ANOTHER PROOF OF THE HARDY INEQUALITY IN A LIMITING CASE WITH THE QUASI-SCALE INVARIANCE

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Abstract. We introduce a new scaling in the logarithmic Hardy inequality. Our inequality has the quasi-scale invariance under the scaling. By using a scaling argument, we also show that there is no minimizer for an associated minimizing problem.

1. Introduction

Let Ω be a smooth bounded domain with $0 \in \Omega$ in $\mathbb{R}^N$(or $\Omega = \mathbb{R}^N$). The classical Hardy inequality

$$\left(\frac{N-p}{p}\right)^p \int_\Omega \frac{|u(x)|^p}{|x|^p}\,dx \le \int_\Omega |\nabla u(x)|^p\,dx \tag{1.1}$$

holds for all $u \in W_0^{1,p}(\Omega)$, where $N \ge 2$, $1 \le p < N$. It is known that, for $1 < p$, the best constant $(\frac{N-p}{p})^p$ is not attained in $W_0^{1,p}(\Omega)$. Furthermore, the inequality (1.1) can be improved by adding remainder terms in (1.1) (see [1], [3], [5], [6], [8], [9], [10], [11], [13], [16] and the references therein). One of the novelties of the inequality (1.1) is its scale invariance under the scaling

$$u_\lambda(x) = \lambda^{-\frac{N-p}{p}} u\left(\frac{x}{\lambda}\right) \tag{1.2}$$

for $\lambda > 0$ when $\Omega = \mathbb{R}^N$. Indeed, one can easily check that

$$\int_{\mathbb{R}^N} |\nabla u_\lambda(x)|^p\,dx = \int_{\mathbb{R}^N} |\nabla u(y)|^p\,dy, \quad \int_{\mathbb{R}^N} \frac{|u_\lambda(x)|^p}{|x|^p}\,dx = \int_{\mathbb{R}^N} \frac{|u(y)|^p}{|y|^p}\,dy.$$

On a bounded domain case, the inequality (1.1) does not have its scale invariance under the scaling (1.2) due to change in the domain of integration by the scaling. Indeed, one can see that

$$\int_{\lambda\Omega} |\nabla u_\lambda(x)|^p\,dx = \int_{\Omega} |\nabla u(y)|^p\,dy, \quad \int_{\lambda\Omega} \frac{|u_\lambda(x)|^p}{|x|^p}\,dx = \int_{\Omega} \frac{|u(y)|^p}{|y|^p}\,dy.$$

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However, if we admit change in the domain of integration, we can say that the inequality (1.1) in a bounded domain also has scale invariance under the scaling (1.2). Here we call this property as *the quasi-scale invariance*.

On the critical case $p = N$, the inequality (1.1) fails for every constant and instead of (1.1) the inequality

$$\left(\frac{N-1}{N}\right)^N \int_\Omega \frac{|u(x)|^N}{|x|^N (\log \frac{eR}{|x|})^N}\, dx \le \int_\Omega \left|\nabla u(x) \cdot \frac{x}{|x|}\right|^N dx \tag{1.3}$$

holds for all $u \in W_0^{1,N}(\Omega)$, where $R := \sup_{x\in\Omega} |x|$ (see [1], [17], Proposition 7 in Appendix). We call (1.3) as *the Hardy inequality in a limiting case*. It is also known that the best constant $(\frac{N-1}{N})^N$ is not attained in $W_0^{1,N}(\Omega)$ by adding remainder terms in (1.3) (see [2], [15], Proposition 7 in Appendix).

On the other hand, the critical Hardy inequality

$$\left(\frac{N-1}{N}\right)^N \int_\Omega \frac{|u(x)|^N}{|x|^N (\log \frac{R}{|x|})^N}\, dx \le \int_\Omega \left|\nabla u(x) \cdot \frac{x}{|x|}\right|^N dx \tag{1.4}$$

holds for all $u \in W_0^{1,N}(\Omega)$ (see [12], [17]). It is also known that the best constant $(\frac{N-1}{N})^N$ is not attained in $W_0^{1,N}(\Omega)$ (see [12], [16]).

In this paper, we focus on the Hardy inequality in a limiting case (1.3). In author's opinion, the attainability of $(\frac{N-1}{N})^N$ in (1.3) have been showed by using adding remainder terms in (1.3) or by using the attainability of $(\frac{N-1}{N})^N$ in (1.4). The main aim of this paper is to provide another proof of the attainability of $(\frac{N-1}{N})^N$ in (1.3) by using the property which the inequality (1.3) has. We do not need remainder terms of (1.3) and results of (1.4).

Note that the inequalities (1.3), (1.4) are not the invariant under the scaling $u_\lambda(x) = u(\lambda x)$ due to the logarithmic term. However Cassani-Ruf-Tarsi [4] and Ioku-Ishiwata [12] introduced the scaling

$$u_\lambda(x) = \lambda^{-\frac{N-1}{N}} u\left(|x|^{\lambda-1} x\right) \tag{1.5}$$

for $\lambda > 0$ when Ω is a unit ball. One can observe that the inequality (1.4) has its scale invariance under the scaling (1.5) when Ω is a unit ball (see [12]). Ioku-Ishiwata [12] proved that the best constant $(\frac{N-1}{N})^N$ in (1.4) is not attained in $W_0^{1,N}(\Omega)$ when Ω is a ball. One of key tools in their proof is its scale invariance in the inequality (1.4). However, our inequality (1.3) does not have its scale invariance under the scaling (1.5). Instead of (1.5), we introduce the following scaling

$$u_\lambda(x) = \lambda^{-\frac{N-1}{N}} u\left(\left(\frac{|x|}{eR}\right)^{\lambda-1} x\right) \tag{1.6}$$

for $\lambda > 0$ to the Hardy inequality in a limiting case (1.3). One of key tools in our proof is its quasi-scale invariance in (1.3) under the scaling (1.6).

Before starting main results let us introduce the following notations: $B(R)$ is a ball centered 0 with radius R in $\mathbb{R}^N$, $|A|$ denotes the measure of a set $A \subset \mathbb{R}^N$, ω_N is the area of the unit sphere in $\mathbb{R}^N$, and $W^{1,N}_{0,\mathrm{rad}}(B(R))$ is a class of radially symmetric functions which is contained in $W^{1,N}_0(B(R))$. Moreover, we set

$$C_H := \inf_{0 \neq u \in W^{1,N}_0(\Omega)} \frac{\int_\Omega \left| \nabla u(x) \cdot \frac{x}{|x|} \right|^N dx}{\int_\Omega \frac{|u(x)|^N}{|x|^N \left(\log \frac{eR}{|x|}\right)^N} dx}. \tag{1.7}$$

Note that $C_H = (\frac{N-1}{N})^N$ from Proposition 7 in Appendix. Our main result is as follows:

Theorem 1. *The optimal constant C_H is not attained for any bounded domain Ω.*

Remark 2. The inequality (1.3) has the quasi-scale invariance under the scaling (1.6) (see Proposition 8 in Appendix).

2. Proof of Theorem 1

For a while, let $\Omega = B(R)$. We assume that C_H is attained by $\tilde{u} \in W^{1,N}_0(B(R))$ and derive a contradiction. At first, we show the existence of a radial minimizer of C_H. This lemma can be proved by using the same method as one of [12].

Lemma 3 ([12])**.** *Let $\tilde{u} \in W^{1,N}_0(B(R))$ be a minimizer of C_H. Then there exists a radial minimizer $u \in W^{1,N}_{0,rad}(B(R))$ of C_H.*

Proof of Lemma 3. Since $\tilde{u} \in W^{1,N}_0(B(R))$ is a minimizer of C_H, then the forth inequality in (3.2) should be an equality:

$$\int_{B(R)} \frac{|\tilde{u}(x)|^{N-1}}{|x|^{N-1}\left(\log \frac{eR}{|x|}\right)^{N-1}} \left| \frac{x}{|x|} \cdot \nabla \tilde{u}(x) \right| dx$$
$$= \frac{N}{N-1} \left(\int_{B(R)} \left| \nabla \tilde{u}(x) \cdot \frac{x}{|x|} \right|^N dx \right)^{\frac{1}{N}} \left(\int_{B(R)} \frac{|\tilde{u}(x)|^N}{|x|^N (\log \frac{eR}{|x|})^N} dx \right)^{\frac{N-1}{N}}.$$

By the equality condition for the Hölder inequality, there exists a constant $\alpha \in \mathbb{R}$ such that

$$\left| \frac{x}{|x|} \cdot \nabla \tilde{u}(x) \right| = \alpha \frac{|\tilde{u}(x)|}{|x| (\log \frac{eR}{|x|})} \tag{2.1}$$

holds for almost every $x \in B(R)$. Since $\tilde{u}$ is a minimizer of C_H, we have $\alpha = (C_H)^{\frac{1}{N}}$.

On the other hand, by using polar coordinates $x = (r, \theta)\,(r \in (0, R), \theta \in \mathbb{S}^{N-1})$ and Fubini's theorem together with the fact $\tilde{u} \in W_0^{1,N}(B(R))$, we obtain

$$\int_0^R |\partial_r \tilde{u}(r, \theta)|^N r^{N-1}\, dr < \infty, \int_0^R \frac{|\tilde{u}(r, \theta)|}{r(\log \frac{eR}{r})^N}\, dr < \infty. \tag{2.2}$$

for almost every $\theta \in \mathbb{S}^{N-1}$. Combining (2.1) and (2.2), we observe that there exists $\theta_0 \in \mathbb{S}^{N-1}$ such that

$$0 < \int_0^R |\partial_r \tilde{u}(r, \theta_0)|^N r^{N-1}\, dr = C_H \int_0^R \frac{|\tilde{u}(r, \theta_0)|}{r(\log \frac{eR}{r})^N}\, dr < \infty.$$

Here we put $u(x) := \tilde{u}(|x|, \theta_0)$. Then we can observe that u satisfies

$$C_H = \frac{\int_{\mathbb{S}^{N-1}} d\theta \int_0^R |\partial_r \tilde{u}(r, \theta_0)|^N\, r^{N-1}\, dr}{\int_{\mathbb{S}^{N-1}} d\theta \int_0^R \frac{|\tilde{u}(r,\theta_0)|^N}{r\left(\log \frac{eR}{r}\right)^N}\, dr} = \frac{\int_{B(R)} \left|\nabla u(x) \cdot \frac{x}{|x|}\right|^N\, dx}{\int_{B(R)} \frac{|u(x)|^N}{|x|^N\left(\log \frac{eR}{|x|}\right)^N}\, dx}$$

which implies that $u \in W_{0,\mathrm{rad}}^{1,N}(B(R))$ is a radial minimizer of C_H. □

Next lemma is concerning the quasi-scale invariance of the inequality (1.3) for radial functions. Recall that the scale invariance of the inequality (1.4) plays a essential role for proving the nonexistence of minimizers of (1.4) in [12]. In our inequality (1.3), the quasi-scale invariance of (1.3) plays a essential role for the proof of Theorem 1.

Lemma 4. *Let $u = u(s) \in W_{0,rad}^{1,N}(B(R))$ and $u_\lambda = u_\lambda(r)$ be defined by (1.6). Then $u_\lambda \in W_{0,rad}^{1,N}(B(e^{1-\frac{1}{\lambda}}R))$ and there hold*

$$\int_0^{e^{1-\frac{1}{\lambda}}R} |\partial_r u_\lambda(r)|^N r^{N-1}\, dr = \int_0^R |\partial_s u(s)|^N s^{N-1}\, ds \tag{2.3}$$

and

$$\int_0^{e^{1-\frac{1}{\lambda}}R} \frac{|u_\lambda(r)|^N}{r(\log \frac{eR}{r})^N}\, dr = \int_0^R \frac{|u(s)|^N}{s(\log \frac{eR}{s})^N}\, ds. \tag{2.4}$$

Especially, if $u \in W_{0,rad}^{1,N}(B(R))$ is a radial minimizer of C_H, then $u_\lambda \in W_{0,rad}^{1,N}(B(e^{1-\frac{1}{\lambda}}R)) \subset W_{0,rad}^{1,N}(B(R))$ is also a radial minimizer of C_H for all $0 < \lambda \le 1$.

Proof of Lemma 4. Since (1.6), there holds

$$\int_0^{e^{1-\frac{1}{\lambda}}R} |\partial_r u_\lambda(r)|^N r^{N-1}\, dr = \lambda^{1-N} \int_0^{e^{1-\frac{1}{\lambda}}R} |\partial_r u(r^\lambda (eR)^{1-\lambda})|^N r^{N-1}\, dr.$$

Now applying the change of variables $s = s(r) = r^\lambda (eR)^{1-\lambda}$(i.e. $r = r(s) = s^{\frac{1}{\lambda}}(eR)^{1-\frac{1}{\lambda}}$), we obtain

$$\int_0^{e^{1-\frac{1}{\lambda}}R} |\partial_r u_\lambda(r)|^N r^{N-1}\, dr = \lambda^{1-N}\int_0^R |u'(s)|^N (s'(r)\, r(s))^{N-1}\, ds$$
$$= \int_0^R |u'(s)|^N s^{N-1}\, ds$$

which implies (2.3). On the other hand, by the change of variables $s = s(r) = r^\lambda (eR)^{1-\lambda}$(i.e. $r = r(s) = s^{\frac{1}{\lambda}}(eR)^{1-\frac{1}{\lambda}}$), we get

$$\int_0^{e^{1-\frac{1}{\lambda}}R} \frac{|u_\lambda(r)|^N}{r(\log\frac{eR}{r})^N}\, dr = \lambda^{1-N}\int_0^{e^{1-\frac{1}{\lambda}}R} \frac{|u(r^\lambda (eR)^{1-\lambda})|^N}{r(\log\frac{eR}{r})^N}\, dr$$
$$= \lambda^{1-N}\int_0^R \frac{|u(s)|^N}{\lambda s(\frac{1}{\lambda}\log\frac{eR}{s})^N}\, ds$$
$$= \int_0^R \frac{|u(s)|^N}{s(\log\frac{eR}{s})^N}\, ds$$

which yields (2.4). □

Remark 5. In fact, we can prove Lemma 4 for even non-radially symmetric functions (see Proposition 8 in Appendix). However, in that case, we must use the Jacobian $J\left(\frac{\partial(x_1,\cdots,x_N)}{\partial(y_1,\cdots,y_N)}\right)$ in (3.7) calculated by Ioku-Ishiwata [12]. In present proof, we do not need the complicated Jacobian thanks to restrict functions to radial ones.

Lemma 6. *Let* $u \in W^{1,N}_{0,rad}(B(R))$ *be a nonnegative minimizer of* C_H. *Then* $u \in C^1(B(R)\setminus\{0\})$ *and* $u > 0$ *in* $B(R)\setminus\{0\}$.

Proof of Lemma 6. For the sake of simplicity, we put

$$E(v) := \int_0^R |\partial_r v(r)|^N r^{N-1}\, dr,\quad G(u) := \int_0^R \frac{|v(r)|^N}{r(\log\frac{eR}{r})^N}\, dr.$$

where $v \in W^{1,N}_{0,\mathrm{rad}}(B(R))$. For $v, \phi \in W^{1,N}_{0,\mathrm{rad}}(B(R))$, we have

$$E^{'}(v)\phi := \frac{d}{dt}\Big|_{t=0} E(v+t\phi) = \int_0^R |\partial_r v(r)|^{N-2}\partial_r v(r)\partial_r\phi(r) r^{N-1}\, dr$$
$$G^{'}(v)\phi := \frac{d}{dt}\Big|_{t=0} G(v+t\phi) = \int_0^R \frac{|v(r)|^{N-2}v(r)\phi(r)}{r(\log\frac{eR}{r})^N}\, dr.$$

By using this notations, the optimal constant C_H is written by

$$C_H = \inf_{0\neq v\in W^{1,N}_{0,\mathrm{rad}}(B(R))} \frac{E(v)}{G(v)} = \inf_{v\in W^{1,N}_{0,\mathrm{rad}}(B(R)),\,G(v)=1} E(v).$$

By the method of Lagrange multiplier, a minimizer u satisfies

$$\int_0^R |\partial_r u(r)|^{N-2}\partial_r u(r)\partial_r \phi(r)\, r^{N-1}\, dr = C_H \int_0^R \frac{|u(r)|^{N-2}u(r)\phi(r)}{r(\log \frac{eR}{r})^N}\, dr$$

for all $\phi \in W^{1,N}_{0,\mathrm{rad}}(B(R))$. This means that u is a weak solution of

$$\begin{cases} -\Delta_N u = C_H \frac{|u|^{N-2}u}{\left(|x| \log \frac{eR}{|x|}\right)^N} & \text{in } B(R) \\ u = 0 & \text{on } \partial B(R) \end{cases}$$

where $-\Delta_N v :=$div $(|\nabla v|^{N-2}\nabla v)$ is the N-Laplacian. Particularly, for $\varepsilon > 0$,

$$\text{the function } \left(|x| \log \frac{eR}{|x|}\right)^{-N} \text{ is bounded in } B(R) \setminus B(\varepsilon). \tag{2.5}$$

Furthermore Sobolev embedding $W^{1,N}(B(R)) \hookrightarrow L^q(B(R))$ $(1 \le {}^\forall q < \infty)$ yields that

$$u \in L^q(B(R)) \text{ for all } 1 \le q < \infty. \tag{2.6}$$

Thus we see that $\Delta_N u \in L^q(B(R) \setminus B(\varepsilon))$ for all $1 \le q < \infty$ by (2.5) and (2.6). If we take $q > \frac{N^2}{N-1}$, then $u \in C^1(B(R) \setminus B(\varepsilon))$. (see [7]) Hence, by applying the strong maximum principle for the distributional solution $u \in C^1(B(R) \setminus B(\varepsilon))$ to the inequality $-\Delta_N u \ge 0$ in $B(R) \setminus B(\varepsilon)$, we obtain $u > 0$ in $B(R) \setminus B(\varepsilon)$. (see [14] Theorem 2.5.1.) Since $\varepsilon > 0$ is arbitrary, we have proved $u \in C^1(B(R) \setminus \{0\})$ and $u > 0$ in $B(R) \setminus \{0\}$. □

At last, we shall prove Theorem 1 by using three Lemmas. Proof of Theorem 1 consists of two Steps. In Step 1, we show the nonexistence of minimizers of C_H when Ω is a ball $B(R)$. In Step 2, we show the nonexistence of minimizers of C_H when Ω is a general bounded domain.

Proof of Theorem 1. (Step 1) : First we will show the optimal constant C_H is not attained when $\Omega = B(R)$. Now we assume that $\tilde{u} \in W^{1,N}_0(B(R))$ is the minimizer of C_H and derive a contradiction. From Lemma 3, there exists a radial minimizer $u \in W^{1,N}_{0,\mathrm{rad}}(B(R))$ of C_H. Furthermore we suppose that u is nonnegative without loss of generality. Let $R_S > 0$ be such that $R_S < R$. From Lemma 4, if we take $\lambda > 0$ as satisfying $e^{1-\frac{1}{\lambda}}R = R_S$, then $u_\lambda \in W^{1,N}_{0,\mathrm{rad}}(B(R_S))$ and there hold

$$\int_0^{R_S} |\partial_r u_\lambda(r)|^N r^{N-1}\, dr = \int_0^R |\partial_s u(s)|^N s^{N-1}\, ds \tag{2.7}$$

and

$$\int_0^{R_S} \frac{|u_\lambda(r)|^N}{r(\log \frac{eR}{r})^N}\, dr = \int_0^R \frac{|u(s)|^N}{s(\log \frac{eR}{s})^N}\, ds. \tag{2.8}$$

Here we put

$$\overline{u}_\lambda(r) := \begin{cases} u_\lambda(r) & \text{if } 0 < r \le R_S \\ 0 & \text{if } R_S < r < R. \end{cases}$$

Thus $\overline{u}_\lambda \in W^{1,N}_{0,\mathrm{rad}}(B(R))$ is also a radial minimizer of C_H. However, by using Lemma 6, we have $\overline{u}_\lambda > 0$ in $0 < r < R$. Since $\overline{u}_\lambda(r) \equiv 0$ in $R_S \le r \le R$, this is a contradiction. Therefore the optimal constant C_H is not attained when Ω is a ball.

(Step 2) : Next we will show the optimal constant C_H is not attained when Ω is a general bounded domain. We assume that C_H is attained by $v \in W^{1,N}_0(\Omega)$. Then we put

$$\overline{v}(x) := \begin{cases} v(x) & \text{if } x \in \Omega \\ 0 & \text{if } x \in B(R) \setminus \Omega. \end{cases}$$

Since $\overline{v} \in W^{1,N}_0(B(R))$, C_H is attained by $\overline{v} \in W^{1,N}_0(B(R))$ when Ω is a ball $B(R)$. This contradicts to Step 1.

The proof of Theorem 1 is now complete. □

3. Appendix

For the reader's convenience, we give a proof of the inequality (1.3) with its optimal constant.

Proposition 7. *Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain with $0 \in \Omega$ and $N \ge 2$. Then the inequality (1.3):*

$$\left(\frac{N-1}{N}\right)^N \int_\Omega \frac{|u(x)|^N}{|x|^N (\log \frac{eR}{|x|})^N}\, dx \le \int_\Omega \left| \nabla u(x) \cdot \frac{x}{|x|} \right|^N dx$$

holds for all $u \in W^{1,N}_0(\Omega)$, where $R = \sup_{x\in\Omega} |x|$. Furthermore, the constant $(\frac{N-1}{N})^N$ is optimal, namely there holds

$$C_H = \inf_{0 \ne u \in W^{1,N}_0(\Omega)} \frac{\int_\Omega \left| \nabla u(x) \cdot \frac{x}{|x|} \right|^N dx}{\int_\Omega \frac{|u(x)|^N}{|x|^N \left(\log \frac{eR}{|x|}\right)^N} dx} = \left(\frac{N-1}{N}\right)^N. \tag{3.1}$$

Proof of Proposition 7. First we will describe the simple proof of (1.3) by F.Takahashi [17]. Since

$$\mathrm{div}\left(\frac{x}{|x|^N \left(\log \frac{eR}{|x|}\right)^{N-1}} \right) = \frac{N-1}{|x|^N \left(\log \frac{eR}{|x|}\right)^N} \quad \text{for } |x| \ne 0, R,$$

we have

$$\begin{aligned}
\int_\Omega \frac{|u(x)|^N}{|x|^N(\log \frac{eR}{|x|})^N} dx &= \frac{1}{N-1}\int_\Omega \mathrm{div}\left(\frac{x}{|x|^N\left(\log\frac{eR}{|x|}\right)^{N-1}}\right)|u(x)|^N dx \\
&= \frac{N}{N-1}\int_\Omega \frac{|u(x)|^{N-2}u(x)}{|x|^{N-1}\left(\log\frac{eR}{|x|}\right)^{N-1}}\left(-\frac{x}{|x|}\cdot\nabla u(x)\right)dx \\
&\le \frac{N}{N-1}\int_\Omega \frac{|u(x)|^{N-1}}{|x|^{N-1}\left(\log\frac{eR}{|x|}\right)^{N-1}}\left|\frac{x}{|x|}\cdot\nabla u(x)\right| dx \\
&\le \frac{N}{N-1}\left(\int_\Omega\left|\nabla u(x)\cdot\frac{x}{|x|}\right|^N dx\right)^{\frac{1}{N}}\left(\int_\Omega \frac{|u(x)|^N}{|x|^N(\log\frac{eR}{|x|})^N}dx\right)^{\frac{N-1}{N}}. \qquad (3.2)
\end{aligned}$$

Hence the inequality (1.3) holds for all $u \in W_0^{1,N}(\Omega)$.

Next we will give a proof of (3.1) by using the quasi-scale invariance in the Hardy inequality in a limiting case (1.3). Let $0 < \ell \ll 1$ and $\frac{N-1}{N} < \delta < 1$. We define the function $u_{\ell,\delta}$ as follows:

$$u_{\ell,\delta}(r) = \begin{cases} \left(\log\frac{e}{\ell}\right)^\delta - 1, & 0 \le r \le \ell R \\ \left(\log\frac{eR}{r}\right)^\delta - 1, & \ell R \le r \le R. \end{cases}$$

where $r = |x|$, and

$$u'_{\ell,\delta}(r) = \begin{cases} 0, & 0 \le r \le \ell R \\ -\delta\left(\log\frac{eR}{r}\right)^{\delta-1} r^{-1}, & \ell R \le r \le R. \end{cases}$$

Therefore, one can easily check that $u \in W_{0,\mathrm{rad}}^{1,N}(B(R))$. Direct calculations show that

$$\begin{aligned}
\int_{B(R)}\left|\nabla u_{\ell,\delta}(x)\cdot\frac{x}{|x|}\right|^N dx &= \omega_N\delta^N\int_{\ell R}^R |u'_{\ell,\delta}(r)|^N r^{N-1}dr \\
&= \omega_N\,\delta^N\int_{\ell R}^R\left(\log\frac{eR}{r}\right)^{N\delta-N}\frac{dr}{r} \\
&= \omega_N\,\delta^N\int_\ell^1\left(\log\frac{e}{r}\right)^{N\delta-N}\frac{dr}{r} \\
&= \omega_N\,\delta^N\int_1^{\log\frac{e}{\ell}} t^{N\delta-N}\,dt \\
&= \omega_N\,\frac{\delta^N}{N\delta-N+1}\left(\left(\log\frac{e}{\ell}\right)^{N\delta-N+1}-1\right). \qquad (3.3)
\end{aligned}$$

On the other hand, we have

$$
\begin{aligned}
&\int_{B(R)} \frac{|u_{\ell,\delta}(x)|^N}{|x|^N \left(\log \frac{eR}{|x|}\right)^N}\,dx \\
&= \omega_N \left(\left(\log \frac{e}{\ell}\right)^{\delta} - 1\right)^N \int_0^{\ell R} \left(\log \frac{eR}{r}\right)^{-N} \frac{dr}{r} \\
&\qquad + \omega_N \sum_{k=0}^{N} (-1)^k \left(\begin{array}{c} N \\ k \end{array}\right) \int_{\ell R}^{R} \left(\log \frac{eR}{r}\right)^{(N-k)\delta - N} \frac{dr}{r} \\
&= \omega_N \int_0^{\ell} \left(\log \frac{e}{r}\right)^{-N} \frac{dr}{r} + \omega_N \sum_{k=0}^{N} (-1)^k \left(\begin{array}{c} N \\ k \end{array}\right) \int_{\ell}^{1} \left(\log \frac{e}{r}\right)^{(N-k)\delta - N} \frac{dr}{r} \\
&= \omega_N \sum_{k=0}^{N} (-1)^k \left(\begin{array}{c} N \\ k \end{array}\right) \left(\log \frac{e}{\ell}\right)^{(N-k)\delta} \int_{\log \frac{e}{\ell}}^{\infty} t^{-N}\,dt \\
&\qquad + \omega_N \sum_{k=0}^{N} (-1)^k \left(\begin{array}{c} N \\ k \end{array}\right) \int_1^{\log \frac{e}{\ell}} t^{(N-k)\delta - N}\,dt \\
&= \omega_N \sum_{k=0}^{N} (-1)^k \left(\begin{array}{c} N \\ k \end{array}\right) \frac{1}{N-1} \left(\log \frac{e}{\ell}\right)^{(N-k)\delta + 1 - N} \\
&\qquad + \omega_N \sum_{k=0}^{N} (-1)^k \left(\begin{array}{c} N \\ k \end{array}\right) \frac{1}{(n-k)\delta - N + 1} \left(\left(\log \frac{e}{\ell}\right)^{(N-k)\delta - N + 1} - 1\right) \\
&= \omega_N \Bigg(\frac{N\delta}{(N-1)(N\delta - N + 1)} \left(\log \frac{e}{\ell}\right)^{N\delta + 1 - N} \\
&\qquad + \sum_{k=1}^{N} C_{k,N,\ell} \left(\log \frac{e}{\ell}\right)^{(N-k)\delta + 1 - N} + \sum_{k=0}^{N} D_{k,N,\delta} \Bigg) \qquad (3.4)
\end{aligned}
$$

where

$$
\begin{aligned}
C_{k,N,\delta} &:= (-1)^k \left(\begin{array}{c} N \\ k \end{array}\right) \frac{(N-k)\delta}{(N-1)(\,(N-k)\delta - N + 1\,)} \\
D_{k,N,\delta} &:= (-1)^k \left(\begin{array}{c} N \\ k \end{array}\right) \frac{1}{(N-k)\delta - N + 1}
\end{aligned}
$$

From (3.3) and (3.4), we obtain

$$\frac{\int_{B(R)} \left| \nabla u_{\ell,\delta}(x) \cdot \frac{x}{|x|} \right|^N dx}{\int_{B(R)} \frac{|u_{\ell,\delta}(x)|^N}{|x|^N \left(\log \frac{eR}{|x|}\right)^N} dx} \to \frac{(N-1)\delta^{N-1}}{N} \text{ as } \ell \to 0$$

$$\to \left(\frac{N-1}{N} \right)^N \text{ as } \delta \to \frac{N-1}{N}. \tag{3.5}$$

Now applying Lemma 4, we have

$$\frac{\int_{B(R)} \left| \nabla u_{\ell,\delta}(x) \cdot \frac{x}{|x|} \right|^N dx}{\int_{B(R)} \frac{|u_{\ell,\delta}(x)|^N}{|x|^N \left(\log \frac{eR}{|x|}\right)^N} dx} = \frac{\int_{B(e^{1-\frac{1}{\lambda}} R)} \left| \nabla (u_{\ell,\delta})_\lambda(x) \cdot \frac{x}{|x|} \right|^N dx}{\int_{B(e^{1-\frac{1}{\lambda}} R)} \frac{|(u_{\ell,\delta})_\lambda(x)|^N}{|x|^N \left(\log \frac{eR}{|x|}\right)^N} dx} \tag{3.6}$$

for all $\lambda > 0$. Here, if we take $\lambda > 0$ as satisfying $B(e^{1-\frac{1}{\lambda}} R) \subset \Omega$, then $(u_{\ell,\delta})_\lambda \in W^{1,N}_{0,\mathrm{rad}}(\Omega)$. Consequently, from (3.5) and (3.6), we can observe that the constant $(\frac{N-1}{N})^N$ is optimal in the inequality (1.3) for any bounded domain Ω. □

Proposition 8. *The inequality (1.3) has quasi-scale invariance under the scaling (1.6).*

Proof of Proposition 8. Let $y = \left(\frac{|x|}{eR} \right)^{\lambda-1} x \left(\text{i.e.} x = \left(\frac{|y|}{eR} \right)^{\frac{1}{\lambda}-1} y \right)$ and

$$\Omega_\lambda := \left\{ x = \left(\frac{|y|}{eR} \right)^{\frac{1}{\lambda}-1} y \;\middle|\; y \in \Omega \right\}.$$

Ioku-Ishiwata [12] calculated the Jacobian $J\left(\frac{\partial(x_1,\cdots,x_N)}{\partial(y_1,\cdots,y_N)} \right)$ as follows:

$$J\left(\frac{\partial(x_1,\cdots,x_N)}{\partial(y_1,\cdots,y_N)} \right) := \det \begin{pmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \cdots & \frac{\partial x_1}{\partial y_N} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} & \cdots & \frac{\partial x_2}{\partial y_N} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_N}{\partial y_1} & \frac{\partial x_N}{\partial y_2} & \cdots & \frac{\partial x_N}{\partial y_N} \end{pmatrix} = \left(\frac{|y|}{eR} \right)^{(\frac{1}{\lambda}-1)N} \frac{1}{\lambda}. \tag{3.7}$$

Since

$$\nabla_x u_\lambda(x) \cdot \frac{x}{|x|} = \lambda^{\frac{1}{N}} \left(\frac{|x|}{eR} \right)^{\lambda-1} \nabla_y u(y) \cdot \frac{x}{|x|},$$

there holds

$$\int_{\Omega_\lambda} \left| \nabla u_\lambda(x) \cdot \frac{x}{|x|} \right|^N dx = \int_\Omega \lambda \left(\frac{|x|}{eR} \right)^{(\lambda-1)N} \left| \nabla_y u(y) \cdot \frac{x}{|x|} \right|^N dx.$$

Thus we obtain

$$\int_{\Omega_\lambda} \left|\nabla u_\lambda(x) \cdot \frac{x}{|x|}\right|^N dx = \int_\Omega \left(\frac{|y|}{eR}\right)^{(1-\frac{1}{\lambda})N} \lambda \left|\nabla_y u(y) \cdot \frac{y}{|y|}\right|^N J\left(\frac{\partial(x_1, \cdots, x_N)}{\partial(y_1, \cdots, y_N)}\right) dy$$

$$= \int_\Omega \left|\nabla u(y) \cdot \frac{y}{|y|}\right|^N dy. \tag{3.8}$$

On the other hand, by the change of variables $x = \left(\frac{|y|}{eR}\right)^{\frac{1}{\lambda}-1} y$, we have

$$\int_{\Omega_\lambda} \frac{|u_\lambda(x)|^N}{|x|^N \left(\log \frac{eR}{|x|}\right)^N} dx = \lambda^{1-N} \int_\Omega \frac{|u(y)|^N}{(eR)^{(1-\frac{1}{\lambda})N} |y|^{\frac{N}{\lambda}} \left(\frac{1}{\lambda} \log \frac{eR}{|y|}\right)^N} J\left(\frac{\partial(x_1, \cdots, x_N)}{\partial(y_1, \cdots, y_N)}\right) dy$$

$$= \int_\Omega \frac{|u(y)|^N}{|y|^N \left(\log \frac{eR}{|y|}\right)^N} dy. \tag{3.9}$$

Therefore we have proved Proposition 8 by (3.8) and (3.9). □

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