknotting number.

2. Criteria on the unknotting number

The signature [12] is the most useful tool for this problem (Proposition ??). For a nontrivial knot K, we have

$$\mathrm{u}(K) \geq \max\{1, |\sigma(K)|/2\}. \tag{1}$$

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Let $\Sigma_n(K)$ be the n-fold cyclic covering space of S^3 branched over a knot K. Then Wendt [**?**] proved:

$$\mathrm{u}(K) \geq e_n(K)/(n-1), \tag{2}$$

where $e_n(K)$ is the minimal number of generators of the first homology group $H_1(\Sigma_n(K); \boldsymbol{Z})$ of $\Sigma_n(K)$.

We let $S_{p,q}$ denote the 2-bridge knot whose 2-fold branched cover is the lens space of type (p, q), where p and q are relatively prime integers and p is odd positive.

Proposition 2.1. *Suppose that K is a 2-bridge knot. Then, $\mathrm{u}(K) = 1$ if and only if: There exist an odd integer p and relatively prime integers m and n with $2mn = p \pm 1$ and K is equivalent to $S_{p,2n^2}$.*

Given a 2-bridge knot $S_{p,q}$, we decide whether its unknotting number is one or greater than one as in Corollary 3 in [**?**]: We list all 2-bridge knots with determinant p and unknotting number one;

$$\left\{ S_{p,2n^2} \,;\, n \text{ is a divisor of } (p \pm 1)/2. \right\}. \tag{3}$$

Then we investigate if $S_{p,q}$ is contained in this set.

The *Jones polynomial* $V(L; t) \in \boldsymbol{Z}[t^{\pm 1/2}]$ [3], and the *HOMFLYPT polynomial* $P(L; v, z) \in \boldsymbol{Z}[v^{\pm 1}, z^{\pm 1}]$ [**?**, 3, **?**] are invariants of the isotopy type of an oriented link L, which are defined by the following formulas:

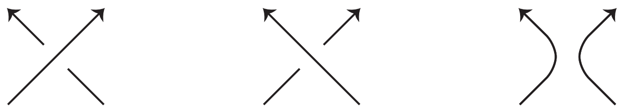
$$V(U; t) = 1; \tag{4}$$

$$t^{-1}V(L_+; t) - tV(L_-; t) = \left(t^{1/2} - t^{-1/2}\right) V(L_0; t); \tag{5}$$

$$P(U; v, z) = 1; \tag{6}$$

$$v^{-1}P(L_+; v, z) - vP(L_-; v, z) = zP(L_0; v, z), \tag{7}$$

where U is the unknot and (L_+, L_-, L_0) is a *skein triple*, an ordered set of three oriented links that are identical except near one point where they are as in Fig. 2.



L_+ L_- L_0

FIGURE 1. A skein triple.

For a link L we have the following [4, **?**]:

$$V(L; \omega) = \pm i^{c(L)-1}(i\sqrt{3})^{\delta}; \tag{8}$$

$$P(L; i, i) = (-2)^{\tau/2}, \tag{9}$$

where $\omega = e^{i\pi/3}$, $V(L; \omega)$ means the value of $V(L; t)$ at $t^{1/2} = e^{i\pi/6}$, $c(L)$ is the number of components of L, $\delta = \dim H_1(\Sigma_2(L); \boldsymbol{Z}_3)$, $P(L; i, i)$ means the value of $P(L; v, z)$ at $v = z = i$, and $\tau = \dim H_1(\Sigma_3(L); \boldsymbol{Z}_2)$. Using these values, we have the criteria on the unknotting number [10, 18]; cf. [**?**].

Proposition 2.2. *Let K be a knot and n be a non-negative integer.*
(i) *If $V(K;\omega) = \pm\sqrt{3}^n$, then $\mathrm{u}(K) \geq n$.*
(ii) *If either $\sigma(K) = -2n$ and $V(K;\omega) = -(i\sqrt{3})^n$, or $\sigma(K) = 2n$ and $V(K;\omega) = -(-i\sqrt{3})^n$, then $\mathrm{u}(K) \geq n+1$.*
(iii) *If $P(K;i,i) = (-2)^n$, then $\mathrm{u}(K) \geq n$.*

The *Q polynomial* $Q(L;z) \in \boldsymbol{Z}[z^{\pm 1}]$ [**?**, **?**] is an invariant of the isotopy type of an unoriented link L, which is defined by the following formulas:

$$Q(U;z) = 1; \tag{10}$$

$$Q(L_+;z) + Q(L_-;z) = z\left(Q(L_0;z) + Q(L_\infty;z)\right), \tag{11}$$

where $(L_+, L_-, L_0, L_\infty)$ is an *unoriented skein quadruple*, an ordered set of four unoriented links that are identical except near one point where they are as in Fig. 2.

FIGURE 2. An unoriented skein quadruple.

Let $\rho(L) = Q\left(L;(\sqrt{5}-1)/2\right)$, the value of the Q polynomial at $z = (\sqrt{5}-1)/2$. Then Jones [4] has shown:

$$\rho(L) = \pm\sqrt{5}^f \tag{12}$$

where $f = \dim H_1(\Sigma_2(L);\boldsymbol{Z}_5)$. Using this value, we have the criteria on the unknotting number [**?**].

Proposition 2.3. *Let K be a knot and n be a non-negative integer.*
(i) *If $\rho(K) = (-\sqrt{5})^n$, then $\mathrm{u}(K) \geq n$.*
(ii) *If $\rho(K) = -(-\sqrt{5})^n$, then $\mathrm{u}(K) \geq n+1$.*

3. TABLES

We consider a knot K which has the property:

(†) $\mathrm{u}(K) > \max\{1, |\sigma(K)|/2\}$, and the lower bound of $\mathrm{u}(K)$ is obtained from one of the criteria given in Sect. 2.

In this section we list all such prime knots with up to 12 crossings together with their unknotting numbers. We indicate unknotting numbers in the middle colums; the upper bounds are taken from the table [1]; 2-3, 2-4, 3-4 mean that the unknotting number is 2 or 3, 2 or 3 or 4, 3 or 4, respectively. If we find a new lower bound for the unknotting, an asterisk is added.

The marks in the right columns in the tables indicate reasons for giving the lower bounds of the unknotting numbers:

r: Since K is a 2-bridge knot, using Proposition 2.1 we may decide $\mathrm{u}(K) \geq 2$.

v: Since $\sigma(K) = 2\epsilon$ and $V(K;\omega) = i\sqrt{3}\epsilon$, $\epsilon = \pm 1$, by Proposition 2.2(ii) we have $\mathrm{u}(K) \geq 2$.
v_1: Since $V(K;\omega) = \pm 3$, by Proposition 2.2(i) we have $\mathrm{u}(K) \geq 2$.
v_2: Since $\sigma(K) = \pm 4$ and $V(K;\omega) = 3$, by Proposition 2.2(ii) we have $\mathrm{u}(K) \geq 3$.
v_3: Since $V(K;\omega) = \pm 3\sqrt{3}i$, by Proposition 2.2(i) we have $\mathrm{u}(K) \geq 3$.
q: Since $\rho(K) = \sqrt{5}$, by Proposition 2.3(ii) we have $\mathrm{u}(K) \geq 2$.
q_1: Since $\rho(K) = 5$, by Proposition 2.3(i) we have $\mathrm{u}(K) \geq 2$.
q_2: Since $\rho(K) = -5$, by Proposition 2.3(ii) we have $\mathrm{u}(K) \geq 3$.
h: Since $P(K;i,i) = 4$, by Proposition 2.2(iii) we have $\mathrm{u}(K) \geq 2$.
w_2**:** Since $e_2(K) = 2$, by Eq. (2) we have $\mathrm{u}(K) \geq 2$.
w_3**:** Since $e_3(K) = 4$, by Eq. (2) $\mathrm{u}(K) \geq 2$.

TABLE 1. Prime knots K with up to 10 crossings.

K	u		K	u		K	u		K	u		K	u		K	u		K	u		K	u	
7_4	2	rvq	9_5	2	r	9_{47}	2	v_1	10_{15}	2	r	10_{34}	2	r	10_{45}	2	r	10_{97}	2	v	10_{121}	2	q
8_3	2	r	9_8	2	r	9_{48}	2	v_1	10_{16}	2	r	10_{35}	2	r	10_{65}	2	v	10_{99}	2	v_1	10_{122}	2	q
8_4	2	r	9_{15}	2	rv	9_{49}	3	q_2	10_{19}	2	rv	10_{36}	2	rv	10_{67}	2	v	10_{103}	3	q_2	10_{123}	3	w_3
8_6	2	r	9_{17}	2	rv	10_3	2	rq	10_{20}	2	rq	10_{37}	2	r	10_{69}	2	v	10_{106}	2	q	10_{140}	2	h
8_8	2	rq	9_{31}	2	rq	10_4	2	r	10_{22}	2	r	10_{38}	2	r	10_{74}	2	v_1	10_{108}	2	v	10_{144}	2	qh
8_{12}	2	r	9_{37}	2	v_1	10_{11}	2	r	10_{24}	2	rq	10_{40}	2	rq	10_{75}	2	v_1	10_{109}	2	q	10_{155}	2	q_1
8_{16}	2	q	9_{40}	2	vq_1	10_{12}	2	r	10_{28}	2	r	10_{41}	2	r	10_{86}	2	q	10_{115}	2	h	10_{163}	2	vh
8_{18}	2	qh	9_{46}	2	v_1	10_{13}	2	r	10_{29}	2	rv	10_{43}	2	r	10_{89}	2	v	10_{116}	2	q	10_{165}	2	v

Remark. $H_1(\Sigma_3(10_{123}); \boldsymbol{Z}) = \boldsymbol{Z}_5 \oplus \boldsymbol{Z}_5 \oplus \boldsymbol{Z}_5 \oplus \boldsymbol{Z}_5$.

TABLE 2. 11 crossing alternating knots 11ak with unknotting number > 1.

k	u		k	u		k	u		k	u		k	u		k	u		k	u		k	u	
3	2	q	47	2	v_1h	90	2	rv	123	3	v_2	157	2	vq	181	2	v_1	229	2	r	297	2	q_1h
7	2	rq	57	2	v_1h	93	2	r	125	2	q	159	2	rv	183	2	rq	231	2	v_1h	314	2	v_1
13	2	r	59	2	r	97	2	h	126	2-3	q	165	2	h	185	2	r	239	2	q	317	2	q_1
16	2	q	65	2	r	99	2	vq	132	2	vq	166	2	r	188	2	r	249	2	v_1	322	2	h
17	2	v	75	2	r	102	2	v	135	2	v_1	170	2	q	193	2	rq	258	2	q	324	2	v
19	2	q	76	2	q	107	2	vh	137	2-3	v	173	2	v_1	199	2	v	274	2	q	332	2	v_1h
21	2	q	84	2	r	110	2	r	145	2	r	174	2	r	202	2-3	v	277	2	v_1	333	2	rq
25	2	q	85	2	r	111	2	r	148	2	q	175	2	rq	205	2	r	281	2	q	347	2	vh
33	2	q	87	2	h	118	2	v	154	2	r	176	2	r	211	2	r	288	2	q	352	2	v_1
44	2	v_1h	89	2	r	119	2	r	155	2	v_1	178	2	rv	219	2	v	296	2	v	363	2-3	rq

Remark 3.1. The unknotting numbers for the following knots compiled in the table [1] are $> \max\{1, \sigma(K)/2\}$ without any references:

$11n76, 11n106$;

$12nk$, $k = 91, 105, 110, 136, 148, 187, 199, 207, 217, 220, 228, 242, 293, 321, 328, 329,$
$366, 374, 402, 417, 426, 433, 472, 518, 528, 537, 574, 575, 591, 594, 624, 627, 640, 647,$
$660, 679, 680, 688, 689, 691, 692, 693, 694, 696, 725, 750, 830, 850, 851, 888.$

TABLE 3. Nonalternating 11 crossing knots 11nk with unknotting number > 1.

k	u		k	u		k	u		k	u		k	u		k	u		k	u		k	u	
11	2	q	49	2^*	h	74	2	$\mathrm{v}_1\mathrm{h}$	83	2^*	h	117	2^*	q	140	2^*	v	155	2^*	v	167	2^*	v_1
15	2^*	q	58	2^*	q	75	2	$\mathrm{v}_1\mathrm{h}$	91	2^*	h	127	2^*	q	146	2^*	v	157	2^*	q	168	2^*	q
29	2^*	v	71	2	$\mathrm{v}_1\mathrm{h}$	79	2^*	vq	92	2^*	vq	132	2^*	q	148	3	q_2	162	2^*	qh	170	2^*	v
37	2	q	73	2	$\mathrm{v}_1\mathrm{h}$	80	2	vq	113	2^*	q	133	3	q_2	150	2^*	q	165	2^*	qh	178	2^*	q

Remark. $H_1(\Sigma_2(12a1019); \boldsymbol{Z}) = \boldsymbol{Z}_{19} \oplus \boldsymbol{Z}_{19}$.
$H_1(\Sigma_3(12a1019); \boldsymbol{Z}) = \boldsymbol{Z}_{11} \oplus \boldsymbol{Z}_{11} \oplus \boldsymbol{Z}_{11} \oplus \boldsymbol{Z}_{11}$.
$H_1(\Sigma_2(12a1105); \boldsymbol{Z}) = \boldsymbol{Z}_{17} \oplus \boldsymbol{Z}_{17}$. $H_1(\Sigma_2(12a1202); \boldsymbol{Z}) = \boldsymbol{Z}_{13} \oplus \boldsymbol{Z}_{13}$.
Remark. $H_1(\Sigma_2(12n397); \boldsymbol{Z}) = H_1(\Sigma_2(12n706); \boldsymbol{Z}) = \boldsymbol{Z}_7 \oplus \boldsymbol{Z}_7$.
$H_1(\Sigma_3(12n651); \boldsymbol{Z}) = H_1(\Sigma_3(12n746); \boldsymbol{Z}) = \boldsymbol{Z}_5 \oplus \boldsymbol{Z}_5 \oplus \boldsymbol{Z}_{10} \oplus \boldsymbol{Z}_{10}$.
$H_1(\Sigma_3(12n781); \boldsymbol{Z}) = \boldsymbol{Z}_5 \oplus \boldsymbol{Z}_5 \oplus \boldsymbol{Z}_5 \oplus \boldsymbol{Z}_5$.

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TABLE 4. 12 crossing alternating knots 12ak with unknotting number > 1.

k	u		k	u		k	u		k	u		k	u		k	u	
10	2^*	v	260	2^*	v	465	2^*	h	650	2^*	rq	907	3	q_2	1127	2-3^*	r
29	2^*	vh	265	2^*	v_1	471	2^*	r	652	2^*	rq	908	2^*	q	1132	2^*	r
30	2^*	h	270	2^*	v_1	475	2^*	vqh	665	2-3^*	v	914	2^*	vq	1133	2^*	rv
33	2^*	h	271	2^*	q	477	2-3^*	r	677	2^*	q	927	2^*	q	1136	2^*	r
38	2^*	r	279	2^*	h	478	2^*	q	682	2^*	r	934	2^*	q	1138	2^*	r
49	2-3^*	v	291	2-3^*	h	481	2-3^*	h	683	3	v_2	939	2^*	v	1142	2-3^*	v_1
67	2^*	v	296	2^*	v	482	2-3^*	r	684	2-3^*	rvq	940	2-3^*	q	1143	2^*	q
71	2^*	v	298	2^*	v_1	488	2^*	q	689	2-3^*	v	941	2^*	vqh	1145	2^*	r
74	2^*	vq	302	2-3^*	r	493	2-3^*	v_1	690	2^*	r	949	2^*	vh	1146	2^*	r
77	2^*	q	306	2^*	r	494	2-3^*	h	691	2^*	r	951	2^*	q	1148	2-3^*	r
81	2^*	v	307	2^*	r	499	2^*	r	703	2^*	vh	959	2^*	q	1149	2^*	rq
86	2-3^*	v	311	3	v_2	500	2^*	r	712	2^*	v_1	960	2^*	h	1151	2-3^*	vq
87	2^*	q	312	2^*	vh	501	2^*	r	713	2^*	r	964	2^*	vq	1152	2^*	q_1h
88	2^*	v	313	2^*	v	503	2^*	v_1	714	2^*	r	972	2^*	v	1154	2^*	q
95	2^*	q	326	2^*	q	506	2^*	r	725	3	v_2	986	2^*	q	1161	2^*	rq
103	2-3^*	v	327	3	q_2	510	2^*	r	728	2^*	r	989	2-3^*	q	1163	2-3^*	r
113	2^*	vh	330	2^*	r	512	2^*	r	735	2^*	v	990	2^*	v_1h	1165	2-3^*	r
116	2^*	vh	332	2^*	v_1	514	2^*	r	738	2^*	r	992	2^*	q	1166	2^*	r
122	2^*	vh	339	2^*	q	518	2^*	r	742	2^*	v_1h	1006	2^*	q	1168	2^*	q
127	2-3^*	v	347	2^*	vh	528	2^*	r	743	2^*	r	1010	2^*	q	1171	2-3^*	q
129	2^*	q	348	2^*	h	532	2^*	r	750	3^*	v_3	1018	2^*	v	1174	2^*	q
136	2^*	v	372	2-3^*	q	534	2^*	r	752	2^*	q	1019	3	w_2w_3	1180	2^*	q
150	2-4^*	v	376	2-3^*	vqh	542	2^*	v	753	2^*	q	1021	2^*	v	1181	2^*	v_1q
155	2^*	v	378	2^*	r	545	2-3^*	r	758	2^*	r	1022	2^*	v_1	1185	2^*	q
157	2^*	h	379	2-3^*	r	549	2^*	rv	760	2^*	v	1024	2^*	r	1189	2^*	v
164	2^*	v_1h	380	2-3^*	r	550	2^*	r	767	2-3^*	v	1025	2^*	q	1194	3	q_2
166	2^*	v_1h	381	2-3^*	q	552	2^*	r	768	2^*	v	1026	2^*	h	1200	2-3^*	vq
169	2^*	r	384	2^*	r	554	3^*	v_3	769	2^*	v_1	1029	2^*	r	1202	2-3^*	hw_2
175	2^*	v	385	2^*	r	561	3	q_2	775	2^*	v	1030	2^*	r	1205	2-3^*	h
177	2^*	v_1	386	3	v_2	562	2^*	v	776	2^*	q	1033	2^*	r	1211	2^*	q
179	2^*	vq	389	2^*	v	563	3	v_2	779	2^*	q	1039	2-3^*	r	1216	2^*	v
182	2^*	vh	395	2^*	v	564	2^*	vq	780	3	q_2	1040	2-3^*	rq	1221	2^*	q
197	2^*	r	396	2^*	v_1	566	2^*	q	787	2^*	v_1	1044	2^*	v	1222	2^*	q
200	2^*	q	401	2^*	v	569	3	v_2	791	2-3^*	r	1067	2^*	q	1225	2^*	v_1qh
203	2^*	v	404	2^*	v	570	2^*	vq	792	2^*	rq	1068	2^*	q	1230	2^*	q
204	2^*	r	406	2^*	r	581	2^*	r	797	2^*	r	1069	2^*	q	1237	2^*	q
215	2^*	q	413	2^*	v_1	582	2^*	r	802	2^*	r	1077	2^*	v	1241	2^*	q
218	2^*	qh	416	2^*	v	583	2^*	r	806	2^*	vh	1079	2^*	vh	1251	2^*	h
221	2^*	r	419	2^*	v	584	2^*	r	808	2^*	h	1087	2^*	q	1260	2^*	v_1h
224	2^*	v	425	2^*	r	594	2^*	v_1	810	2^*	v_1	1090	2^*	q	1265	2^*	q
230	2^*	q	427	2^*	v_1q_1h	595	2^*	r	818	2-3^*	q	1092	2^*	v_1	1267	2^*	q
239	2^*	r	428	2^*	q	596	2^*	r	827	2-3^*	v	1093	2^*	v_1	1269	2-3^*	h
241	2^*	r	429	2^*	vh	597	2-3^*	rv	845	2-3^*	v	1102	2^*	qh	1270	2^*	q
243	2^*	r	433	3	v_2	601	2^*	r	855	2^*	v	1103	2-3^*	vq	1274	2^*	r
244	2-3^*	v_1	434	2-3^*	q	619	2^*	q	866	2^*	q	1104	2^*	q	1275	2^*	r
245	2^*	v_1	435	2^*	v_1h	621	2^*	q	873	2^*	v_1	1105	2^*	hw_2	1276	2^*	r
247	2^*	r	437	2^*	r	628	2^*	v	883	2^*	q	1106	2^*	q	1277	2^*	r
248	2^*	qh	447	2^*	r	633	2^*	v	886	2-3^*	v_1	1108	2^*	q	1279	2-3^*	r
249	2^*	h	448	2^*	qh	634	2-3^*	v_1q	890	2^*	q	1111	2^*	v	1281	2-3^*	r
251	2^*	rv	450	2^*	v	642	2^*	q	895	2^*	v_1	1118	2-3^*	q	1282	2-3^*	r
255	2^*	r	454	2-3^*	r	643	2-3^*	rv	904	2^*	vh	1122	2^*	q	1283	2	v_1h
257	2^*	r	456	2^*	q	644	2^*	r	905	2^*	v_1	1123	2^*	v_1h	1287	2-3^*	r
259	2^*	r	464	2^*	q	649	2-3^*	r	906	2^*	q	1124	2-3^*	qh	1288	2-3^*	v_1h

TABLE 5. 12 crossing nonalternating knots 12nk with unknotting number > 1.

k	u		k	u		k	u		k	u		k	u		k	u	
10	2^*	v	171	2^*	v	333	2^*	v_1q	460	2^*	v_1	598	2^*	v_1	757	2^*	qh
15	2^*	v	173	2^*	v	334	2	v_1	462	2^*	qh	601	2-3^*	v_1	759	2^*	v
27	2^*	v	174	2^*	q	335	2^*	q	469	2^*	q	602	2-3^*	v_1	769	2-3^*	v
33	2^*	v	202	2-3^*	vq	339	2^*	q	480	2^*	v_1	605	2^*	v_1h	779	2^*	vh
40	2^*	v	206	2^*	q	355	2^*	h	481	2^*	q	611	2^*	vq_1	781	3^*	w_3
52	2^*	v	211	2^*	v	356	2^*	h	486	2^*	q	612	2^*	q	798	2^*	h
55	2^*	vh	219	2^*	vh	357	2^*	v	494	3	v_2	615	2^*	v	805	2-3^*	q
56	2^*	h	221	2^*	h	364	2^*	v	495	2^*	v_1	621	2^*	q	810	2^*	q
57	2^*	h	223	2^*	vh	365	2^*	v	496	3	v_2	622	2^*	v_1	813	2^*	v_1
60	2^*	vh	224	2^*	h	379	2^*	v_1h	497	2^*	q	626	3	v_2	814	2-3^*	q
61	2^*	vh	225	2^*	vh	380	2^*	v_1h	498	2^*	h	630	2-3^*	vq	838	2^*	q_1h
62	2^*	h	227	2^*	q	388	2^*	v_1	505	2^*	v_1	636	2^*	v_1	840	2^*	h
63	2^*	vh	247	2^*	v	389	2^*	v_1	533	2^*	vh	637	2^*	v_1	844	2-3^*	vq_1
66	2^*	h	248	2-3^*	q	391	2^*	q	543	2^*	v	642	3-4^*	v_3	845	2^*	v
73	2^*	q	253	2-3^*	q	393	2^*	h	546	2^*	v_1	651	3^*	w_3	846	2^*	v_1
78	2^*	q	256	2^*	q	394	2^*	qh	553	3^*	v_3	654	3	v_2	847	2^*	vh
85	2^*	q	264	2^*	q	397	3^*	w_2	554	3^*	v_3	665	2^*	v	856	2-3^*	q
95	2^*	q	266	2^*	q	401	2^*	v	555	3^*	v_3	669	2^*	v_1	869	2^*	v_1
99	2^*	vq	267	2^*	v	404	2-3	v	556	3^*	v_3	672	2^*	v_1	870	2^*	q
109	2^*	v	268	2^*	v_1	409	2^*	v	558	2^*	q	701	2-3^*	v_1	873	2-3^*	h
126	2^*	vq	269	2^*	v_1	410	2^*	v	561	2^*	q	706	2^*	hw_2	874	2^*	vh
130	2^*	v	270	2-3^*	v_1	413	2^*	q	562	2^*	q	717	2^*	v	876	2^*	v_1
144	2^*	q	274	2^*	qh	420	2^*	v_1	567	2^*	v_1h	726	2^*	v	877	2^*	qh
145	2^*	qh	276	3	q_2	423	2^*	vq	571	2^*	v_1h	737	2^*	v_1	878	2^*	h
147	3	q_2	278	2^*	q	439	2-3^*	v	580	2^*	q	743	2^*	q	883	2^*	v_1
151	2^*	v	283	2^*	v	440	2^*	v_1h	582	2^*	v_1	746	3^*	w_3	886	2^*	v
157	2^*	q	297	2^*	qh	442	2^*	vh	583	2^*	v_1	752	2^*	vh			
164	2^*	q	317	2^*	q	451	2^*	q	596	2^*	q	756	2^*	v_1			