REMARKS ON QUANTUM UNIPOTENT SUBGROUP AND DUAL CANONICAL BASIS

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Abstract. We prove the tensor product decomposition of the half of quantized universal enveloping algebra associated with a Weyl group element which was conjectured by Berenstein and Greenstein [3, Conjecture 5.5] using the theory of the dual canonical basis.

1. Introduction

Let $\mathfrak{g}$ be a symmetrizable Kac-Moody Lie algebra and w be a Weyl group element. In [7], we have studied the compatibility of the dual canonical basis and the quantum coordinate ring of the unipotent subgroup associated with a finite subset $\Delta_+ \cap w\Delta_-$. The purpose of this paper is to study the compatibility of the dual canonical basis and the "quantum coordinate ring" of the pro-unipotent subgroup associated with a co-finite subset $\Delta_+ \cap w\Delta_+$ and show the multiplicity-free property of the multiplications of the dual canonical basis between finite part and co-finite part.

The following tensor product decomposition of the half $\mathbf{U}_q^-$ was conjectured by Berenstein and Greenstein [3, Conjecture 5.5] in general. We also prove the decomposition in the dual integral form $\mathbf{U}_q^-(\mathfrak{g})_{\mathcal{A}}^{\mathrm{up}}$ of Lusztig integral form $\mathbf{U}_q^-(\mathfrak{g})_{\mathcal{A}}$ with respect to Kashiwara's non-degenerate bilinear form.

Theorem 1.1. *Let $T_w = T_{i_1}T_{i_2}\cdots T_{i_\ell} \colon \mathbf{U}_q \to \mathbf{U}_q$ be the Lusztig braid group action associated with a Weyl group element w, where $\boldsymbol{i} = (i_1, \cdots, i_\ell)$ is a reduced word of w.*

(1) *For a Weyl group element $w \in W$, multiplication in $\mathbf{U}_q^-$ defines an isomorphism of vector spaces over $\mathbb{Q}(q)$:*

$$\left(\mathbf{U}_q^- \cap T_w\mathbf{U}_q^{\geq 0}\right) \otimes \left(\mathbf{U}_q^- \cap T_w\mathbf{U}_q^-\right) \xrightarrow{\sim} \mathbf{U}_q^-.$$

(2) *For a Weyl group element $w \in W$, we set $\left(\mathbf{U}_q^- \cap T_w\mathbf{U}_q^{\geq 0}\right)_{\mathcal{A}}^{\mathrm{up}} := \mathbf{U}_q^-(\mathfrak{g})_{\mathcal{A}}^{\mathrm{up}} \cap T_w\mathbf{U}_q^{\geq 0}$ and $\left(\mathbf{U}_q^- \cap T_w\mathbf{U}_q^-\right)_{\mathcal{A}}^{\mathrm{up}} := \mathbf{U}_q^-(\mathfrak{g})_{\mathcal{A}}^{\mathrm{up}} \cap T_w\mathbf{U}_q^-$. Then the multiplication in $\mathbf{U}_q^-(\mathfrak{g})_{\mathcal{A}}^{\mathrm{up}}$ defines an isomorphism of free $\mathcal{A}$-modules:*

$$\left(\mathbf{U}_q^- \cap T_w\mathbf{U}_q^{\geq 0}\right)_{\mathcal{A}}^{\mathrm{up}} \otimes_{\mathcal{A}} \left(\mathbf{U}_q^- \cap T_w\mathbf{U}_q^-\right)_{\mathcal{A}}^{\mathrm{up}} \xrightarrow{\sim} \mathbf{U}_q^-(\mathfrak{g})_{\mathcal{A}}^{\mathrm{up}}.$$

Remark 1.2. (1) Theorem 1.1 (1) can be shown directly in finite type cases using the Poincaré-Birkhoff-Witt bases of $\mathbf{U}_q^-$ (see [3, Proposition 5.3]). Hence it is a new result only in infinite type cases.

(2) For the proof of Theorem 1.1 (1), we use the dual canonical basis and the multiplication formula among them, in particular we will prove Theorem 1.1 (2). After finishing this work, the proof which does not involve the theory of the dual canonical basis was informed to the author by Toshiyuki Tanisaki [13, Proposition 2.10]. He also proves the tensor product decomposition in Lusztig form, De Concini-Kac form and De Concini-Procesi form.

We note that De Concini-Kac form (resp. De Concini-Procesi form) is related with the dual integral form of Lusztig's integral form with respect to the Kashiwara (resp. Lusztig) non-degenerate bilinear form on $\mathbf{U}_q^-$ respectively. Since the multiplicative structure among the dual canonical basis does not depend on a choice of the non-degenerate bilinear form (and hence the definition

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of the dual canonical basis), our argument yields results for the tensor product decompositions of the De Concini-Kac form and De Concini-Procesi form.

(3) We note that the fact that $\mathbf{U}_q^- \cap T_w \mathbf{U}_q^{\geq 0}$ has a Poincaré-Birkhoff-Witt bases is known by Beck-Chari-Pressley [2, Proposition 2.3] in general. The injectivity in Theorem 1.1 can be easily proved by the linear independence of the Poincaré-Birkhoff-Witt monomials (see [9, Theorem 40.2.1 (a)]) and the triangular decomposition of the quantized enveloping algebra (see [9, 3.2]). Hence the non-trivial assertion is the surjectivities in Theorem 1.1.

Theorem 1.3. (1) *For a Weyl group element $w \in W$ and a reduced word $\boldsymbol{i} = (i_1, \cdots, i_\ell)$ of w, we have*

$$\mathbf{U}_q^- \cap T_w \mathbf{U}_q^- = \mathbf{U}_q^- \cap T_{i_1} \mathbf{U}_q^- \cap T_{i_1} T_{i_2} \mathbf{U}_q^- \cap \cdots \cap T_{i_1} \cdots T_{i_\ell} \mathbf{U}_q^-.$$

(2) *$\mathbf{U}_q^- \cap T_w \mathbf{U}_q^-$ is compatible with the dual canonical basis, that is $\mathbf{B}^{\mathrm{up}} \cap \mathbf{U}_q^- \cap T_w \mathbf{U}_q^-$ is a $\mathbb{Q}(q)$-basis of $\mathbf{U}_q^- \cap T_w \mathbf{U}_q^-$. In fact there exists a subset $\mathscr{B}\left(\mathbf{U}_q^- \cap T_w \mathbf{U}_q^-\right) \subset \mathscr{B}(\infty)$ such that*

$$\mathbf{U}_q^- \cap T_w \mathbf{U}_q^- = \bigoplus_{b \in \mathscr{B}\left(\mathbf{U}_q^- \cap T_w \mathbf{U}_q^-\right)} \mathbb{Q}(q)\, G^{\mathrm{up}}(b).$$

Using the theory of crystal basis, we can obtain the characterization of the subset $\mathscr{B}\left(\mathbf{U}_q^- \cap T_w \mathbf{U}_q^-\right)$.

1.0.1. For $w \in W$, we have the decomposition theorem of the crystal basis $\mathscr{B}(\infty)$ of $\mathbf{U}_q^-$ associated with a Weyl group element (and a reduced word) and the corresponding multiplication formula. We consider the map Ω_w associated with a Weyl group element which is introduced in Saito [12] (and Baumann-Kamnitzer-Tingley [1]) :

$$\Omega_w := (\tau_{\leq w}, \tau_{>w}) : \mathscr{B}(\infty) \to \mathscr{B}\left(\mathbf{U}_q^- \cap T_w \mathbf{U}_q^{\geq 0}\right) \times \mathscr{B}\left(\mathbf{U}_q^- \cap T_w \mathbf{U}_q^-\right),$$

where $\tau_{\leq w}(b)$ and $\tau_{>w}(b)$ are defined by crystal basis as follows:

$$\begin{aligned}
L(b, \boldsymbol{i}) &:= \left(\varepsilon_{i_1}(b), \varepsilon_{i_2}\left(\hat{\sigma}_{i_1}^* b\right), \cdots, \varepsilon_{i_\ell}\left(\hat{\sigma}_{i_{\ell-1}}^* \cdots \hat{\sigma}_{i_1}^* b\right)\right) \in \mathbb{Z}_{\geq 0}^{\ell}, \\
b(\boldsymbol{c}, \boldsymbol{i}) &:= f_{i_1}^{(c_1)} T_{i_1}\left(f_{i_2}^{(c_2)}\right) \cdots T_{i_1} \cdots T_{i_{\ell-1}}\left(f_{i_\ell}^{(c_\ell)}\right) \bmod q\mathscr{L}(\infty) \in \mathscr{B}(\infty), \\
\tau_{\leq w}(b) &:= b\left(L(b, \boldsymbol{i}), \boldsymbol{i}\right) \in \mathscr{B}(\infty), \\
\tau_{>\boldsymbol{i}}(b) &:= \sigma_{i_1} \cdots \sigma_{i_\ell} \hat{\sigma}_{i_\ell}^* \cdots \hat{\sigma}_{i_1}^* b \in \mathscr{B}(\infty).
\end{aligned}$$

The following is the multiplicity-free result of the multiplication among the dual canonical basis elements in finite part and co-finite part.

Theorem 1.4. *Let w be a Weyl group element and $\boldsymbol{i} = (i_1, \cdots, i_\ell)$ be a reduced word of w.*
For a crystal basis element $b \in \mathscr{B}(\infty)$, we have

$$G^{\mathrm{up}}(\tau_{\leq w}(b))\, G^{\mathrm{up}}(\tau_{>w}(b)) \in G^{\mathrm{up}}(b) + \sum_{L\left(b', \boldsymbol{i}\right) < L(b, \boldsymbol{i})} q\mathbb{Z}[q]\, G^{\mathrm{up}}\left(b'\right)$$

where $L\left(b', \boldsymbol{i}\right) < L(b, \boldsymbol{i})$ is the left lexicographic order on $\mathbb{Z}_{\geq 0}^{\ell}$ associated with a reduced word $\boldsymbol{i}$.

Using the induction on the lexicographic order on each root space, we obtain the surjectivity in Theorem 1.1 (2). In particular, 1.1 (1) can be shown.

Since the subalgebras $\mathbf{U}_q^- \cap T_w \mathbf{U}_q^{\geq 0}$ and $\mathbf{U}_q^- \cap T_w \mathbf{U}_q^-$ are compatible with the dual canonical base and the dual bar-involution σ which characterizes the dual canonical base is an (twisted) anti-involution, we obtain the tensor product factorization in the opposite order.

Corollary 1.5. *For a Weyl group element $w \in W$, multiplication in $\mathbf{U}_q^-$ defines an isomorphism of vector spaces:*

$$\left(\mathbf{U}_q^- \cap T_w \mathbf{U}_q^-\right) \otimes \left(\mathbf{U}_q^- \cap T_w \mathbf{U}_q^{\geq 0}\right) \xrightarrow{\sim} \mathbf{U}_q^-.$$

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2. Review on Quantum unipotent subgroup and Dual canonical basis

2.1. **Quantum universal enveloping algebra.** In this subsection, we give a brief review of the definition of quantum universal enveloping algebra.

2.1.1. Let I be a finite index set.

Definition 2.1. A *root datum* is a quintuple $(A, P, \Pi, P^\vee, \Pi^\vee)$ which consists of

1. a square matrix $(a_{ij})_{i,j\in I}$, called the symmetrizable generalized Cartan matrix, that is an I-indexed $\mathbb{Z}$-valued matrix which satisfies
   (a) $a_{ii} = 2$ for $i \in I$
   (b) $a_{ij} \in \mathbb{Z}_{\leq 0}$ for $i \neq j$
   (c) there exists a diagonal matrix $\operatorname{diag}(d_i)_{i\in I}$ such that $(d_i a_{ij})_{i,j\in I}$ is symmetric and d_i are positive integers.
2. P : a free abelian group (the weight lattice),
3. $\Pi = \{\alpha_i \mid i \in I\} \subset P$: the set of simple roots such that $\Pi \subset P\otimes_{\mathbb{Z}}\mathbb{Q}$ is linearly independent,
4. $P^\vee = \operatorname{Hom}_{\mathbb{Z}}(P, \mathbb{Z})$: the dual lattice (the coweight lattice) of P with perfect paring $\langle \cdot, \cdot\rangle : P^\vee \otimes_{\mathbb{Z}} P \to \mathbb{Z}$,
5. $\Pi^\vee = \{h_i \mid i \in I\} \subset P^\vee$: the set of simple coroots, satisfying the following properties:
   (a) $a_{ij} = \langle h_i, \alpha_j\rangle$ for all $i, j \in I$,
   (b) There exists $\{\Lambda_i\}_{i\in I} \subset P$, called the set of fundamental weights, satisfying $\langle h_i, \Lambda_j\rangle = \delta_{ij}$ for $i, j \in I$.

We say $\Lambda \in P$ is *dominant* if $\langle h_i, \Lambda\rangle \geq 0$ for any $i \in I$ and denote by P_+ the set of dominant integral weights. Let $Q = \bigoplus_{i\in I} \mathbb{Z}\alpha_i \subset P$ be the root lattice. Let $Q_\pm = \pm\sum_{i\in I} \mathbb{Z}_{\geq 0}\alpha_i$. For $\xi = \sum_{i\in I} \xi_i\alpha_i \in Q$, we set $|\xi| = \sum_{i\in I} \xi_i$.

2.1.2. Let $(A, P, \Pi, P^\vee, \Pi^\vee)$ be a root datum. We set $\mathfrak{h} := P \otimes_{\mathbb{Z}} \mathbb{C}$. A triple $(\mathfrak{h}, \Pi, \Pi^\vee)$ is called a Cartan datum or a realization of a generalized Cartan matrix A.

It is known that there exists a symmetric bilinear form $(\cdot, \cdot)$ on $\mathfrak{h}^*$ satisfying

1. $(\alpha_i, \alpha_i) = d_i a_{ij}$,
2. $\langle h_i, \lambda\rangle = 2(\alpha_i, \lambda) / (\alpha_i, \alpha_i)$ for $i \in I$ and $\lambda \in \mathfrak{h}^*$.

Definition 2.2. Let $\mathfrak{g}$ be the *symmetrizable Kac-Moody Lie algebra* associated with a realization $(\mathfrak{h}, \Pi, \Pi^\vee)$ of a symmetrizable generalized Cartan matrix $A = (a_{ij})_{i,j\in I}$, that is a Lie algebra which is generated by $\{e_i\}_{i\in I} \cup \{f_i\}_{i\in I} \cup \mathfrak{h}$ with the following relations:

1. $[h_1, h_2] = 0$ for $h_1, h_2 \in \mathfrak{h}$,
2. $[h, e_i] = \langle h, \alpha_i\rangle e_i$, $[h, f_i] = -\langle h, \alpha_i\rangle f_i$ for $h \in \mathfrak{h}$ and $i \in I$,
3. $[e_i, f_j] = \delta_{ij}\alpha_i^\vee$ for $i, j \in I$,
4. $\operatorname{ad}(e_i)^{1-a_{ij}}(e_j) = 0$ and $\operatorname{ad}(f_i)^{1-a_{ij}}(f_j) = 0$ for $i \neq j$, where $\operatorname{ad}(x)(y) = [x, y]$.

Let $\mathfrak{n}_+$ (resp. $\mathfrak{n}_-$) be the Lie subalgebra which is generated by $\{e_i\}_{i\in I}$ (resp. $\{f_i\}_{i\in I}$). We have the triangular decomposition and the root space decomposition

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+ = \mathfrak{h} \oplus \bigoplus_{\alpha\in\mathfrak{h}^*\setminus\{0\}} \mathfrak{g}_\alpha$$

where $\mathfrak{g}_\alpha = \left\{x \in \mathfrak{g} \mid [h, x] = \langle h, \alpha\rangle x \ {}^\forall h \in \mathfrak{h}\right\}$. The set $\Delta := \{\alpha \in \mathfrak{h}^* \setminus \{0\} \mid \mathfrak{g}_\alpha \neq 0\}$ is called the root system of $\mathfrak{g}$.

2.1.3. We fix a root datum $(A, P, \Pi, P^\vee, \Pi^\vee)$. We introduce an indeterminate q. For $i \in I$, we set $q_i = q^{d_i}$. For $\xi = \sum \xi_i \alpha_i \in Q$, we set $q_\xi := \prod_{i \in I} q_i^{\xi_i}$.

For $n \in \mathbb{Z}$ and $i \in I$, we set.

$$[n]_i := \frac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}}$$

and $[n]_i! = [n]_i [n-1]_i \cdots [1]_i$ for $n > 0$ and $[0]! = 1$.

Definition 2.3. The *quantized enveloping algebra* $\mathbf{U}_q(\mathfrak{g})$ associated with a root datum $(A, P, \Pi, P^\vee, \Pi^\vee)$ is the $\mathbb{Q}(q)$-algebra which is generated by $\{e_i\}_{i \in I}$, $\{f_i\}_{i \in I}$ and $\left\{q^h \mid h \in P^\vee\right\}$ with the following relations:

(1) $q^0 = 1$ and $q^{h+h'} = q^h q^{h'}$ for $h, h' \in P^\vee$,
(2) $q^h e_i q^{-h} = q^{\langle h, \alpha_i \rangle} e_i$, $q^h f_i q^{-h} = q^{-\langle h, \alpha_i \rangle} f_i$ for $i \in I$ and $h \in P^\vee$,
(3) $e_i f_j - f_j e_i = \delta_{ij} \left(k_i - k_i^{-1}\right) / \left(q_i - q_i^{-1}\right)$ where $k_i = q^{d_i h_i}$,
(4) $\displaystyle\sum_{k=0}^{1-a_{ij}} (-1)^k e_i^{(1-a_{ij}-k)} e_j e_i^{(k)} = \sum_{k=0}^{1-a_{ij}} (-1)^k f_i^{(1-a_{ij}-k)} f_j f_i^{(k)} = 0$ (q-Serre relations),

where $e_i^{(k)} = e_i^k / [k]_i!$, $f_i^{(k)} = f_i^k / [k]_i!$ for $i \in I$ and $k \in \mathbb{Z}_{>0}$.

2.1.4. Let $\mathbf{U}_q^0$ be the subalgebra which is generated by $\left\{q^h \mid h \in P^\vee\right\}$, it is isomorphic to the group algebra $\mathbb{Q}(q)[P^\vee] := \bigoplus_{h \in P^\vee} \mathbb{Q}(q) q^h$ over $\mathbb{Q}(q)$. For $\xi = \sum_{i \in I} \xi_i \alpha_i \in Q$, we set $k_\xi := \prod_{i \in I} k_i^{\xi_i} = \prod_{i \in I} q^{d_i \xi_i h_i}$.

Let $\mathbf{U}_q^+$ be the $\mathbb{Q}(q)$-subalgebra generated by $\{e_i\}_{i \in I}$, $\mathbf{U}_q^-$ be the $\mathbb{Q}(q)$-subalgebra generated by $\{f_i\}_{i \in I}$, $\mathbf{U}_q^{\geq 0}$ be the $\mathbb{Q}(q)$-subalgebra generated by $\mathbf{U}_q^0$ and $\mathbf{U}_q^+$, and $\mathbf{U}_q^{\leq 0}$ be the $\mathbb{Q}(q)$-subalgebra generated by $\mathbf{U}_q^0$ and $\mathbf{U}_q^-$.

Theorem 2.4 ([9, Corollary 3.2.5])**.** *The multiplication of $\mathbf{U}_q$ induces the triangular decomposition of $\mathbf{U}_q(\mathfrak{g})$ as vector spaces over $\mathbb{Q}(q)$:*

$$\mathbf{U}_q(\mathfrak{g}) \cong \mathbf{U}_q^+ \otimes \mathbf{U}_q^0 \otimes \mathbf{U}_q^- \cong \mathbf{U}_q^- \otimes \mathbf{U}_q^0 \otimes \mathbf{U}_q^+. \tag{2.1}$$

2.1.5. For $\xi \in \pm Q$, we define its root space $\mathbf{U}_q^\pm(\mathfrak{g})_\xi$ by

$$\mathbf{U}_q^\pm(\mathfrak{g})_\xi := \left\{x \in \mathbf{U}_q^\pm(\mathfrak{g}) \middle| q^h x q^{-h} = q^{\langle h, \xi \rangle} x \text{ for } h \in P^\vee\right\}. \tag{2.2}$$

Then we have a root space decomposition $\mathbf{U}_q^\pm(\mathfrak{g}) = \bigoplus_{\xi \in Q_\pm} \mathbf{U}_q^\pm(\mathfrak{g})_\xi$.

An element $x \in \mathbf{U}_q^\pm(\mathfrak{g})$ is called homogenous if $x \in \mathbf{U}_q^\pm(\mathfrak{g})_\xi$ for some $\xi \in Q_\pm$.

2.1.6. We define a $\mathbb{Q}(q)$-algebra anti-involution $*\colon \mathbf{U}_q(\mathfrak{g}) \to \mathbf{U}_q(\mathfrak{g})$ by

$$*(e_i) = e_i, \qquad *(f_i) = f_i, \qquad *(q^h) = q^{-h}. \tag{2.3}$$

We call this *star involution.*

We define a $\mathbb{Q}$-algebra automorphism $^-\colon \mathbf{U}_q(\mathfrak{g}) \to \mathbf{U}_q(\mathfrak{g})$ by

$$\overline{e_i} = e_i, \qquad \overline{f_i} = f_i, \qquad \overline{q} = q^{-1}, \qquad \overline{q^h} = q^{-h}. \tag{2.4}$$

We call this the *bar involution.*

We remark that these two involutions preserve $\mathbf{U}_q^+(\mathfrak{g})$ and $\mathbf{U}_q^-(\mathfrak{g})$, and we have $^- \circ * = * \circ {}^-$.

2.1.7. In this article, we choose the following comultiplication $\Delta = \Delta_-$ on $\mathbf{U}_q(\mathfrak{g})$:

$$\Delta(q^h) = q^h \otimes q^h, \qquad \Delta(e_i) = e_i \otimes k_i^{-1} + 1 \otimes e_i, \qquad \Delta(f_i) = f_i \otimes 1 + k_i \otimes f_i. \tag{2.5}$$

2.1.8. We have a symmetric non-degenerate bilinear form $(\cdot,\cdot)=(\cdot,\cdot)_{-}:\mathbf{U}_q^-\otimes\mathbf{U}_q^-\to\mathbb{Q}(q)$. We define a $\mathbb{Q}(q)$-algebra structure on $\mathbf{U}_q^-\otimes\mathbf{U}_q^-$ by

$$(x_1\otimes y_1)(x_2\otimes y_2)=q^{-(\mathrm{wt}(x_2),\mathrm{wt}(y_1))}x_1x_2\otimes y_1y_2, \tag{2.6}$$

where x_i, y_i $(i=1,2)$ are homogenous elements.

Let $r=r_-:\mathbf{U}_q^-\to\mathbf{U}_q^-\otimes\mathbf{U}_q^-$ be the $\mathbb{Q}(q)$-algebra homomorphism defined by

$$r(f_i)=f_i\otimes 1+1\otimes f_i\ \ (i\in I).$$

We call this the twisted comultiplication.

Then it is known that there exists a unique $\mathbb{Q}(q)$-valued non-degenerate symmetric bilinear form $(\cdot,\cdot):\mathbf{U}_q^-\otimes\mathbf{U}_q^-\to\mathbb{Q}(q)$ with the following properties:

$$(1,1)=1,(f_i,f_j)=\delta_{ij},\qquad (r(x),y_1\otimes y_2)=(x,y_1y_2),(x_1\otimes x_2,r(y))=(x_1x_2,y)$$

for homogenous $x,y_1,y_2\in\mathbf{U}_q^-$ where the form $(\cdot\otimes\cdot,\cdot\otimes\cdot):\left(\mathbf{U}_q^-\otimes\mathbf{U}_q^-\right)\otimes\left(\mathbf{U}_q^-\otimes\mathbf{U}_q^-\right)\to\mathbb{Q}(q)$ is defined by $(x_1\otimes x_2,y_1\otimes y_2)=(x_1,y_1)(x_2\otimes y_2)$ for $x_1,x_2,y_1,y_2\in\mathbf{U}_q^-$.

2.1.9. For $i\in I$, we define the unique $\mathbb{Q}(q)$-linear map ${}_ir:\mathbf{U}_q^-\to\mathbf{U}_q^-$ (resp. $r_i:\mathbf{U}_q^-\to\mathbf{U}_q^-$) defined by

$$\begin{aligned}({}_ir(x),y)&=(x,f_iy),\\ (r_i(x),y)&=(x,yf_i).\end{aligned}$$

Lemma 2.5. *For $x,y\in\mathbf{U}_q^-$, we have q-Boson relations:*

$$\begin{aligned}{}_ir(xy)&={}_ir(x)\,y+q^{(\mathrm{wt}\,x,\alpha_i)}x\,{}_ir(y),\\ r_i(xy)&=q^{(\mathrm{wt}\,y,\alpha_i)}r_i(x)\,y+xr_i(y).\end{aligned}$$

Lemma 2.6. *We have*

$$[e_i,x]=\frac{r_i(x)\,k_i-k_i^{-1}\,{}_ir(x)}{q_i-q_i^{-1}} \tag{2.7}$$

for $x\in\mathbf{U}_q^-$.

Using the q-Boson relation, we obtain the following proposition.

Lemma 2.7. *For each $i\in I$, any element $x\in\mathbf{U}_q^-$ can be written uniquely as*

$$x=\sum_{c\geq 0}f_i^{(c)}x_c\ \textit{with}\ x_c\in\mathrm{Ker}\,({}_ir).$$

2.2. **Canonical basis and dual canonical basis.** In this subsection, we give a brief review of the theory of the canonical basis and the dual canonical basis following Kashiwara. Note that Kashiwara called it the lower global basis and the upper global basis.

2.2.1. We define $\mathbb{Q}$-subalgebras $\mathcal{A}_0$, $\mathcal{A}_\infty$ and $\mathcal{A}$ of $\mathbb{Q}(q)$ by

$$\begin{aligned}\mathcal{A}_0&:=\{f\in\mathbb{Q}(q)\,;f\text{ is regular at }q=0\},\\ \mathcal{A}_\infty&:=\{f\in\mathbb{Q}(q)\,;f\text{ is regular at }q=\infty\},\\ \mathcal{A}&:=\mathbb{Q}[q^{\pm1}].\end{aligned}$$

2.2.2. We define the Kashiwara operator $\tilde{e}_i$, $\tilde{f}_i$ on $\mathbf{U}_q^-$ by

$$\tilde{e}_i x = \sum_{c \geq 1} f_i^{(c-1)} x_c,$$
$$\tilde{f}_i x = \sum_{c \geq 0} f_i^{(c+1)} x_c,$$

and we set

$$\mathscr{L}(\infty) := \sum_{\substack{\ell \geq 0 \\ i_1, \cdots, i_\ell \in i}} \mathcal{A}_0 \tilde{f}_{i_1} \cdots \tilde{f}_{i_\ell} 1 \subset \mathbf{U}_q^-,$$
$$\mathscr{B}(\infty) := \left\{ \tilde{f}_{i_1} \cdots \tilde{f}_{i_\ell} 1 \bmod q\mathscr{L}(\infty) \middle| l \geq 0, i_1, \cdots, i_\ell \in I \right\} \subset \mathscr{L}(\infty) / q\mathscr{L}(\infty).$$

Then $\mathscr{L}(\infty)$ is a $\mathcal{A}_0$-lattice with $\mathbb{Q}(q) \otimes_{\mathcal{A}_0} \mathscr{L}(\infty) \simeq \mathbf{U}_q^-$ and stable under $\tilde{e}_i$ and $\tilde{f}_i$. $\mathscr{B}(\infty)$ is a $\mathbb{Q}$-basis of $\mathscr{L}(\infty)/q\mathscr{L}(\infty)$. We also have induced maps $\tilde{f}_i \colon \mathscr{B}(\infty) \to \mathscr{B}(\infty)$ and $\tilde{e}_i \colon \mathscr{B}(\infty) \to \mathscr{B}(\infty) \sqcup \{0\}$ with the property that $\tilde{f}_i \tilde{e}_i b = b$ for $b \in \mathscr{B}(\infty)$ with $\tilde{e}_i b \neq 0$. We call $(\mathscr{B}(\infty), \mathscr{L}(\infty))$ the (lower) crystal basis of $\mathbf{U}_q^-$ and call $\mathscr{L}(\infty)$ the (lower) crystal lattice. We denote $1 \bmod q\mathscr{L}(\infty)$ by u_∞.

2.2.3. It is also known that $* \colon \mathbf{U}_q^- \to \mathbf{U}_q^-$ induces $* \colon \mathscr{L}(\infty) \to \mathscr{L}(\infty)$ and $* \colon \mathscr{B}(\infty) \to \mathscr{B}(\infty)$. We setand $\tilde{f}_i^* := * \circ \tilde{f}_i \circ * \colon \mathscr{B}(\infty) \to \mathscr{B}(\infty)$ and $\tilde{e}_i^* := * \circ \tilde{e}_i \circ * \colon \mathscr{B}(\infty) \to \mathscr{B}(\infty) \sqcup \{0\}$

2.2.4. Let $\overline{\mathscr{L}(\infty)} = \{\overline{x} \mid x \in \mathscr{L}(\infty)\}$. Then the natural map

$$\mathscr{L}(\infty) \cap \overline{\mathscr{L}(\infty)} \cap \mathbf{U}_q^-(\mathfrak{g})_{\mathcal{A}} \to \mathscr{L}(\infty)/q\mathscr{L}(\infty)$$

is an isomorphism of $\mathbb{Q}$-vector spaces, let G^{low} be the inverse of this isomorphism. The image

$$\mathbf{B}^{\mathrm{low}} = \left\{ G^{\mathrm{low}}(b) \mid b \in \mathscr{B}(\infty) \right\} \subset \mathscr{L}(\infty) \cap \overline{\mathscr{L}(\infty)} \cap \mathbf{U}_q^-(\mathfrak{g})_{\mathcal{A}}$$

is an $\mathcal{A}$-basis of $\mathbf{U}_q^-(\mathfrak{g})_{\mathcal{A}}$ and is called the canonical basis or the lower global basis of $\mathbf{U}_q^-$.

2.2.5. The important property of the canonical basis is the following compatibility with the left and right ideal which are generated by Chevalley generators $\{f_i\}_{i \in I}$.

Theorem 2.8 ([9, Theorem 14.3.2, Theorem 14.4.3],[4, Theorem 7])**.** *For $i \in I$ and $n \geq 1$, $f_i^n \mathbf{U}_q^-$ and $\mathbf{U}_q^- f_i^n$ is compatible with the canonical base, that is $f_i^n \mathbf{U}_q^- \cap \mathbf{B}^{\mathrm{low}}$ (resp. $\mathbf{U}_q^- f_i^n \cap \mathbf{B}^{\mathrm{low}}$) is a basis of $f_i^n \mathbf{U}_q^-$ (resp. $\mathbf{U}_q^- f_i^n$). In fact, we have*

$$f_i^n \mathbf{U}_q^- \cap \mathbf{U}_q^-(\mathfrak{g})_{\mathcal{A}} = \bigoplus_{\substack{b \in \mathscr{B}(\infty) \\ \varepsilon_i(b) \geq m}} \mathcal{A} G^{\mathrm{low}}(b),$$
$$\mathbf{U}_q^- f_i^n \cap \mathbf{U}_q^-(\mathfrak{g})_{\mathcal{A}} = \bigoplus_{\substack{b \in \mathscr{B}(\infty) \\ \varepsilon_i^*(b) \geq m}} \mathcal{A} G^{\mathrm{low}}(b)$$

2.2.6. Let $\sigma \colon \mathbf{U}_q^- \to \mathbf{U}_q^-$ be the $\mathbb{Q}$-linear map defined by

$$(\sigma(x), y) = \overline{(x, \overline{y})}$$

for arbitrary $x, y \in \mathbf{U}_q^-$. Let $\sigma(\mathscr{L}(\infty)) := \{\sigma(x) \mid x \in \mathscr{L}(\infty)\}$ and set the dual integral form:

$$\mathbf{U}_q^-(\mathfrak{g})_{\mathcal{A}}^{\mathrm{up}} := \left\{ x \in \mathbf{U}_q^- \mid \left(x, \mathbf{U}_q^-(\mathfrak{g})_{\mathcal{A}}\right) \subset \mathcal{A} \right\}.$$

$\mathbf{U}_q^-(\mathfrak{g})_{\mathcal{A}}^{\mathrm{up}}$ has an $\mathcal{A}$-subalgebra of $\mathbf{U}_q^-$. The natural map

$$\mathscr{L}(\infty) \cap \sigma(\mathscr{L}(\infty)) \cap \mathbf{U}_q^-(\mathfrak{g})_{\mathcal{A}}^{\mathrm{up}} \to \mathscr{L}(\infty)/q\mathscr{L}(\infty)$$

is also an isomorphism of $\mathbb{Q}$-vector spaces, so let G^{up} be the inverse of the above isomorphism.

$$\mathbf{B}^{\mathrm{up}} = \{G^{\mathrm{up}}(b) \mid b \in \mathscr{B}(\infty)\} \subset \mathscr{L}(\infty) \cap \sigma(\mathscr{L}(\infty)) \cap \mathbf{U}_q^-(\mathfrak{g})_{\mathcal{A}}^{\mathrm{up}}$$

is an $\mathcal{A}$-basis of $\mathbf{U}_q^-(\mathfrak{g})_{\mathcal{A}}^{\mathrm{up}}$ and is called the dual canonical basis or the upper global basis of $\mathbf{U}_q^-$.

Remark 2.9. We note that this definition of the dual canonical basis $\mathbf{B}^{\mathrm{up}}$ does depend on a choice of a non-degenerate bilinear form on $\mathbf{U}_q^-(\mathfrak{g})$.

Proposition 2.10 ([7, Proposition 4.26 (1)]). *For $i \in I$ and $c \geq 1$, let*

$$f_i^{\{c\}} = f_i^{(c)} / \left(f_i^{(c)}, f_i^{(c)}\right),$$

then we have $f_i^{\{c\}} = q_i^{c(c-1)/2}\left(f_i/\left(f_i, f_i\right)\right)^c = q_i^{c(c-1)/2} f_i^c \in \mathbf{B}^{\mathrm{up}}$.

We note that we have used the normalization $(f_i, f_i) = 1$ in the above proposition.

2.2.7. For the dual canonical basis, we have the following expansion of left and right multiplication with respect to the Chevalley generators and its (shifted) powers.

Theorem 2.11 ([5, Proposition 2.2], [11, Proposition 4.14 (ii)]). *For $b \in \mathscr{B}(\infty)$, $i \in I$ and $c \geq 1$, we have*

$$f_i^{\{c\}} G^{\mathrm{up}}(b) = q_i^{-c\varepsilon_i(b)} G^{\mathrm{up}}\left(\tilde{f}_i^c b\right) + \sum_{\varepsilon_i\left(b'\right) < \varepsilon_i(b)+c} F_{i;b,b'}^{\{c\}}(q)\, G^{\mathrm{up}}\left(b'\right), \tag{2.8a}$$

$$G^{\mathrm{up}}(b)\, f_i^{\{c\}} = q_i^{-c\varepsilon_i^*(b)} G^{\mathrm{up}}\left(\tilde{f}_i^{*c} b\right) + \sum_{\varepsilon_i^*\left(b'\right) < \varepsilon_i^*(b)+c} F_{i;b,b'}^{*\{c\}}(q)\, G^{\mathrm{up}}\left(b'\right), \tag{2.8b}$$

where

$$F_{i;b,b'}^{\{c\}}(q) := \left(f_i^{\{c\}} G^{\mathrm{up}}(b), G^{\mathrm{low}}\left(b'\right)\right) = q_i^{c(c-1)/2}\left(G^{\mathrm{up}}(b), \left({}_i r\right)^c G^{\mathrm{low}}\left(b'\right)\right) \in q_i^{-c\varepsilon_i(b)} q\mathbb{Z}[q],$$
$$F_{i;b,b'}^{*\{c\}}(q) := \left(G^{\mathrm{up}}(b)\, f_i^{\{c\}}, G^{\mathrm{low}}\left(b'\right)\right) = q_i^{c(c-1)/2}\left(G^{\mathrm{up}}(b), \left(r_i\right)^c G^{\mathrm{low}}\left(b'\right)\right) \in q_i^{-c\varepsilon_i^*(b)} q\mathbb{Z}[q].$$

2.3. **Braid group action and the (dual) canonical basis.** In this subsection, we recall the compatibility between Lusztig's braid symmetry and the (dual) canonical basis (for more details, see [7, Section 4.4, Section 4,6])

2.3.1. *Braid group action on quantized enveloping algebra.* Following Lusztig [9, Section 37.1.3], we define the $\mathbb{Q}(q)$-algebra automorphisms $T'_{i,\epsilon} \colon \mathbf{U}_q(\mathfrak{g}) \to \mathbf{U}_q(\mathfrak{g})$ and $T''_{i,\epsilon} \colon \mathbf{U}_q(\mathfrak{g}) \to \mathbf{U}_q(\mathfrak{g})$ for $i \in I$ and $\epsilon \in \{\pm 1\}$ by the following formulae:

$$T'_{i,\epsilon}\left(q^h\right) = q^{s_i(h)}, \tag{2.9a}$$

$$T'_{i,\epsilon}\left(e_j\right) = \begin{cases} -k_i^{\epsilon} e_i & \text{for } j = i, \\ \displaystyle\sum_{r+s=-\langle h_i, \alpha_j\rangle} (-1)^r q_i^{\epsilon r} e_i^{(r)} e_j e_i^{(r)} & \text{for } j \neq i, \end{cases} \tag{2.9b}$$

$$T'_{i,\epsilon}\left(f_j\right) = \begin{cases} -e_i k_i^{-\epsilon} & \text{for } j = i, \\ \displaystyle\sum_{r+s=-\langle h_i, \alpha_j\rangle} (-1)^r q_i^{-\epsilon r} f_i^{(s)} f_j f_i^{(r)} & \text{for } j \neq i, \end{cases} \tag{2.9c}$$

and

$$T''_{i,-\epsilon}\left(q^h\right) = q^{s_i(h)}, \tag{2.10a}$$

$$T''_{i,-\epsilon}\left(e_j\right) = \begin{cases} -f_i k_i^{-\epsilon} & \text{for } j = i, \\ \displaystyle\sum_{r+s=-\langle h_i, \alpha_j\rangle} (-1)^r q_i^{\epsilon r} e_i^{(s)} e_j e_i^{(r)} & \text{for } j \neq i, \end{cases} \tag{2.10b}$$

$$T''_{i,-\epsilon}\left(f_j\right) = \begin{cases} -k_i^{\epsilon} e_i & \text{for } j = i, \\ \displaystyle\sum_{r+s=-\langle h_i, \alpha_j\rangle} (-1)^r q_i^{-\epsilon r} f_i^{(r)} f_j f_i^{(s)} & \text{for } j \neq i. \end{cases} \tag{2.10c}$$

It is known that $\left\{T'_{i,\epsilon}\right\}_{i\in I}$ and $\left\{T''_{i,\epsilon}\right\}_{i\in I}$ satisfy the braid relation.

Lemma 2.12 ([9, Proposition 37.1.2 (d), Section 37.2.4])**.** (1) *We have* $T'_{i,\epsilon} \circ T''_{i,-\epsilon} = T''_{i,-\epsilon} \circ T'_{i,\epsilon} = \mathrm{id}$.

(2) *We have* $* \circ T'_{i,\epsilon} \circ * = T''_{i,-\epsilon}$ *for* $i \in I$ *and* $\epsilon \in \{\pm 1\}$.

In the following, we write $T_i = T''_{i,1}$ and $T_i^{-1} = T'_{i,-1}$ as in [12, Proposition 1.3.1].

2.3.2. We have the following orthogonal decomposition with respect to the bilinear form $(\cdot,\cdot) : \mathbf{U}_q^- \otimes \mathbf{U}_q^- \to \mathbb{Q}(q)$ and the compatibility with the canonical basis.

Proposition 2.13 ([9, Proposition 38.1.6, Lemma 38.1.5])**.** (1) *For* $i \in I$*, we have*

$$\mathbf{U}_q^- \cap T_i \mathbf{U}_q^- = \left\{ x \in \mathbf{U}_q^- \mid {}_i r(x) = 0 \right\},$$
$$\mathbf{U}_q^- \cap T_i^{-1} \mathbf{U}_q^- = \left\{ x \in \mathbf{U}_q^- \mid r_i(x) = 0 \right\}.$$

(2) *For* $i \in I$*, we have the following orthogonal decomposition with respect to* $(\cdot,\cdot)_-$*:*

$$\mathbf{U}_q^- = \left(\mathbf{U}_q^- \cap T_i \mathbf{U}_q^- \right) \oplus f_i \mathbf{U}_q^- = \left(\mathbf{U}_q^- \cap T_i^{-1} \mathbf{U}_q^- \right) \oplus \mathbf{U}_q^- f_i.$$

Corollary 2.14. *For* $i \in I$*, the subalgebra* $\mathbf{U}_q^- \cap T_i \mathbf{U}_q^-$ *(resp.* $\mathbf{U}_q^- \cap T_i \mathbf{U}_q^-$*) is compatible with the dual canonical base, that is we have*

$$\mathbf{U}_q^- \cap T_i \mathbf{U}_q^- \cap \mathbf{U}_q^-(\mathfrak{g})_{\mathcal{A}}^{\mathrm{up}} = \bigoplus_{\substack{b \in \mathscr{B}(\infty) \\ \varepsilon_i(b) = 0}} \mathcal{A} G^{\mathrm{up}}(b),$$
$$\mathbf{U}_q^- \cap T_i^{-1} \mathbf{U}_q^- \cap \mathbf{U}_q^-(\mathfrak{g})_{\mathcal{A}}^{\mathrm{up}} = \bigoplus_{\substack{b \in \mathscr{B}(\infty) \\ \varepsilon_i^*(b) = 0}} \mathcal{A} G^{\mathrm{up}}(b).$$

2.3.3. The following result is due to Saito [12].

Proposition 2.15 ([12, Proposition 3.4.7, Corollary 3.4.8])**.** (1) *Let* $x \in \mathbf{U}_q^- \in \mathscr{L}(\infty) \cap T_i^{-1} \mathbf{U}_q^-$ *with* $b := x \bmod q\mathscr{L}(\infty) \in \mathscr{B}(\infty)$*, we have*

$$T_i(x) \in \mathscr{L}(\infty) \cap T_i \mathbf{U}_q^-,$$
$$T_i(x) \equiv \tilde{f}_i^{*\varphi_i(b)} \tilde{e}_i^{\varepsilon_i(b)} b \ \mathit{mod}\ q\mathscr{L}(\infty) \in \mathscr{B}(\infty).$$

(2) *Let* $\sigma_i \colon \{ b \in \mathscr{B}(\infty) \mid \varepsilon_i^*(b) = 0 \} \to \{ b \in \mathscr{B}(\infty) \mid \varepsilon_i(b) = 0 \}$ *be the map defined by* $\sigma_i(b) = \tilde{f}_i^{*\varphi_i(b)} \tilde{e}_i^{\varepsilon_i(b)} b$*. Then* σ_i *is bijective and its inverse is given by* $\sigma_i^*(b) = (* \circ \sigma_i \circ *)(b) = \tilde{f}_i^{\varphi_i^*(b)} \tilde{e}_i^{*\varepsilon_i^*(b)} b$*.*

The bijections σ_i and σ_i^* is called Saito crystal reflections. In [12, Corollary 3.4.8], σ_i and σ_i^* are denoted by Λ_i and Λ_i^{-1}.

Following Baumann-Kamnitzer-Tingley [1, Section 5.5], for convenience, we extend σ_i and σ_i^* to $\mathscr{B}(\infty)$ by setting

$$\hat{\sigma}_i(b) := \sigma_i\left(\tilde{e}_i^{*\max}(b)\right),$$
$$\hat{\sigma}_i^*(b) := \sigma_i^*\left(\tilde{e}_i^{\max}(b)\right),$$

so we can consider as maps from $\mathscr{B}(\infty)$ to itself.

2.3.4. Let ${}^i\pi \colon \mathbf{U}_q^- \to \mathbf{U}_q^- \cap T_i \mathbf{U}_q^-$ (resp. $\pi^i \colon \mathbf{U}_q^- \to \mathbf{U}_q^- \cap T_i^{-1} \mathbf{U}_q^-$) be the orthogonal projection whose kernel is $f_i \mathbf{U}_q^-$ (resp. $\mathbf{U}_q^- f_i$). We have the following relations among the braid group action and the (dual) canonical basis.

Theorem 2.16 ([10, Theorem 1.2], [7, Theorem 4.23])**.** (1) *For* $b \in \mathscr{B}(\infty)$ *with* $\varepsilon_i^*(b) = 0$*, we have*

$$T_i\left(\pi^i G^{\mathrm{low}}(b)\right) = {}^i\pi\left(G^{\mathrm{low}}(\sigma_i(b))\right),$$
$$\left(1 - q_i^2\right)^{\langle h_i, \mathrm{wt}\, b \rangle} T_i G^{\mathrm{up}}(b) = G^{\mathrm{up}}(\sigma_i b).$$

(2) *For $b \in \mathscr{B}(\infty)$ with $\varepsilon_i(b) = 0$, we have*

$$T_i^{-1}\left({}^i\pi G^{\mathrm{low}}(b)\right) = \pi^i\left(G^{\mathrm{low}}\left(\sigma_i^*(b)\right)\right),$$
$$\left(1-q_i^2\right)^{\langle h_i, \mathrm{wt}\, b\rangle} T_i^{-1} G^{\mathrm{up}}(b) = G^{\mathrm{up}}\left(\sigma_i^* b\right).$$

We note that the constant term $\left(1-q_i^2\right)^{\langle h_i, \mathrm{wt}\, b\rangle}$ depends on a choice of the non-degenerate bilinear form on $\mathbf{U}_q^-(\mathfrak{g})$.

2.4. **Poincaré-Birkhoff-Witt bases.** Let $W = \langle s_i \mid i \in I\rangle$ be the Weyl group of $\mathfrak{g}$ where $\{s_i \mid i \in I\}$ is the set of simple reflections associated with $i \in I$ and $\ell\colon W \to \mathbb{Z}_{\geq 0}$ be the length function. For a Weyl group element w, let

$$I(w) := \left\{\left(i_1, i_2, \cdots, i_{\ell(w)}\right) \in I^{\ell(w)} \mid s_{i_1} \cdots s_{i_{\ell(w)}} = w\right\}$$

be the set of reduced words of w.

2.4.1. Let $\Delta = \Delta_+ \sqcup \Delta_-$ be the root system of the Kac-Moody Lie algebra $\mathfrak{g}$ and decomposition into positive and negative roots.

For a Weyl group element $w \in W$, we set

$$\Delta_+(\leq w) := \Delta_+ \cap w\Delta_- = \left\{\beta \in \Delta_+ \mid w^{-1}\beta \in \Delta_-\right\},$$
$$\Delta_+(> w) := \Delta_+ \cap w\Delta_+ = \left\{\beta \in \Delta_+ \mid w^{-1}\beta \in \Delta_+\right\}.$$

It is well-known that $\Delta_+(\leq w)$ and $\Delta_+(> w)$ are bracket closed, that is, for $\alpha, \beta \in \Delta_+(\leq w)$ (resp. $\alpha, \beta \in \Delta_+(> w)$) with $\alpha + \beta \in \Delta_+$, we have $\alpha + \beta \in \Delta_+(w)$ (resp. $\in \Delta_+(> w)$).

For a reduced word $\boldsymbol{i} = (i_1, i_2, \cdots, i_\ell) \in I(w)$, we define positive roots $\beta_{\boldsymbol{i},k}$ $(1 \leq k \leq \ell)$ by the following formula:

$$\beta_{\boldsymbol{i},k} = s_{i_1} \cdots s_{i_{k-1}}\left(\alpha_{i_k}\right) \ (1 \leq k \leq \ell).$$

It is well known that $\Delta_+(\leq w) = \{\beta_{\boldsymbol{i},k}\}_{1 \leq k \leq \ell}$ and we put a total order on $\Delta_+(w)$. We note that the total order on $\Delta_+(\leq w)$ does depends on a choice of a reduced word $\boldsymbol{i} \in I(w)$.

2.4.2. For a Weyl group element $w \in W$, a reduced word $\boldsymbol{i} = (i_1, i_2, \cdots, i_\ell) \in I(w)$, we define the root vector $f_\epsilon(\beta_{\boldsymbol{i},k})$ associated with $\beta_{\boldsymbol{i},k} \in \Delta_+(w)$ and a sign $\epsilon \in \{\pm 1\}$ by

$$f_\epsilon(\beta_{\boldsymbol{i},k}) := T_{i_1}^\epsilon T_{i_2}^\epsilon \cdots T_{i_{k-1}}^\epsilon\left(f_{i_k}\right),$$

and its divided power

$$f_\epsilon(\beta_{\boldsymbol{i},k})^{(c)} := T_{i_1}^\epsilon T_{i_2}^\epsilon \cdots T_{i_{k-1}}^\epsilon\left(f_{i_k}^{(c)}\right)$$

for $c \in \mathbb{Z}_{\geq 0}$.

Theorem 2.17 ([9, Proposition 40.2.1, 41.1.3])**.** *For $w \in W$, $\boldsymbol{w} = (i_1, \cdots, i_\ell) \in I(w)$, $\epsilon \in \{\pm 1\}$ and $\boldsymbol{c} \in \mathbb{Z}_{\geq 0}^\ell$, we set*

$$f_\epsilon(\boldsymbol{c}, \boldsymbol{i}) := \begin{cases} f_\epsilon(\beta_{\boldsymbol{i},1})^{(c_1)} f_\epsilon(\beta_{\boldsymbol{i},2})^{(c_2)} \cdots f_\epsilon(\beta_{\boldsymbol{i},\ell})^{(c_\ell)} & \text{if } \epsilon = +1, \\ f_\epsilon(\beta_{\boldsymbol{i},\ell})^{(c_\ell)} f_\epsilon(\beta_{\boldsymbol{i},\ell-1})^{(c_{\ell-1})} \cdots f_\epsilon(\beta_{\boldsymbol{i},1})^{(c_1)} & \text{if } \epsilon = -1. \end{cases}$$

Then $\{f_\epsilon(\boldsymbol{c}, \boldsymbol{i})\}_{\boldsymbol{c} \in \mathbb{Z}_{\geq 0}^\ell}$ forms a basis of a subspace defined to be $\mathbf{U}_q^-(\leq w, \epsilon)$ of $\mathbf{U}_q^-(\mathfrak{g})$ which does not depend on a choice of a reduced word $\boldsymbol{i} \in I(w)$. $\{f_\epsilon(\boldsymbol{c}, \boldsymbol{i})\}_{\boldsymbol{c} \in \mathbb{Z}_{\geq 0}^\ell}$ is called the Poincaré-Birkhoff-Witt basis or the lower Poincaré-Birkhoff-Witt basis.

Definition 2.18. For a Weyl group element $w \in W$, a reduced word $i = (i_1, \cdots, i_\ell) \in I(w)$ and $c = (c_1, \cdots, c_\ell) \in \mathbb{Z}_{\geq 0}^\ell$, we set

$$\xi(\boldsymbol{c}, \boldsymbol{i}) := -\sum_{1 \leq k \leq \ell} c_k \beta_{k,\boldsymbol{i}} \in Q_-.$$

We also have the following characterization of $\mathbf{U}_q^-(\leq w, \epsilon)$.

Theorem 2.19 ([2, Proposition 2.3])**.** *For $w \in W$ and $\epsilon \in \{\pm 1\}$, we have*

$$\mathbf{U}_q^-(\leq w, \epsilon) = \mathbf{U}_q^- \cap T_{w^\epsilon}^{\epsilon} \mathbf{U}_q^{\geq 0}.$$

In particular, $\mathbf{U}_q^-(\leq w, \epsilon)$ is a $\mathbb{Q}(q)$-subalgebra of $\mathbf{U}_q^-$.

In fact, it can be shown that $\mathbf{U}_q^-(\leq w, \epsilon)$ is a $\mathbb{Q}(q)$-subalgebra of $\mathbf{U}_q^-$ which is generated by $\{f_\epsilon(\beta_{\boldsymbol{i},k})\}_{1 \leq k \leq l}$ can be shown by the Levendorskii-Soibelman formula. For more details, see [7, Section 4.3].

2.4.3. *Poincaré-Birkhoff-Witt basis and crystal basis.*

Theorem 2.20. *For $w \in W$, $\boldsymbol{i} \in (i_1, \cdots, i_\ell) \in I(w)$ and $\epsilon \in \{\pm 1\}$,*

(1) *we have $f_\epsilon(\boldsymbol{c}, \boldsymbol{i}) \in \mathscr{L}(\infty)$ and*

$$b_\epsilon(\boldsymbol{c}, \boldsymbol{i}) := f_\epsilon(\boldsymbol{c}, \boldsymbol{i}) \mod q\mathscr{L}(\infty) \in \mathscr{B}(\infty).$$

(2) *The map $\mathbb{Z}_{\geq 0}^{\ell} \to \mathscr{B}(\infty)$ which is defined by $\boldsymbol{c} \mapsto b_\epsilon(\boldsymbol{c}, \boldsymbol{i})$ is injective. We denote the image by $\mathscr{B}(w, \epsilon)$, and this does not depend on a choice of a reduced word $\boldsymbol{i} \in I(w)$.*

2.4.4.

Proposition 2.21 ([7, Proposition 4.26 (2)])**.** *For $c \geq 1$ and $1 \leq k \leq \ell$, let*

$$f_\epsilon^{\mathrm{up}}(\beta_{\boldsymbol{i},k})^{\{c\}} = f_\epsilon(\beta_{\boldsymbol{i},k})^{(c)} / \left(f_\epsilon(\beta_{\boldsymbol{i},k})^{(c)}, f_\epsilon(\beta_{\boldsymbol{i},k})^{(c)}\right),$$

then we have $f_\epsilon^{\mathrm{up}}(\beta_{\boldsymbol{i},k})^{\{c\}} = q_{i_k}^{c(c-1)/2} f_\epsilon^{\mathrm{up}}(\beta_{\boldsymbol{i},k})^{c} \in \mathbf{B}^{\mathrm{up}}$.

Definition 2.22 (dual Poincare-Birkhoff-Witt basis)**.** For $w \in W$, $\boldsymbol{i} \in I(w)$ and $c \in \mathbb{Z}_{\geq 0}^{\ell}$, we set

$$f_\epsilon^{\mathrm{up}}(\boldsymbol{c}, \boldsymbol{i}) := \frac{f_\epsilon(\boldsymbol{c}, \boldsymbol{i})}{(f_\epsilon(\boldsymbol{c}, \boldsymbol{i}), f_\epsilon(\boldsymbol{c}, \boldsymbol{i}))}$$

and $\{f_\epsilon^{\mathrm{up}}(\boldsymbol{c}, \boldsymbol{i})\}_{\boldsymbol{c} \in \mathbb{Z}_{\geq 0}^{\ell}}$ is called the dual Poincaré-Birkhoff-Witt basis or upper Poincaré-Birkhoff-Witt basis.

By the definition of the dual Poincare-Birkhoff-Witt basis and the computation of $(f_\epsilon(\boldsymbol{c}, \boldsymbol{i}), f_\epsilon(\boldsymbol{c}, \boldsymbol{i}))$, we have

$$\begin{aligned}
f_\epsilon^{\mathrm{up}}(\boldsymbol{c}, \boldsymbol{i}) &= \begin{cases} f_\epsilon^{\mathrm{up}}(\beta_{\boldsymbol{i},1})^{[c_1]} f_\epsilon^{\mathrm{up}}(\beta_{\boldsymbol{i},2})^{[c_2]} \cdots f_\epsilon^{\mathrm{up}}(\beta_{\boldsymbol{i},\ell})^{[c_\ell]} & \text{if } \epsilon = +1, \\ f_\epsilon^{\mathrm{up}}(\beta_{\boldsymbol{i},\ell})^{[c_\ell]} f_\epsilon^{\mathrm{up}}(\beta_{\boldsymbol{i},\ell-1})^{[c_{\ell-1}]} \cdots f_\epsilon^{\mathrm{up}}(\beta_{\boldsymbol{i},1})^{[c_1]} & \text{if } \epsilon = -1. \end{cases} \\
&= \begin{cases} \left(1 - q_{i_1}^2\right)^{\langle h_{i_1}, \xi(c_{\geq 2}, i_{\geq 2})\rangle} f_\epsilon^{\mathrm{up}}(\beta_{\boldsymbol{i},1})^{[c_1]} T_{i_1}^{\epsilon}\left(f_\epsilon^{\mathrm{up}}\left(\beta_{\boldsymbol{i}_{\geq 2},2}\right)^{[c_2]} \cdots f_\epsilon^{\mathrm{up}}\left(\beta_{\boldsymbol{i}_{\geq 2},\ell}\right)^{[c_\ell]}\right) & \text{if } \epsilon = +1, \\ \left(1 - q_{i_1}^2\right)^{\langle h_{i_1}, \xi(c_{\geq 2}, i_{\geq 2})\rangle} T_{i_1}^{\epsilon}\left(f_\epsilon^{\mathrm{up}}\left(\beta_{\boldsymbol{i}_{\geq 2},2}\right)^{[c_2]} \cdots f_\epsilon^{\mathrm{up}}\left(\beta_{\boldsymbol{i}_{\geq 2},\ell}\right)^{[c_\ell]}\right) f_\epsilon^{\mathrm{up}}(\beta_{\boldsymbol{i},1})^{[c_1]} & \text{if } \epsilon = -1. \end{cases}
\end{aligned}$$

where $\boldsymbol{c}_{\geq 2} = (c_2, \cdots, c_\ell) \in \mathbb{Z}_{\geq 0}^{\ell-1}$, $w_{\geq 2} = s_{i_2} \cdots s_{i_\ell}$ and $i_{\geq 2} = (i_2, \cdots, i_\ell) \in I(w_{\geq 2})$.

Using the Levendorskii-Soibelman formula (see [7, Theorem 4.27]) and the definition of the dual canonical basis, we have the following result.

Theorem 2.23 ([7, Theorem 4.25, Theorem 4.29])**.** *Let $w \in W$ and $\boldsymbol{i} \in I(w)$, the Poincaré-Birkhoff-Witt basis satisfying the following properties*

(1) *The subalgebra $\mathbf{U}_q^-(\leq w, \epsilon)$ is compatible with the dual canonical basis, that is there exists a subset $\mathscr{B}(\leq w, \epsilon) := \mathscr{B}\left(\mathbf{U}_q^-(\leq w, \epsilon)\right) \subset \mathscr{B}(\infty)$ such that*

$$\mathbf{U}_q^-(\leq w, \epsilon) = \bigoplus_{b \in \mathscr{B}(\leq w, \epsilon)} \mathbb{Q}(q) G^{\mathrm{up}}(b).$$

(2) *The transition matrix between the dual Poincaré-Birkhoff-Witt basis and the dual canonical basis is triangular with 1's on the diagonal with respect to the (left) lexicographic order $\leq$ on $\mathbb{Z}_{\geq 0}^{\ell}$. More precisely, we have*

$$f_\epsilon^{\mathrm{up}}(\boldsymbol{c}, \boldsymbol{i}) = G^{\mathrm{up}}(b_\epsilon(\boldsymbol{c}, \boldsymbol{i})) + \sum_{\boldsymbol{c}' < \boldsymbol{c}} d_{\boldsymbol{c},\boldsymbol{c}'}^{\boldsymbol{i}}(q) G^{\mathrm{up}}\left(b_\epsilon\left(\boldsymbol{c}', \boldsymbol{i}\right)\right)$$

with $d^{\boldsymbol{i}}_{\boldsymbol{c},\boldsymbol{c}'}(q) := \left(f^{\mathrm{up}}_{\epsilon}(\boldsymbol{c},\boldsymbol{i}), G^{\mathrm{low}}\left(b_{\epsilon}\left(\boldsymbol{c}',\boldsymbol{i}\right)\right)\right) \in q\mathbb{Z}[q]$.

Remark 2.24. In symmetric case, we note that it can be shown that

$$d^{\boldsymbol{i}}_{\boldsymbol{c},\boldsymbol{c}'}(q) = \left(f^{\mathrm{up}}_{\epsilon}(\boldsymbol{c},\boldsymbol{i}), G^{\mathrm{low}}\left(b_{\epsilon}\left(\boldsymbol{c}',\boldsymbol{i}\right)\right)\right) \in q\mathbb{Z}_{\geq 0}[q],$$

by the positivity of the (twisted) comultiplication with respect to the canonical basis and Proposition 2.21.

In particular we obtain a proof of the positivity of the transition matrix from the canonical basis into the lower Poincaré-Birkhoff-Witt basis in simply-laced type for arbitrary reduced word of the longest element w_0 using the orthogonality of the the (lower) Poincaré-Birkhoff-Witt basis.

For "adapted" reduced words, it was proved by Lusztig [8, Corollary 10.7]. For an arbitrary reduced word, it is proved by Kato [6, Theorm 4.17] using the categorification of Poincaré-Birkhoff-Witt basis via Khovanov-Lauda-Rouquier algebra. It is also proved by Oya [11, Theorem 5.2].

3. Proof of the surjectivity

3.1. **Multiplication formula for $\mathbf{U}_q^-(\leq w,\epsilon)$.** For a Weyl group element w, a reduced word $\boldsymbol{i} \in I(w)$ and $0 \leq p < \ell$, we consider a subalgebra which is generated by $\{f^{\mathrm{up}}_{\epsilon}(\beta_{\boldsymbol{i},k})\}_{p+1\leq k\leq \ell}$, then it can be shown that this subalgebra is also compatible with the dual canonical basis. This can be proved using the transition matrix between the dual Poincaré-Birkhoff-Witt basis and the dual canonical basis.

In this subsection, we give statements for the $\epsilon = +1$ case. We can obtain the corresponding claims for $\epsilon = -1$ case by applying the $*$-involution. So we denote $f^{\mathrm{up}}_{\epsilon}(\beta_{\boldsymbol{i},k})$, $f^{\mathrm{up}}_{\epsilon}(\boldsymbol{c},\boldsymbol{i})$, $b_{\epsilon}(\boldsymbol{c},\boldsymbol{i})$ by $f^{\mathrm{up}}(\beta_{\boldsymbol{i},k})$, $f^{\mathrm{up}}(\boldsymbol{c},\boldsymbol{i})$, $b(\boldsymbol{c},\boldsymbol{i})$ by omitting ϵ.

Proposition 3.1. *Let $w \in W$ and $\boldsymbol{i} \in I(w)$. For $\boldsymbol{c} \in \mathbb{Z}^{\ell}_{\geq 0}$ and $0 \leq p < \ell$, we set*

$$\tau_{\leq p}(\boldsymbol{c}) := (c_1, \cdots, c_p, 0, \cdots, 0) \in \mathbb{Z}^{\ell}_{\geq 0},$$
$$\tau_{>p}(\boldsymbol{c}) := (0, \cdots, 0, c_{p+1}, \cdots, c_{\ell}) \in \mathbb{Z}^{\ell}_{\geq 0},$$

then we have

$$G^{\mathrm{up}}(b(\tau_{\leq p}(\boldsymbol{c})), \boldsymbol{i})\, G^{\mathrm{up}}(b(\tau_{>p}(\boldsymbol{c}), \boldsymbol{i})) \in G^{\mathrm{up}}(b(\boldsymbol{c},\boldsymbol{i})) + \sum_{\boldsymbol{d}<\boldsymbol{c}} q\mathbb{Z}[q]\, G^{\mathrm{up}}(b(\boldsymbol{d},\boldsymbol{i})).$$

Proof. By the transition from the dual canonical basis to the dual Poincaré-Birkhoff-Witt basis, we have

$$G^{\mathrm{up}}(b(\tau_{\leq p}(\boldsymbol{c})), \boldsymbol{i}) \in f^{\mathrm{up}}(\tau_{\leq p}(\boldsymbol{c}), \boldsymbol{i}) + \sum_{\boldsymbol{d}_{\leq p} \leq \tau_{\leq p}(\boldsymbol{c})} q\mathbb{Z}[q]\, f^{\mathrm{up}}(\boldsymbol{d}_{\leq p}, \boldsymbol{i}),$$
$$G^{\mathrm{up}}(b(\tau_{>p}(\boldsymbol{c}), \boldsymbol{i})) \in f^{\mathrm{up}}(\tau_{>p}(\boldsymbol{c}), \boldsymbol{i}) + \sum_{\boldsymbol{d}_{>p} \leq \tau_{>p}(\boldsymbol{c})} q\mathbb{Z}[q]\, f^{\mathrm{up}}(\boldsymbol{d}_{>p}, \boldsymbol{i}).$$

and we note that we have $\boldsymbol{d}_{\leq p} = \tau_{\leq p}(\boldsymbol{d}_{\leq p})$ and $\boldsymbol{d}_{>p} = \tau_{>p}(\boldsymbol{d}_{>p})$ by the Levendorskii-Soibelman formula in the right hand sides.

Hence in the product of the right hand side, we have the following 4 kinds of terms:

$$f^{\mathrm{up}}(\boldsymbol{c},\boldsymbol{i}) = f^{\mathrm{up}}(\tau_{\leq p}(\boldsymbol{c}), \boldsymbol{i})\, f^{\mathrm{up}}(\tau_{>p}(\boldsymbol{c}), \boldsymbol{i}),$$
$$f^{\mathrm{up}}(\tau_{\leq p}(\boldsymbol{c}) + \boldsymbol{d}_{>p}, \boldsymbol{i}) = f^{\mathrm{up}}(\tau_{\leq p}(\boldsymbol{c}), \boldsymbol{i})\, f^{\mathrm{up}}(\boldsymbol{d}_{>p}, \boldsymbol{i}),$$
$$f^{\mathrm{up}}(\tau_{>p}(\boldsymbol{c}) + \boldsymbol{d}_{\leq p}, \boldsymbol{i}) = f^{\mathrm{up}}(\boldsymbol{d}_{\leq p}, \boldsymbol{i})\, f^{\mathrm{up}}(\tau_{>p}(\boldsymbol{c}), \boldsymbol{i}),$$
$$f^{\mathrm{up}}(\boldsymbol{d}_{\leq p} + \boldsymbol{d}_{>p}, \boldsymbol{i}) = f^{\mathrm{up}}(\boldsymbol{d}_{\leq p}, \boldsymbol{i})\, f^{\mathrm{up}}(\boldsymbol{d}_{>p}, \boldsymbol{i}).$$

We note that $\tau_{\leq p}(\boldsymbol{c}) + \boldsymbol{d}_{>p} <_{\boldsymbol{w}} \boldsymbol{c}$, $\tau_{>p}(\boldsymbol{c}) + \boldsymbol{d}_{\leq p} <_{\boldsymbol{w}} \boldsymbol{c}$, $\boldsymbol{d}_{\leq p} + \boldsymbol{d}_{>p} <_{\boldsymbol{w}} \boldsymbol{c}$ by construction, so we have $\tau_{\leq p}(\boldsymbol{c}) + \boldsymbol{d}_{>p} < \boldsymbol{c}$, $\tau_{>p}(\boldsymbol{c}) + \boldsymbol{d}_{\leq p} < \boldsymbol{c}$ and $\boldsymbol{d}_{\leq p} + \boldsymbol{d}_{>p} <_{\boldsymbol{w},p} \boldsymbol{c}$. Hence, using the transition from the dual Poincaré-Birkhoff-Witt basis to the dual canonical basis, we obtain the claim. □

3.2. **Compatibility of $\mathbf{U}_q^-(>w,\epsilon)$.** For a Weyl group element, we consider the co-finite subset $\Delta_+\cap w\Delta_+$ and corresponding quantum coordinate ring $\mathbf{U}_q^-(>w,\epsilon)$.

Definition 3.2. For $w\in W$ and $\epsilon\in\{\pm1\}$, we set

$$\mathbf{U}_q^-(>w,\epsilon)=\mathbf{U}_q^-\cap T_{w^\epsilon}^{\epsilon}\mathbf{U}_q^-.$$

Using Proposition 2.13 iteratively, we obtain the following lemma.
The following is the main result in this subsection.

Theorem 3.3. *For $w\in W$ and $\epsilon\in\{\pm1\}$, $\mathbf{U}_q^-(>w,\epsilon)$ is compatible with the dual canonical basis, $\mathbf{B}^{\mathrm{up}}(>w,\epsilon):=\mathbf{B}^{\mathrm{up}}\cap\mathbf{U}_q^-(>w,\epsilon)$ is a $\mathbb{Q}(q)$-basis of $\mathbf{U}_q^-(>w,\epsilon)$.*

The proof of this theorems occupies the rest of this subsection and we give the characterization of the subset $\mathbf{B}^{\mathrm{up}}(>w,\epsilon)$.

3.2.1. First, we give a variant of definition of $\mathbf{U}_q^-(>w,\epsilon)$ which depends on $\boldsymbol{i}=(i_1,\cdots,i_\ell)\in I(w)$ which is suitable for the description of the dual canonical basis.

Proposition 3.4. *For $w\in W$, $\boldsymbol{i}=(i_1,\cdots,i_\ell)\in I(w)$ and $\epsilon\in\{\pm1\}$, we have*

$$\mathbf{U}_q^-(>w,\epsilon)=\mathbf{U}_q^-\cap T_{i_1}^{\epsilon}\mathbf{U}_q^-\cap T_{i_1}^{\epsilon}T_{i_2}^{\epsilon}\mathbf{U}_q^-\cap\cdots\cap T_{i_1}^{\epsilon}\cdots T_{i_\ell}^{\epsilon}\mathbf{U}_q^-.$$

In fact, the right hand side does not depend on a choice of a reduced word $\boldsymbol{i}\in I(w)$. The above proposition can be shown clearly by the following Lemmas.

Lemma 3.5. *For a Weyl group element $w\in W$ and a reduced word $\boldsymbol{i}=(i_1,\cdots,i_\ell)\in I(w)$ and homogenous element $x\in\mathbf{U}_q^-$, there exists $x_{\boldsymbol{c}}\in\mathbf{U}_q^-\cap T_{i_\ell}\mathbf{U}_q^-$ for $\boldsymbol{c}\in\mathbb{Z}_{\geq0}^{\ell}$ with*

$$\begin{aligned}T_w^{-1}(x)&=\sum_{\boldsymbol{c}\in\mathbb{Z}_{\geq0}^{\ell}}T_{i_\ell}^{-1}\cdots T_{i_1}^{-1}\left(f_{i_1}^{(c_1)}\right)\cdots T_{i_\ell}^{-1}T_{i_{\ell-1}}^{-1}\left(f_{i_{\ell-1}}^{(c_{\ell-1})}\right)T_{i_\ell}^{-1}\left(f_{i_\ell}^{(c_\ell)}\right)T_{i_\ell}^{-1}(x_{\boldsymbol{c}})\\&\in\sum_{\boldsymbol{c}\in\mathbb{Z}_{\geq0}^{\ell}}T_{i_\ell}^{-1}\cdots T_{i_2}^{-1}\left(e_{i_1}^{(c_1)}\right)\cdots T_{i_\ell}^{-1}\left(e_{i_{\ell-1}}^{(c_{\ell-1})}\right)e_{i_\ell}^{(c_\ell)}\mathbf{U}_q^{\leq0}.\end{aligned}\tag{3.1}$$

Remark 3.6. We note that it is not clear that $T_{i_\ell}^{-1}(x_{\boldsymbol{c}})\in\mathbf{U}_q^-\cap T_w^{-1}\mathbf{U}_q^-$ in the right hand side of (3.1). But, in fact, it can be proved by the surjectivity.

Lemma 3.7. *If $\ell(s_iw)>\ell(w)$, we have $\mathbf{U}_q^-\cap T_{s_iw}\mathbf{U}_q^-\subset\mathbf{U}_q^-\cap T_i\mathbf{U}_q^-\cap T_w\mathbf{U}_q^-$.*

Proof. For a homogenous element $x\in\mathbf{U}_q^-$, we decompose $x=\sum_{c\geq0}f_i^{(c)}x_c$ with $x_c\in\mathbf{U}_q^-\cap T_i\mathbf{U}_q^-$. So we have

$$T_i^{-1}x=\sum_{c\geq0}T_i^{-1}\left(f_i^{(c)}\right)T_i^{-1}(x_c)\in\sum_{c\geq0}e_i^{(c)}\mathbf{U}_q^{\leq0}$$

with $T_i^{-1}(x_c)\in\mathbf{U}_q^-\cap T_i^{-1}\mathbf{U}_q^-$. Apply T_w^{-1} in the both side, we have

$$\begin{aligned}T_w^{-1}T_i^{-1}x&=\sum_{c\geq0}T_w^{-1}\left(f_i^{(c)}\right)T_w^{-1}(x_c)\\&\in\sum_{c\geq0}T_{i_\ell}^{-1}\cdots T_{i_1}^{-1}\left(e_i^{(c)}\right)T_{i_\ell}^{-1}\cdots T_{i_2}^{-1}\left(e_{i_1}^{(c_1)}\right)\cdots T_{i_\ell}^{-1}\left(e_{i_{\ell-1}}^{(c_{\ell-1})}\right)e_{i_\ell}^{(c_\ell)}\mathbf{U}_q^{\leq0}\end{aligned}$$

For $x\in\mathbf{U}_q^-\cap T_{s_iw}\mathbf{U}_q^-$, we have $T_w^{-1}T_i^{-1}x\in\mathbf{U}_q^-\cap T_{s_iw}^{-1}\mathbf{U}_q^-$. Since

$$\left\{T_{i_\ell}^{-1}\cdots T_{i_1}^{-1}\left(e_i^{(c)}\right)T_{i_\ell}^{-1}\cdots T_{i_2}^{-1}\left(e_{i_1}^{(c_1)}\right)\cdots T_{i_\ell}^{-1}\left(e_{i_{\ell-1}}^{(c_{\ell-1})}\right)e_{i_\ell}^{(c_\ell)}\middle|(c,c_1,\cdots,c_\ell)\in\mathbb{Z}_{\geq0}^{\ell+1}\right\}$$

is linearly independent by the assumption $\ell(s_iw)>\ell(w)$, hence we should have $x_c=0$ for $c>0$. In particular $x=x_0\in\mathbf{U}_q^-\cap T_i\mathbf{U}_q^-$ and also $T_w^{-1}(X_0)\in\mathbf{U}_q^-\cap T_w^{-1}\mathbf{U}_q^-$. So $x=x_0\in\mathbf{U}_q^-\cap T_i\mathbf{U}_q^-\cap T_w\mathbf{U}_q^-$. ∎

3.2.2. Let w be a Weyl group element and $\boldsymbol{i} = (i_1, \cdots, i_\ell) \in I(w)$ be a reduced word. Following Saito [12, Lemma 4.1.3] and Baumann-Kamnitzer-Tingley [1, Proposition 5.24], we define Lusztig datum of $b \in \mathscr{B}(\infty)$ in direction $\boldsymbol{i} \in I(w)$ and $\epsilon \in \{\pm 1\}$ ($(\boldsymbol{i}, \epsilon)$-Lusztig datum for short).

Definition 3.8 ($(\boldsymbol{i}, \epsilon)$-Lusztig datum)**.** For $w \in W$, $\boldsymbol{i} \in I(w)$ and $\epsilon \in \{\pm 1\}$, we define

$$L_\epsilon(b, \boldsymbol{i}) = \begin{cases} \left(\varepsilon_{i_1}(b), \varepsilon_{i_2}\left(\hat{\sigma}^*_{i_1} b\right), \cdots, \varepsilon_{i_\ell}\left(\hat{\sigma}^*_{i_{\ell-1}} \cdots \hat{\sigma}^*_{i_1} b\right)\right) \in \mathbb{Z}^\ell_{\geq 0} & \epsilon = +1 \\ \left(\varepsilon^*_{i_1}(b), \varepsilon^*_{i_2}\left(\hat{\sigma}_{i_1} b\right), \cdots, \varepsilon^*_{i_\ell}\left(\hat{\sigma}_{i_{\ell-1}} \cdots \hat{\sigma}_{i_1} b\right)\right) \in \mathbb{Z}^\ell_{\geq 0} & \epsilon = -1 \end{cases}$$

By construction in 2.20, we have

$$\boldsymbol{c} = L_\epsilon\left(b_\epsilon(\boldsymbol{c}, \boldsymbol{i}), \boldsymbol{i}\right)$$

for $\boldsymbol{c} \in \mathbb{Z}^\ell_{\geq 0}$, that is the map $b_\epsilon(-, \boldsymbol{i}) : \mathbb{Z}^\ell_{\geq 0} \to \mathscr{B}(\infty)$ is a section of $(\boldsymbol{i}, \epsilon)$-Lusztig datum $L_\epsilon(-, \boldsymbol{i}) : \mathscr{B}(\infty) \to \mathbb{Z}^\ell_{\geq 0}$.

3.2.3. The following gives a characterization of $\mathbf{B}^{\mathrm{up}}(> w, \epsilon)$ in terms of the $(\boldsymbol{i}, \epsilon)$-Lusztig data.

Theorem 3.9. *For $w \in W$ and $\boldsymbol{i} = (i_1, \cdots, i_\ell) \in I(w)$, we set*

$$\mathscr{B}(> w, \epsilon) = \{b \in \mathscr{B}(\infty) \mid L_\epsilon(b, \boldsymbol{i}) = 0\},$$

then we have

$$\mathbf{U}^-_q(> w, \epsilon) = \bigoplus_{b \in \mathscr{B}(> w, \epsilon)} \mathbb{Q}(q)\, G^{\mathrm{up}}(b).$$

Proof. By the Proposition 3.4, it suffices for us to prove the compatibility for the intersection $\mathbf{U}^-_q \cap T^\epsilon_{i_1}\mathbf{U}^-_q \cap T^\epsilon_{i_1}T^\epsilon_{i_2}\mathbf{U}^-_q \cap \cdots \cap T^\epsilon_{i_1} \cdots T^\epsilon_{i_\ell}\mathbf{U}^-_q$.

Since $\epsilon = -1$ can be obtained by applying the $*$-involution, we only prove $\epsilon = 1$ case. We prove the claim by the induction on the length $\ell(w)$. For $\ell(w) = 1$, it is the claim in Corollary 2.14. We consider the following intersection:

$$\mathbf{U}^-_q \cap T^{-1}_{i_1}\mathbf{U}^-_q \cap T_{i_2}\mathbf{U}^-_q \cap \cdots \cap T_{i_2} \cdots T_{i_\ell}\mathbf{U}^-_q;$$

By the assumption of the induction on length, we know that $\mathbf{U}^-_q \cap T_{i_2}\mathbf{U}^-_q \cap \cdots \cap T_{i_2} \cdots T_{i_\ell}\mathbf{U}^-_q$ is compatible with the dual canonical basis and also $\mathbf{U}^-_q \cap T^{-1}_{i_1}\mathbf{U}^-_q$ is compatible with the dual canonical basis, hence the intersection $\mathbf{U}^-_q \cap T^{-1}_{i_1}\mathbf{U}^-_q \cap T_{i_2}\mathbf{U}^-_q \cap \cdots \cap T_{i_2} \cdots T_{i_\ell}\mathbf{U}^-_q$ is compatible with the dual canonical basis. Applying Theorem 2.16, we obtain the claim for $\mathbf{U}^-_q \cap T_{i_1}\mathbf{U}^-_q \cap T_{i_1}T_{i_2}\mathbf{U}^-_q \cap \cdots \cap T_{i_1} \cdots T_{i_\ell}\mathbf{U}^-_q$. Since $\mathbf{U}^-_q \cap T_{i_1}\mathbf{U}^-_q \cap T_{i_1}T_{i_2}\mathbf{U}^-_q \cap \cdots \cap T_{i_1} \cdots T_{i_\ell}\mathbf{U}^-_q = \mathbf{U}^-_q \cap T_{i_1}\left(\mathbf{U}^-_q \cap T_{i_2}\mathbf{U}^-_q \cap \cdots \cap T_{i_2} \cdots T_{i_\ell}\mathbf{U}^-_q\right)$, we obtain the description of $\mathbf{B}^{\mathrm{up}}(> w, +1)$. $\square$

3.3. Multiplication formula between $\mathbf{B}^{\mathrm{up}}(\leq w, \epsilon)$ and $\mathbf{B}^{\mathrm{up}}(> w, \epsilon)$.

3.3.1. We generalize the (special cases of) formula in Theorem 2.11 using the dual canonical basis $\mathbf{B}^{\mathrm{up}}(> w, \epsilon)$.

Theorem 3.10. *For $b \in \mathscr{B}(> w, \epsilon)$ and $\boldsymbol{c} \in \mathbb{Z}_{\geq 0}$, we have*

$$f^{\mathrm{up}}_\epsilon(\boldsymbol{c}, \boldsymbol{i})\, G^{\mathrm{up}}(b) \in G^{\mathrm{up}}\left(\nabla^{\boldsymbol{c}}_{\boldsymbol{i},\epsilon}(b)\right) + \sum_{L_\epsilon\left(b', \boldsymbol{i}\right) < \boldsymbol{c}} q\mathbb{Z}[q]\, G^{\mathrm{up}}\left(b'\right) \text{ if } \epsilon = +1,$$

$$G^{\mathrm{up}}(b)\, f^{\mathrm{up}}_\epsilon(\boldsymbol{c}, \boldsymbol{i}) \in G^{\mathrm{up}}\left(\nabla^{\boldsymbol{c}}_{\boldsymbol{i},\epsilon}(b)\right) + \sum_{L_\epsilon\left(b', \boldsymbol{i}\right) < \boldsymbol{c}} q\mathbb{Z}[q]\, G^{\mathrm{up}}\left(b'\right) \text{ if } \epsilon = -1,$$

where

$$\nabla^{\boldsymbol{c}}_{\boldsymbol{i},\epsilon}(b) = \begin{cases} \tilde{f}^{c_1}_{i_1}\sigma_{i_1} \cdots \tilde{f}^{c_{\ell-1}}_{i_{\ell-1}}\sigma_{i_{\ell-1}}\tilde{f}^{c_\ell}_{i_\ell}\sigma_{i_\ell}\sigma^*_{i_\ell} \cdots \sigma^*_{i_1}(b) & \text{if } \epsilon = +1, \\ \tilde{f}^{*c_1}_{i_1}\sigma^*_{i_1} \cdots \tilde{f}^{*c_{\ell-1}}_{i_{\ell-1}}\sigma^*_{i_{\ell-1}}\tilde{f}^{*c_\ell}_{i_\ell}\sigma^*_{i_\ell}\sigma_{i_\ell} \cdots \sigma_{i_1}(b) & \text{if } \epsilon = -1. \end{cases}$$

Proof. We only proof for $\epsilon = +1$ case. The $\epsilon = -1$ can be proved by applying the $*$-involution. We prove by induction on the length $\ell(w)$. Let $w_{\geq 2} = s_{i_2} \cdots s_{i_\ell} \in W$ and $\boldsymbol{i}_{\geq 2} = (i_2, \cdots, i_\ell) \in I(w_{\geq 2})$. Let $b \in \mathscr{B}(\infty)$ with $L_{+1}(b, \boldsymbol{i}) = 0$, that is we have

$$\left(\varepsilon_{i_1}(b), \varepsilon_{i_2}\left(\sigma_{i_1}^* b\right), \cdots, \varepsilon_{i_\ell}\left(\sigma_{i_\ell}^* \cdots \sigma_{i_1}^* b\right)\right) = (0, \cdots, 0),$$

so let $b_{\geq 2} := \sigma_{i_1}^* b$, then we have

$$\begin{aligned} L_{+1}(b_{\geq 2}, \boldsymbol{i}_{\geq 2}) &= \left(\varepsilon_{i_2}(b_{\geq 2}), \cdots, \varepsilon_{i_\ell}\left(\sigma_{i_\ell}^* \cdots \sigma_{i_2}^* b_{\geq 2}\right)\right) \\ &= \left(\varepsilon_{i_2}\left(\sigma_{i_1}^* b\right), \cdots, \varepsilon_{i_\ell}\left(\sigma_{i_\ell}^* \cdots \sigma_{i_2}^* \sigma_{i_1}^* b\right)\right) = (0, \cdots, 0) \in \mathbb{Z}_{\geq 0}^{\ell-1} \end{aligned}$$

by definition of the Lusztig datum.

By induction hypothesis, we have

$$f_\epsilon^{\mathrm{up}}(\boldsymbol{c}_{\geq 2}, \boldsymbol{i}_{\geq 2}) G^{\mathrm{up}}(b_{\geq 2}) - G^{\mathrm{up}}\left(\nabla_{\boldsymbol{i}_{\geq 2}, \epsilon}^{\boldsymbol{c}_{\geq 2}}(b_{\geq 2})\right) \in \sum_{L_\epsilon\left(b'_{\geq 2}, \boldsymbol{i}_{\geq 2}\right) < \boldsymbol{c}_{\geq 2}} q\mathbb{Z}[q] G^{\mathrm{up}}\left(b'_{\geq 2}\right).$$

with $c_{\geq 2} = (c_2, \cdots, c_\ell) \in \mathbb{Z}_{\geq 0}^{\ell-1}$. Since $\mathbf{U}_q^- \cap T_{i_1}^{-1} \mathbf{U}_q^-$ is spanned by the dual canonical basis $\left\{G^{\mathrm{up}}(b) \mid \varepsilon_{i_1}^*(b) = 0\right\}$ and $f_\epsilon^{\mathrm{up}}(\boldsymbol{c}_{\geq 2}, \boldsymbol{i}_{\geq 2}) \in \mathbf{U}_q^- \cap T_{i_1}^{-1} \mathbf{U}_q^-$ and $G^{\mathrm{up}}(b_{\geq 2}) \in \mathbf{U}_q^- \cap T_{i_1}^{-1} \mathbf{U}_q^-$, so we obtain that $\varepsilon_{i_1}^*\left(\nabla_{\boldsymbol{i}_{\geq 2}, \epsilon}^{\boldsymbol{c}_{\geq 2}}(b_{\geq 2})\right) = 0$ and $\varepsilon_{i_1}^*\left(b'_{\geq 2}\right) = 0$.

We have

$$\begin{aligned} f_{+1}^{\mathrm{up}}(\boldsymbol{c}, \boldsymbol{i}) G^{\mathrm{up}}(b) &= \left(1 - q_{i_1}^2\right)^{\langle h_{i_1}, \xi(c_{\geq 2}, i_{\geq 2}) + \mathrm{wt}(b)\rangle} f_{i_1}^{\{c_1\}} T_{i_1}\left(f_\epsilon^{\mathrm{up}}(\boldsymbol{c}_{\geq 2}, \boldsymbol{i}_{\geq 2}) G^{\mathrm{up}}(b_{\geq 2})\right) \\ &\in \left(1 - q_{i_1}^2\right)^{\langle h_{i_1}, \xi(c_{\geq 2}, i_{\geq 2}) + \mathrm{wt}(b)\rangle} f_{i_1}^{\{c_1\}} \\ &\quad \times T_{i_1}\left(G^{\mathrm{up}}\left(\nabla_{\boldsymbol{i}_{\geq 2}, +1}^{\boldsymbol{c}_{\geq 2}}(b_{\geq 2})\right) + \sum_{L_{+1}\left(b'_{\geq 2}, i_{\geq 2}\right) < c_{\geq 2}} q\mathbb{Z}[q] G^{\mathrm{up}}\left(b'_{\geq 2}\right)\right) \\ &= f_{i_1}^{\{c_1\}}\left(G^{\mathrm{up}}\left(\sigma_{i_1} \nabla_{\boldsymbol{i}_{\geq 2}, +1}^{\boldsymbol{c}_{\geq 2}}(b_{\geq 2})\right) + \sum_{L_{+1}\left(b'_{\geq 2}, i_{\geq 2}\right) < c_{\geq 2}} q\mathbb{Z}[q] G^{\mathrm{up}}\left(\sigma_{i_1} b'_{\geq 2}\right)\right). \end{aligned}$$

We note that $\tilde{f}_{i_1}^{c_1} \sigma_{i_1} \nabla_{\boldsymbol{i}_{\geq 2}, +1}^{\boldsymbol{c}_{\geq 2}}(b_{\geq 2}) = \tilde{f}_{i_1}^{c_1} \sigma_{i_1} \tilde{f}_{i_2}^{c_1} \sigma_{i_2} \cdots \tilde{f}_{i_{\ell-1}}^{c_{\ell-1}} \sigma_{i_{\ell-1}} \tilde{f}_{i_\ell}^{c_\ell} \sigma_{i_\ell} \sigma_{i_\ell}^* \cdots \sigma_{i_2}^*(b_{\geq 2}) = \nabla_{\boldsymbol{i}, +1}^{\boldsymbol{c}}(b)$ and

$$\begin{aligned} f_{i_1}^{\{c_1\}} G^{\mathrm{up}}\left(\sigma_{i_1} \nabla_{\boldsymbol{i}_{\geq 2}, +1}^{\boldsymbol{c}_{\geq 2}}(b_{\geq 2})\right) &\in G^{\mathrm{up}}\left(\nabla_{\boldsymbol{i}, +1}^{\boldsymbol{c}}(b)\right) + \sum_{\varepsilon_{i_1}(b'') < c_1} q\mathbb{Z}[q] G^{\mathrm{up}}(b''), \\ f_{i_1}^{\{c_1\}} G^{\mathrm{up}}\left(\sigma_{i_1} b'_{\geq 2}\right) &\in G^{\mathrm{up}}\left(\tilde{f}_{i_1}^{c_1} \sigma_{i_1} b'_{\geq 2}\right) + \sum_{\varepsilon_{i_1}(b'') < c_1} q\mathbb{Z}[q] G^{\mathrm{up}}(b'') \end{aligned}$$

by Theorem 2.11, $f_{i_1}^{\{c_1\}}\left(G^{\mathrm{up}}\left(\sigma_{i_1} \nabla_{\boldsymbol{i}_{\geq 2}, +1}^{\boldsymbol{c}_{\geq 2}}(b_{\geq 2})\right) + \sum_{L_{+1}\left(b'_{\geq 2}, i_{\geq 2}\right) < c_{\geq 2}} q\mathbb{Z}[q] G^{\mathrm{up}}\left(\sigma_{i_1} b'_{\geq 2}\right)\right)$ can be written in the following form:

$$G^{\mathrm{up}}\left(\nabla_{\boldsymbol{i}, +1}^{\boldsymbol{c}}(b)\right) + \sum_{L_{+1}\left(b'_{\geq 2}, i_{\geq 2}\right) < \boldsymbol{c}_{\geq 2}} q\mathbb{Z}[q] G^{\mathrm{up}}\left(\tilde{f}_{i_1}^{c_1} \sigma_{i_1} b'_{\geq 2}\right) + \sum_{\varepsilon_{i_1}(b'') < c_1} q\mathbb{Z}[q] G^{\mathrm{up}}(b'').$$

Since we have $(c'_2, \cdots, c'_\ell) = L_{+1}\left(b'_{\geq 2}, i_{\geq 2}\right) < \boldsymbol{c}_{\geq 2}$, we obtain $L_{+1}\left(\tilde{f}_{i_1}^{c_1} \sigma_{i_1} b'_{\geq 2}, \boldsymbol{i}\right) = (c_1, c'_2, \cdots, c'_\ell) < \boldsymbol{c}$ and we have $L_{+1}(b'', \boldsymbol{i}) = (\varepsilon_{i_1}(b''), \cdots) < L_{+1}(b, \boldsymbol{i}) = (c_1, c_2, \cdots, c_\ell)$ by $\varepsilon_{i_1}\left(b''\right) < c_1$. We we obtain the claim. $\square$

Using the transition Theorem 2.23 (2) from Poincaré-Birkhoff-Witt basis to the dual canonical basis, we obtain the following multiplicity-free result.

Theorem 3.11. *Let $w \in W$, $\boldsymbol{i} = (i_1, \cdots, i_\ell) \in I(w)$ and $\epsilon \in \{\pm 1\}$.*
For $\boldsymbol{c} \in \mathbb{Z}_{\geq 0}^{\ell}$ and $b \in \mathscr{B}(> w, \epsilon)$, we have

$$G^{\mathrm{up}}(b_\epsilon(\boldsymbol{c}, \boldsymbol{i}))\, G^{\mathrm{up}}(b) \in G^{\mathrm{up}}\left(\nabla^{\boldsymbol{c}}_{\boldsymbol{i},\epsilon}(b)\right) + \sum_{L_\epsilon\left(b', \boldsymbol{i}\right) < c} q\mathbb{Z}[q]\, G^{\mathrm{up}}\left(b^{'}\right) \text{ if } \epsilon = +1,$$

$$G^{\mathrm{up}}(b)\, G^{\mathrm{up}}(b_\epsilon(\boldsymbol{c}, \boldsymbol{i})) \in G^{\mathrm{up}}\left(\nabla^{\boldsymbol{c}}_{\boldsymbol{i},\epsilon}(b)\right) + \sum_{L_\epsilon\left(b', \boldsymbol{i}\right) < c} q\mathbb{Z}[q]\, G^{\mathrm{up}}\left(b^{'}\right) \text{ if } \epsilon = -1.$$

3.3.2.

Definition 3.12. Let $w \in W$, $\boldsymbol{i} = (i_1, \cdots, i_\ell) \in I(w)$ and $\epsilon \in \{\pm 1\}$. We define maps $\tau_{\leq w, \epsilon} \colon \mathscr{B}(\infty) \to \mathscr{B}(\leq w, \epsilon)$ and $\tau_{> w, \epsilon} \colon \mathscr{B}(\infty) \to \mathscr{B}(> w, \epsilon)$ by

$$\tau_{\leq w, \epsilon}(b) = b_\epsilon(L_\epsilon(b, \boldsymbol{i}), \boldsymbol{i}),$$

$$\tau_{> w, \epsilon}(b) = \begin{cases} \sigma_{i_1} \cdots \sigma_{i_\ell} \hat{\sigma}^*_{i_\ell} \cdots \hat{\sigma}^*_{i_1}(b) & \text{if } \epsilon = +1, \\ \sigma^*_{i_1} \cdots \sigma^*_{i_\ell} \hat{\sigma}_{i_\ell} \cdots \hat{\sigma}_{i_1}(b) & \text{if } \epsilon = -1. \end{cases}$$

We note that the independence of the maps $\tau_{\leq w, \epsilon}$ and $\tau_{> w, \epsilon}$ on a reduced word $\boldsymbol{i} \in I(w)$ of a Weyl group element w can be shown in [1, Theorem 4.4] using representation theory of the preprojective algebra and the torsion pair in symmetric case. Using the folding argument , it also can be proved in general.

Proposition 3.13. *We have a bijection as sets:*

$$\Omega_w := (\tau_{\leq w, \epsilon}, \tau_{> w, \epsilon}) : \mathscr{B}(\infty) \to \mathscr{B}(\leq w, \epsilon) \times \mathscr{B}(> w, \epsilon).$$

We prove the multiplication property of the dual canonical basis element between $\mathbf{U}_q^-(\leq w, \epsilon)$ and $\mathbf{U}_q^-(> w, \epsilon)$.

Theorem 3.14. *Let $w \in W$, $\boldsymbol{i} = (i_1, \cdots, i_\ell) \in I(w)$ and $\epsilon \in \{\pm 1\}$. For $b \in \mathscr{B}(\infty)$, we have*

$$G^{\mathrm{up}}(\tau_{\leq w, \epsilon}(b))\, G^{\mathrm{up}}(\tau_{> w, \epsilon}(b)) \in G^{\mathrm{up}}(b) + \sum_{L_\epsilon(b', \boldsymbol{i}) < L_\epsilon(b, \boldsymbol{i})} q\mathbb{Z}[q]\, G^{\mathrm{up}}\left(b^{'}\right) \text{ if } \epsilon = +1,$$

$$G^{\mathrm{up}}(\tau_{> w, \epsilon}(b))\, G^{\mathrm{up}}(\tau_{\leq w, \epsilon}(b)) \in G^{\mathrm{up}}(b) + \sum_{L_\epsilon(b', \boldsymbol{i}) < L_\epsilon(b, \boldsymbol{i})} q\mathbb{Z}[q]\, G^{\mathrm{up}}\left(b^{'}\right) \text{ if } \epsilon = -1.$$

Proof. We prove $\epsilon = +1$ case. We prove by the induction on the length $\ell(w)$.
First we have

$$\begin{aligned} & G^{\mathrm{up}}(b) - f_{i_1}^{\{\varepsilon_{i_1}(b)\}} G^{\mathrm{up}}\left(\tilde{e}_{i_1}^{\varepsilon_{i_1}(b)} b\right) \\ = & G^{\mathrm{up}}(b) - \left(1 - q_{i_1}^2\right)^{\langle h_{i_1}, \mathrm{wt}\left(\hat{\sigma}^*_{i_1} b\right)\rangle} f_{i_1}^{\{\varepsilon_{i_1}(b)\}} T_{i_1} G^{\mathrm{up}}\left(\hat{\sigma}^*_{i_1} b\right) \in \sum_{\varepsilon_{i_1}(b') < \varepsilon_{i_1}(b)} q\mathbb{Z}[q]\, G^{\mathrm{up}}(b') \end{aligned}$$

By Theorem 3.11, we only have to compute the product $\left(1 - q_i^2\right)^{\langle h_i, \mathrm{wt}(\hat{\sigma}^*_i b)\rangle} f_i^{\{\varepsilon_i(b)\}} T_i G^{\mathrm{up}}(\hat{\sigma}^*_i b) \times G^{\mathrm{up}}(\tau_{> w, +1}(b))$.
We note that

$$G^{\mathrm{up}}(\tau_{> w, +1}(b)) = \left(1 - q_i^2\right)^{\left\langle h_{i_1}, \mathrm{wt}\left(\tau_{> w_{\geq 2}}\left(\hat{\sigma}^*_{i_1} b\right)\right)\right\rangle} T_{i_1} G^{\mathrm{up}}\left(\tau_{> w_{\geq 2}}\left(\hat{\sigma}^*_{i_1} b\right)\right)$$

where $w_{\geq 2} = s_{i_2} \cdots s_{i_\ell}$.
By the induction hypothesis, we have

$$G^{\mathrm{up}}\left(\hat{\sigma}^*_{i_1} b\right) - G^{\mathrm{up}}\left(\tau_{\leq w_{\geq 2}}\left(\hat{\sigma}^*_{i_1} b\right)\right) G^{\mathrm{up}}\left(\tau_{> w_{\geq 2}}\left(\hat{\sigma}^*_{i_1} b\right)\right) \in \sum_{L_{+1}\left(b'', \boldsymbol{i}_{\geq 2}\right) < L_{+1}\left(\hat{\sigma}^*_{i_1} b, \boldsymbol{i}_{\geq 2}\right)} q\mathbb{Z}[q]\, G^{\mathrm{up}}(b'')$$

where $i_{\geq 2} = (i_2, \cdots, i_\ell) \in I(s_{i_2} \cdots s_{i_\ell})$.

Applying $\left(1-q_{i_1}^2\right)^{\left\langle h_{i_1},\mathrm{wt}\left(\hat{\sigma}_{i_1}^* b\right)\right\rangle} T_{i_1}$, we obtain

$$G^{\mathrm{up}}\left(\sigma_{i_1}\hat{\sigma}_{i_1}^* b\right)-G^{\mathrm{up}}\left(\sigma_{i_1}\tau_{\le w_{\ge 2}}\left(\hat{\sigma}_{i_1}^* b\right)\right)G^{\mathrm{up}}\left(\sigma_{i_1}\tau_{>w_{\ge 2}}\left(\hat{\sigma}_{i_1}^* b\right)\right)\in\sum_{L_{+1}\left(b'',\boldsymbol{i}_{\ge 2}\right)<L_{+1}\left(\hat{\sigma}_{i_1}^* b,\boldsymbol{i}_{\ge 2}\right)} q\mathbb{Z}\left[q\right]G^{\mathrm{up}}\left(\sigma_{i_1}b''\right).$$

We note that $\tilde{e}_{i_1}^{\varepsilon_{i_1}(b)}b=\sigma_{i_1}\hat{\sigma}_{i_1}^* b$. Multiplying $f_{i_1}^{\left\{\varepsilon_{i_1}(b)\right\}}$ from left to the second term, we have

$$\begin{aligned}&f_{i_1}^{\left\{\varepsilon_{i_1}(b)\right\}}G^{\mathrm{up}}\left(\sigma_{i_1}\tau_{\le w_{\ge 2}}\left(\hat{\sigma}_{i_1}^* b\right)\right)G^{\mathrm{up}}\left(\sigma_{i_1}\tau_{>w_{\ge 2}}\left(\hat{\sigma}_{i_1}^* b\right)\right)\\ \in& G^{\mathrm{up}}\left(\tilde{f}_{i_1}^{\varepsilon_{i_1}(b)}\sigma_{i_1}\tau_{\le w_{\ge 2}}\left(\hat{\sigma}_{i_1}^* b\right)\right)G^{\mathrm{up}}\left(\sigma_{i_1}\tau_{>w_{\ge 2}}\left(\hat{\sigma}_{i_1}^* b\right)\right)+\sum_{\varepsilon_{i_1}(b')<\varepsilon_{i_1}(b)} q\mathbb{Z}\left[q\right]G^{\mathrm{up}}\left(b'\right),\end{aligned}$$

then we obtain

$$\begin{aligned}&f_{i_1}^{\left\{\varepsilon_{i_1}(b)\right\}}\left(G^{\mathrm{up}}\left(\tilde{e}_{i_1}^{\varepsilon_{i_1}(b)}b\right)-G^{\mathrm{up}}\left(\sigma_{i_1}\tau_{\le w_{\ge 2}}\left(\hat{\sigma}_{i_1}^* b\right)\right)G^{\mathrm{up}}\left(\sigma_{i_1}\tau_{>w_{\ge 2}}\left(\hat{\sigma}_{i_1}^* b\right)\right)\right)\\ \in&\sum_{\varepsilon_{i_1}(b')<\varepsilon_{i_1}(b)} q\mathbb{Z}\left[q\right]G^{\mathrm{up}}\left(b'\right)+\sum_{L_{+1}\left(b'',\boldsymbol{i}_{\ge 2}\right)<L_{+1}\left(\hat{\sigma}_{i_1}^* b,\boldsymbol{i}_{\ge 2}\right)} q\mathbb{Z}\left[q\right]G^{\mathrm{up}}\left(\tilde{f}_{i_1}^{\varepsilon_{i_1}(b)}\sigma_{i_1}b''\right).\end{aligned}$$

By the construction, we have $\tilde{f}_{i_1}^{\varepsilon_{i_1}(b)}\sigma_{i_1}\tau_{\le w_{\ge 2}}\left(\hat{\sigma}_{i_1}^* b\right)=\tau_{\le w}\left(b\right)$ and $\sigma_{i_1}\tau_{>w_{\ge 2}}\left(\hat{\sigma}_{i_1}^* b\right)=\tau_{>w}\left(b\right)$, hence we obtain the claim of the theorem.

□

3.4. **Application.** We give a slight refinement of Lusztig's result [10, Proposition 8.3] in the dual canonical basis. The following can be shown in a similar manner using the multiplicity-free property of the multiplications of a triple of the dual canonical basis elements, so we only state the claims.

Theorem 3.15. *Let w be a Weyl group element, $\boldsymbol{i}=\left(i_1,\cdots,i_\ell\right)\in I\left(w\right)$ and $p\in\left[0,\ell\right]$ be an integer. We consider the following intersection:*

$$\mathbf{U}_q^-\cap T_{s_{i_{p+1}}\cdots s_{i_\ell}}\mathbf{U}_q^-\cap T_{s_{i_1}\cdots s_{i_p}}^{-1}\mathbf{U}_q^-=\left(\mathbf{U}_q^-\cap T_{s_{i_{p+1}}\cdots s_{i_\ell}}\mathbf{U}_q^-\right)\cap\left(\mathbf{U}_q^-\cap T_{s_{i_1}\cdots s_{i_p}}^{-1}\mathbf{U}_q^-\right)$$

(1) *The subalgebra $\mathbf{U}_q^-\cap T_{s_{i_{p+1}}\cdots s_{i_\ell}}\mathbf{U}_q^-\cap T_{s_{i_1}\cdots s_{i_p}}^{-1}\mathbf{U}_q^-$ is compatible with the dual canonical basis, that is there exists a subset $\mathscr{B}\left(\mathbf{U}_q^-\cap T_{i_{p+1}}\cdots T_{i_\ell}\mathbf{U}_q^-\cap T_{s_{i_1}\cdots s_{i_p}}^{-1}\mathbf{U}_q^-\right)\subset\mathscr{B}\left(\infty\right)$ such that*

$$\mathbf{U}_q^-\cap T_{s_{i_{p+1}}\cdots s_{i_\ell}}\mathbf{U}_q^-\cap T_{s_{i_1}\cdots s_{i_p}}^{-1}\mathbf{U}_q^-=\bigoplus_{b\in\mathscr{B}\left(\mathbf{U}_q^-\cap T_{s_{i_{p+1}}\cdots s_{i_\ell}}\mathbf{U}_q^-\cap T_{s_{i_1}\cdots s_{i_p}}^{-1}\mathbf{U}_q^-\right)}\mathbb{Q}\left(q\right)G^{\mathrm{up}}\left(b\right).$$

(2) *Multiplication in $\mathbf{U}_q^-\left(\mathfrak{g}\right)_{\mathcal{A}}^{\mathrm{up}}$ defines an isomorphism of free $\mathcal{A}$-modules:*

$$\left(\mathbf{U}_q^-\left(s_{i_{p+1}}\cdots s_{i_\ell},+1\right)\right)_{\mathcal{A}}^{\mathrm{up}}\otimes_{\mathcal{A}}\left(\mathbf{U}_q^-\cap T_{s_{i_{p+1}}\cdots s_{i_\ell}}\mathbf{U}_q^-\cap T_{s_{i_1}\cdots s_{i_\ell}}^{-1}\mathbf{U}_q^-\right)_{\mathcal{A}}^{\mathrm{up}}\otimes_{\mathcal{A}}\left(\mathbf{U}_q^-\left(s_{i_p}\cdots s_{i_1},-1\right)\right)_{\mathcal{A}}^{\mathrm{up}}\to\mathbf{U}_q^-$$

where $\left(\mathbf{U}_q^-\cap T_{i_{p+1}}\cdots T_{i_\ell}\mathbf{U}_q^-\cap T_{i_p}^{-1}\cdots T_{i_1}^{-1}\mathbf{U}_q^-\right)_{\mathcal{A}}^{\mathrm{up}}=\mathbf{U}_q^-\left(\mathfrak{g}\right)_{\mathcal{A}}^{\mathrm{up}}\cap T_{i_{p+1}}\cdots T_{i_\ell}\mathbf{U}_q^-\cap T_{i_p}^{-1}\cdots T_{i_1}^{-1}\mathbf{U}_q^-$.

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