Minimization problems on the Hardy-Sobolev inequality

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Abstract We study minimization problems on Hardy-Sobolev type inequality. We consider the case where singularity is in interior of bounded domain $\Omega \subset \mathbb{R}^N$. The attainability of best constants for Hardy-Sobolev type inequalities with boundary singularities have been studied so far, for example [5] [6] [9] etc. . . . According to their results, the mean curvature of $\partial \Omega$ at singularity affects the attainability of the best constants. In contrast with the case of boundary singularity, it is well known that the best Hardy-Sobolev constant

$$
\mu_s(\Omega) := \left\{ \int_\Omega \left| \nabla u \right|^2 dx \bigg| u \in H^1_0(\Omega), \int_\Omega \frac{|u|^{2^*(s)}}{|x|^s} = 1 \right\}
$$

is never achieved for all bounded domain $\Omega$ if $0 \in \Omega$. We see that the position of singularity on domain is related to the existence of minimizer. In this paper, we consider the attainability of the best constant for the embedding $H^1(\Omega) \hookrightarrow L^{2^*(s)}(\Omega)$ for bounded domain $\Omega$ with $0 \in \partial \Omega$. In this problem, scaling invariance doesn’t hold and we can not obtain information of singularity like mean curvature.

Keywords critical exponent · Hardy-Sobolev inequality · minimization problem · Neumann

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1 Introduction

We study the minimization problems for the Hardy-Sobolev type inequalities. Let $N \geq 3$, $\Omega$ is bounded domain in $\mathbb{R}^N$, $0 \in \Omega$, $0 < s < 2$, and $2^*(s) := 2(N - s)/(N - 2)$. The Hardy-Sobolev inequality asserts that there exists a positive constant $C$ such that

$$
C \left( \int_\Omega \frac{|u|^{2^*(s)}}{|x|^s} dx \right)^{\frac{2^*(s)}{2N}} \leq \int_\Omega |\nabla u|^2 dx
$$

(1)

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for all \( u \in H^1_0(\Omega) \). For \( s = 0 \), the inequality (1) is called Sobolev inequality and for \( s = 2 \), the inequality (1) is called Hardy inequality.

In the non-singular case \((s = 0)\), it is well known that the best Sobolev constant \( S \) is independent of domain \( \Omega \) and \( S \) is never achieved for all bounded domains. But if \( \Omega = \mathbb{R}^N \) and \( H^1(\Omega) \) is replaced by the function space of \( u \in L^{2N/(N-2)}(\Omega) \) with \( \nabla u \in L^2(\Omega) \), then \( S \) is achieved by the function \( u(x) = c(1 + |x|^2)^{(2-N)/2} \) and hence the value \( S = N(N-2)\pi N/(N+2) \) explicitly (see \([\ref{1}, \ref{13} \) and \([\ref{16}]\)).

In the case of \( s = 2 \), the best constant for the Hardy inequality is \( [(N-2)/2]^2 \) and this constant is never achieved for all bounded domains and \( \mathbb{R}^N \). This fact suggests that it is possible to improve this inequality. For example Brezis and Vazquez \([\ref{2}]\), many people research the optimal inequality of (1). In other words, the best remainder term for (1) is studied actively.

In the case of \( 0 < s < 2 \), the best Hardy-Sobolev constant is defined by

\[
\mu_s(\Omega) := \left\{ \int_\Omega |\nabla u|^2 \, dx \left| \begin{array}{l} u \in H^1_0(\Omega), \\
\int_\Omega \frac{|u|^{2^*(s)}}{|x|^s} \, dx = 1 \end{array} \right. \right\}.
\]

This constant has some similar properties to these of the best Sobolev constant. Indeed, due to scaling invariance, \( \mu_s(\Omega) \) is independent of \( \Omega \), and thus \( \mu_s := \mu_s(\Omega) = \mu_s(\mathbb{R}^N) \) is not attained for all bounded domains. If \( \Omega = \mathbb{R}^N \), then \( \mu_s \) is attained by

\[
y_s(x) = \left[ a(N-s)(N-2) \right]^{\frac{N-2}{N-2-s}} \left( a + |x|^{2-s} \right)^{\frac{s-N}{N-2}}
\]

for some \( a > 0 \) and hence

\[
\mu_s = (N-2)(N-s) \left( \frac{\omega_N-1}{2-s} \left( \frac{\Gamma^2(N-s)}{\Gamma^2(N-2)} \right) \right)^{\frac{2-s}{N-2}}
\]

(see \([\ref{9}] \) and \([\ref{13}]\) where \( \omega_N-1 \) is the area of the unit sphere in \( \mathbb{R}^N \).

On the other hand, for \( 0 \in \partial \Omega \), the result of the attainability for \( \mu_s(\Omega) \) is quite different from that in the situation of \( 0 \in \Omega \). By Ghoussoub-Robert \([\ref{6}]\), it has proved that if \( \Omega \) has smooth boundary and the mean curvature of \( \partial \Omega \) at \( 0 \) is negative, then the extremal of \( \mu_s(\Omega) \) exists for all \( N \geq 3 \). Recently, Lin and Wadade \([\ref{14}]\) have studied the following minimization problem;

\[
\mu_{s,p}^\lambda(\Omega) := \inf \left\{ \int_\Omega |\nabla u|^2 \, dx + \lambda \left( \int_\Omega |u|^p \, dx \right)^{\frac{2}{p}} \left| \begin{array}{l} u \in H^1_0(\Omega), \\
\int_\Omega \frac{|u|^{2^*(s)}}{|x|^s} \, dx = 1 \end{array} \right. \right\}
\]

where \( \lambda \in \mathbb{R} \) and \( 2 \leq p \leq 2N/(N-2) \). Furthermore, as related results, Hsia, Lin and Wadade \([\ref{10}]\) studied the existence of the solution of double critical elliptic equations related with \( \mu_{s,p}^\lambda(\Omega) \), that is, they have showed the existence of the solution for

\[
\begin{cases}
-\Delta u + \lambda u^{2^*-1} + \frac{u^{2^*(s)-1}}{|x|^s} = 0, & u > 0, \quad \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases}
\]

under the appropriate conditions where \( 2^* = 2N/(N-2) \). To prove these results, we use the theorem of Egnell \([\ref{4}]\). He showed that the existence of the extremal for \( \mu_s(\Omega) \) if \( \Omega \) is a half space \( \mathbb{R}^N_+ \) or an open cone. The open cone \( \mathcal{C} \) is written of the form \( \mathcal{C} := \{ x \in \mathbb{R}^N \mid x = \)}
\[ r\theta, \theta \in \Sigma \} \] where \( \Sigma \) is connected domain on the unit sphere \( \mathbb{S}^{N-1} \) in \( \mathbb{R}^N \). By this result, \( \mu_s(\mathcal{C}) > \mu_s(\mathbb{R}^N) \) and there is a positive solution for

\[
\begin{align*}
-\Delta u &= \frac{|u|^{2+\sigma(s)-1}}{|x|} \quad \text{in } \mathcal{C}, \\
u &= 0 \quad \text{on } \partial \mathcal{C}, \quad \text{and} \quad u(x) = o(|x|^{2-N}) \quad \text{as } x \to \infty.
\end{align*}
\]

The Neumann case also has been studied. Let \( \Omega \) has \( C^2 \) boundary and the mean curvature of \( \partial \Omega \) at 0 is positive. Ghoussoub and Kang [5] have showed that there is a least energy solution for

\[
\begin{align*}
-\Delta u + \lambda u &= \frac{|u|^{2+\sigma(s)-1}}{|x|} \quad \text{in } \Omega, \\
\frac{\partial u}{\partial n} &= 0 \quad \text{on } \partial \Omega
\end{align*}
\]

for \( N \geq 3, \lambda > 0 \).

Like these results, if \( 0 \in \partial \Omega \), we can use the benefit of the mean curvature of \( \partial \Omega \) at 0 to show the results. However if \( 0 \in \Omega \), we cannot obtain the information of singularity such the mean curvature, and the fact causes some technical difficulties.

In this paper, we consider the attainability for the following minimization problem

\[
\mu^N_s(\Omega) := \inf \left\{ \int_{\Omega} (|\nabla u|^2 + u^2) \, dx \left| u \in H^1(\Omega), \int_{\Omega} \frac{|u|^{2+\sigma(s)}}{|x|^3} \, dx = 1 \right. \right\}.
\]

The main theorem is as follows.

**Theorem 1** Let \( \partial \Omega \) has a smoothness which the Sobolev embeddings hold, then the following statements hold true.

(I) If \( \Omega \) is sufficiently small, then \( \mu^N_s(\Omega) \) is attained. Especially, if \( \Omega \) satisfies the following:

\[
|\Omega| \left( \int_{\Omega} \frac{1}{|x|^{-s} \, dx} \right)^{-\frac{2}{2+\sigma(s)}} \leq \mu_s
\]

then \( \mu^N_s(\Omega) \) is attained, where \( |\Omega| \) is the \( N \)-dimensional Lebesgue measure of domain \( \Omega \).

(II) There is a positive constant \( M \) which depends on only \( \Omega \) such that \( \mu^N_s(r\Omega) \) is never attained if \( r > M \).

Eventually, the size of domain affects the attainability of \( \mu^N_s(\Omega) \).

The rest of the paper is organized as follows. In Section 2 we introduce three lemmas to prove Theorem 1. Then in Section 3 we prove Theorem 1 using the lemmas in Section 2. In Section 4, as an application, we consider the case when singularity is in the boundary of domain. Then we introduce a new result concerning the attainability of \( \mu^N_s(\Omega) \) with boundary singularity.

**2 Preparation**

In this section, we prepare some lemmas to prove Theorem 1.
Lemma 1 For $r > 0$, the value $\mu_{s, r}^N(\Omega)$ is defined by
\[
\mu_{s, r}^N(\Omega) := \inf \left\{ \left. \int_{\Omega} (|\nabla u|^2 + ru^2) \, dx \right| u \in H^1(\Omega), \int_{\Omega} \frac{|u|^{2^*(s)}}{|x|^s} \, dx = 1 \right\}.
\]

We have
\[
\mu_{s, r}^N(\Omega) = \mu_s^N(r\Omega).
\]

Proof For $r > 0$ and $u \in H^1(\Omega)$, $u_r$ is defined by the scaling of $u$, that is, $u_r(x) := r^{\frac{N-2}{2}} u(x/r) \in H^1(r\Omega)$. Note that
\[
\int_{r\Omega} |\nabla u_r|^2 \, dx = \int_{\Omega} |\nabla u|^2 \, dx,
\]
\[
\int_{r\Omega} u_r^2 \, dx = r^2 \int_{\Omega} u^2 \, dx,
\]
\[
\int_{r\Omega} \frac{u_r^{2^*(s)}}{|x|^s} \, dx = \int_{\Omega} \frac{u^{2^*(s)}}{|x|^s} \, dx.
\]

With these facts in mind, taking $u \in H^1(\Omega)$ such that
\[
\int_{\Omega} \frac{u^{2^*(s)}}{|x|^s} \, dx = 1, \quad \int_{\Omega} (|\nabla u|^2 + r^2 u^2) \, dx \leq \mu_{s, r}^N(\Omega) + \varepsilon
\]
for $\varepsilon > 0$ sufficiently small, we have
\[
\mu_s^N(r\Omega) \leq \int_{r\Omega} (|\nabla u_r|^2 + u_r^2) \, dx = \int_{\Omega} (|\nabla u|^2 + r^2 u^2) \, dx \leq \mu_{s, r}^N(\Omega) + \varepsilon.
\]

Hence we have $\mu_s^N(r\Omega) \leq \mu_{s, r}^N(\Omega)$.

The inverse also holds by replacing $\Omega$ with $r\Omega$.

Lemma 2 There exists a positive constant $C$ which depends on only $\Omega$ such that
\[
\mu_s \left( \int_{\Omega} \frac{|u|^{2^*(s)}}{|x|^s} \, dx \right)^{\frac{2}{2^*(s)}} \leq \int_{\Omega} |\nabla u|^2 \, dx + C \int_{\Omega} u^2 \, dx \quad (u \in H^1(\Omega)). \tag{2}
\]

Before beginning the proof, we make a remark. H. Jaber [12] has shown that the following theorem.

Theorem 2 ([12]) If $(M, g)$ is a compact Riemannian manifold without boundary and $0 \in M$, there is a constant $C = C(M, g)$ such that
\[
\mu_s \left( \int_M \frac{|u|^{2^*(s)}}{d_g(x, 0)^s} \, dv_g \right)^{\frac{2}{2^*(s)}} \leq \int_M |\nabla u|^2 \, dv_g + C \int_{\Omega} u^2 \, dv_g \quad (u \in H^1(M))
\]
where $d_g$ is the Riemannian distance on $M$.

Different from Theorem 2, $\Omega$ is bounded domain of $\mathbb{R}^N$ and therefore $\Omega$ has a boundary, thus we can show the inequality (2) simply.
Proof Let \( 0 \in \Omega_1 \subset \Omega_2 \subset \Omega \) and these two subdomain are taken suitable again later. A cut-off function is defined by \( \phi \) which satisfies
\[
\phi \in C_0^\infty(\mathbb{R}^N), \quad 0 \leq \phi \leq 1 \text{ in } \Omega, \quad \phi = 1 \text{ on } \Omega_1, \quad \phi = 0 \text{ on } \Omega \setminus \Omega_2.
\]
Here, we construct a partition of unity \( \eta_1, \eta_2 \) defined by
\[
\eta_1 := \frac{\phi^2}{\phi^2 + (1 - \phi)^2}, \quad \eta_2 := \frac{(1 - \phi)^2}{\phi^2 + (1 - \phi)^2}.
\]
Note that \( \eta_1^{\frac{1}{2}}, \eta_2^{\frac{1}{2}} \in C^2(\Omega) \) by the definition. We may assume that \( u \in C^\infty(\Omega) \cap H^1(\Omega) \) by density. We have
\[
\mu_s \left( \int_\Omega \frac{|u|^{2^*(s)}}{|x|^s} \, dx \right)^{\frac{2}{2^*(s)}} \leq \mu_s \left( \int_\Omega \frac{|u|^{2^*(s)}}{|x|^s} \, dx \right)^{\frac{2}{2^*(s)}} = \mu_s \left( \int_\Omega \frac{1}{|x|^s} \, dx \right)^{\frac{2}{2^*(s)}} \leq \mu_s \left( \int_\Omega \frac{1}{|x|^s} \, dx \right)^{\frac{2}{2^*(s)}}.
\]
We estimate \( I_1, I_2 \) for each.

For \( I_1 \), since \( \text{supp}\eta_1 \subset \Omega \) we can use the Hardy-Sobolev inequality. We get that
\[
I_1 = \mu_s \left( \int_\Omega \frac{|\eta_1^{\frac{1}{2}} u|^{2^*(s)}}{|x|^s} \, dx \right)^{\frac{2}{2^*(s)}} \leq \int_\Omega \frac{|\nabla (\eta_1^{\frac{1}{2}} u)|^2}{|x|^s} \, dx
\]
\[
= \int_\Omega |\nabla u|^2 \eta_1 \, dx + \int_\Omega (\nabla (\eta_1^{\frac{1}{2}} u)) \cdot (\nabla (\eta_1^{\frac{1}{2}} u))^2 \, dx.
\]
Since \( \eta_1^{\frac{1}{2}} \in C^2(\Omega) \) we may integrate by parts the second term and hence we obtain
\[
I_1 \leq \int_\Omega |\nabla u|^2 \eta_1 \, dx - \int_\Omega \Delta (\eta_1^{\frac{1}{2}}) \eta_1^{\frac{1}{2}} u^2 \, dx \quad \text{(3)}
\]

For \( I_2 \), since \( 0 \not\subset \text{supp}\eta_2 \) and taking account to that \( \eta = 0 \) on \( \Omega_1 \) we have
\[
I_2 = \mu_s \left( \int_\Omega \frac{|\eta_2^{\frac{1}{2}} u|^{2^*(s)}}{|x|^s} \, dx \right)^{\frac{2}{2^*(s)}} = \mu_s \left( \int_{\Omega \setminus \Omega_1} \frac{|\eta_2^{\frac{1}{2}} u|^{2^*(s)}}{|x|^s} \, dx \right)^{\frac{2}{2^*(s)}} \leq \mu_s \cdot a \left( \int_{\Omega \setminus \Omega_1} \frac{|\eta_2^{\frac{1}{2}} u|^{2^*(s)}}{|x|^s} \, dx \right)^{\frac{2}{2^*(s)}} \leq \mu_s \cdot a \cdot |\Omega \setminus \Omega_1|^{\frac{2}{2^*(s)} - \frac{2}{2^*(s)}} \left( \int_{\Omega \setminus \Omega_1} |\eta_2^{\frac{1}{2}} u|^{2^*(s)} \, dx \right)^{\frac{2}{2^*(s)}} \leq \mu_s \cdot a \cdot |\Omega \setminus \Omega_1|^{\frac{2}{2^*(s)} - \frac{2}{2^*(s)}} S(\Omega, \Omega_1)^{-1} \int_{\Omega \setminus \Omega_1} |\nabla (\eta_2^{\frac{1}{2}} u)|^2 \, dx = \mu_s \cdot a \cdot |\Omega \setminus \Omega_1|^{\frac{2}{2^*(s)} - \frac{2}{2^*(s)}} S(\Omega, \Omega_1)^{-1} \int_\Omega |\nabla (\eta_2^{\frac{1}{2}} u)|^2 \, dx
\]
where \( a := \text{dist}(0, \partial \Omega_1)^{-2s/2^*} \) and

\[
S(\Omega, \Omega_1) := \inf \left\{ \int_{\Omega \setminus \Omega_1} |\nabla u|^2 dx \middle| u \in H^1(\Omega), \ u = 0 \text{ on } \partial \Omega_1, \int_{\Omega \setminus \Omega_1} |u|^{2^*} = 1 \right\}.
\]

Here, let us take \( \Omega_0 \subset \Omega_1 \). It is clearly that \( a \leq \text{dist}(0, \partial \Omega_0)^{-2s/2^*} \). On the other hand, for \( u \in H^1(\Omega \setminus \Omega_1) \) such that \( u = 0 \) on \( \partial \Omega_1 \), we define \( v \in H^1(\Omega \setminus \Omega_0) \) by

\[
v := \begin{cases} u & \text{in } \Omega \setminus \Omega_1 \\ 0 & \text{in } \Omega_1 \setminus \Omega_0 \end{cases}
\]

By identifying \( u \in H^1(\Omega \setminus \Omega_1) \) with \( v \in H^1(\Omega \setminus \Omega_0) \) concerning the calculation of the Sobolev quotient, we may see that

\[
\{ u \in H^1(\Omega \setminus \Omega_1) | u = 0 \text{ on } \partial \Omega_1 \} \subset \{ u \in H^1(\Omega \setminus \Omega_0) | u = 0 \text{ on } \partial \Omega_0 \}.
\]

Hence we obtain \( S(\Omega, \Omega_1) \geq S(\Omega, \Omega_0) \). Consequently, if \( \Omega_1 \) is sufficiently large, \( a \) and \( S(\Omega, \Omega_1)^{-1} \) is bounded from above uniformly. By choosing \( \Omega_1 \) and \( \Omega_2 \) close to \( \Omega \) we obtain

\[
I_2 \leq \frac{1}{2} \int_{\Omega} |\nabla (\eta_1^2 u)|^2 dx.
\]

Therefore

\[
I_2 \leq \int_{\Omega} |\nabla u|^2 \eta_2 dx + \int_{\Omega} |\nabla \eta_2^\frac{1}{2} u|^2 dx. \tag{4}
\]

Here, since \( \eta_1^\frac{1}{2}, \eta_2^\frac{1}{2} \in C^2(\Omega) \) there is a positive constant \( C \) such that

\[
\max_{x \in \Omega} |\nabla \eta_1^\frac{1}{2}| \leq \frac{C}{2}, \quad \max_{x \in \Omega} |\nabla \eta_2^\frac{1}{2}|^2 \leq \frac{C}{2}. \tag{5}
\]

This constant depends on only \( \Omega \).

Consequently (3), (4) and (5) yield that

\[
\mu_s \left( \int_{\Omega} \frac{|u|^{2^*(s)}}{|x|^s} dx \right)^{-\frac{2}{2^*(s)}} \leq I_1 + I_2 \leq \int_{\Omega} |\nabla u|^2 dx + C \int_{\Omega} u^2 dx.
\]

**Lemma 3** \( \mu_s^N(\Omega) \leq \mu_s \) holds (see [9], Lemma 11.1). Furthermore, the following statements hold true:

(I) If \( \mu_s^N(\Omega) < \mu_s \), then \( \mu_s^N(\Omega) \) is attained.

(II) If \( \mu_s^N(\Omega) = \mu_s \), then \( \mu_s^N(r\Omega) \) is not attained for all \( r > 1 \).

Firstly, we prove Lemma 3 (I).

**Proof** (Proof of Lemma 3 (I))

Assume \( \{ u_n \}_{n=1}^\infty \subset H^1(\Omega) \) is a minimizing sequence of \( \mu_s^N(\Omega) \). Without loss of generality, we may assume

\[
\int_{\Omega} \frac{|u_n|^{2^*(s)}}{|x|^s} dx = 1 \tag{6}
\]

for all \( n \in \mathbb{N} \) and which implies

\[
\int_{\Omega} (|\nabla u_n|^2 + u_n^2) dx = \mu_s^N(\Omega) + o(1) \quad (n \to \infty). \tag{7}
\]
Thus \( u_n \) is bounded in \( H^1(\Omega) \). So we can suppose, up to a subsequence,\(^\star\)

\[
\begin{align*}
  u_n &\to u \quad \text{in } H^1(\Omega) \\
  u_n &\to u \quad \text{in } L^p(\Omega) \quad (1 \leq p < 2^*) \\
  u_n &\to u \quad \text{in } L^q(\Omega, |x|^{-s}) \quad (1 \leq q < 2^*(s)) \\
  u_n &\to u \quad \text{a.e. in } \Omega
\end{align*}
\]

as \( n \to \infty \).

For this limit function \( u \), we show that \( u \not\equiv 0 \) a.e. in \( \Omega \). Assume that \( u \equiv 0 \) a.e. in \( \Omega \). By the inequality (2) in Lemma 2,

\[
\mu_s \left( \int_\Omega \frac{|u_n|^{2^*(s)}}{|x|^s} \, dx \right)^{\frac{2}{2^*(s)}} \leq \int_\Omega |\nabla u_n|^2 \, dx + C \int_\Omega u_n^2 \, dx \quad (8)
\]

holds for all \( n \). Thus (6), (7), (8) and \( u_n \to u \) in \( L^2(\Omega) \) yield

\[\mu_s \leq \mu_s^N(\Omega) + o(1).\]

Letting \( n \) tend to infinity, we obtain \( \mu_s \leq \mu_s^N(\Omega) \) and which is contradiction in the assumption of \( \mu_s^N(\Omega) < \mu_s \). Consequently \( u \not\equiv 0 \).

By the theorem of Brezis and Lieb (see [3]), we obtain

\[
\int_\Omega \frac{|u_n|^{2^*(s)}}{|x|^s} \, dx = \int_\Omega \frac{|u|^{2^*(s)}}{|x|^s} \, dx + \int_\Omega \frac{|u_n - u|^{2^*(s)}}{|x|^s} \, dx + o(1)
\]

and it follows that

\[
1 = \left( \int_\Omega \frac{|u_n|^{2^*(s)}}{|x|^s} \, dx \right)^{\frac{2}{2^*(s)}} = \left( \int_\Omega \frac{|u|^{2^*(s)}}{|x|^s} \, dx + \int_\Omega \frac{|u_n - u|^{2^*(s)}}{|x|^s} \, dx \right)^{\frac{2}{2^*(s)}} + o(1)
\]

\[
\leq \left( \int_\Omega \frac{|u|^{2^*(s)}}{|x|^s} \, dx \right)^{\frac{2}{2^*(s)}} + \left( \int_\Omega \frac{|u_n - u|^{2^*(s)}}{|x|^s} \, dx \right)^{\frac{2}{2^*(s)}} + o(1).
\]

On the other hand, we have

\[
\left( \int_\Omega \frac{|u|^{2^*(s)}}{|x|^s} \, dx \right)^{\frac{2}{2^*(s)}} + \left( \int_\Omega \frac{|u_n - u|^{2^*(s)}}{|x|^s} \, dx \right)^{\frac{2}{2^*(s)}} \leq \frac{\int_\Omega (|\nabla u|^2 + u^2) \, dx}{\mu_s^N(\Omega)} + \frac{\int_\Omega (|\nabla (u_n - u)|^2 + (u_n - u)^2) \, dx}{\mu_s^N(\Omega)}
\]

\[
= \frac{\int_\Omega (|\nabla u_n|^2 + u_n^2) \, dx}{\mu_s^N(\Omega)} + o(1)
\]

\[
= 1 + o(1).
\]
Hence there exist a limit and we obtain

\[
\lim_{n \to \infty} \left( \int_{\Omega} \frac{|u|^{2^*(s)}}{|x|^s} \, dx + \int_{\Omega} \frac{|u_n - u|^{2^*(s)}}{|x|^s} \, dx \right)^{\frac{2}{2^*(s)}} = \lim_{n \to \infty} \left[ \left( \int_{\Omega} \frac{|u|^{2^*(s)}}{|x|^s} \, dx \right)^{\frac{2}{2^*(s)}} + \left( \int_{\Omega} \frac{|u_n - u|^{2^*(s)}}{|x|^s} \, dx \right)^{\frac{2}{2^*(s)}} \right] = 1.
\]

By the equality condition of the above, we get either

\[ u \equiv 0 \quad \text{a.e. in } \Omega \quad \text{or} \quad u_n \to u \neq 0 \quad \text{in } L^{2^*(s)}(\Omega, |x|^{-s}). \]

Since \( u \neq 0 \) we obtain \( u_n \to u \neq 0 \) in \( L^{2^*(s)}(\Omega, |x|^{-s}) \) and hence this \( u \) is the minimizer of \( \mu_N^s(\Omega) \).

Next, we prove Lemma 3 (II).

Proof (Proof of Lemma 3 (II)) We assume the existence of the minimizer of \( \mu_N^s(r\Omega) \) and derive a contradiction. Let \( u \in H^1(\Omega) \) be a minimizer of \( \mu_N^s(r\Omega) \), then we have

\[ \mu_N^s(r\Omega) = \int_{r\Omega} (|\nabla u|^2 + u^2) \, dx > \int_{r\Omega} (|\nabla u|^2 + \frac{1}{r^2} u^2) \, dx \geq \mu_N^{s,1/r}(r\Omega). \]

By Lemma 1, the assumption \( \mu_N^s(\Omega) = \mu_s \) and \( \mu_N^s(r\Omega) \leq \mu_s \), we have

\[ \mu_s \geq \mu_N^s(r\Omega) = \mu_N^{s,1/r}(r\Omega) = \mu_N^s(\Omega) = \mu_s. \]

This is a contradiction.

3 Proof of Theorem 1

In this section, we prove Theorem 1.

Proof (Proof of Theorem 1 (I)) We recall that

\[ \mu_N^s(\Omega) := \inf \left\{ \int_{\Omega} (|\nabla u|^2 + u^2) \, dx \mid u \in H^1(\Omega), \int_{\Omega} \frac{|u|^{2^*(s)}}{|x|^s} \, dx = 1 \right\}. \]

Taking a constant \( C \) such that \( \int_{\Omega} \frac{C^{2^*(s)}}{|x|^s} = 1 \) and \( u \equiv C \) as a test function, it follows that

\[ \mu_N^s(\Omega) \leq |\Omega| \left( \int_{\Omega} |x|^{-s} \right)^{-\frac{2}{2^*(s)}}. \]

If this \( C \) is a minimizer of \( \mu_N^s(\Omega) \), then by Lagrange multiplier theorem \( C \) is a classical solution of

\[
\begin{cases}
-\Delta u + u = \mu_N^s(\Omega) \frac{u^{2^*(s)}}{|x|^s} & \text{in } \Omega \\
\frac{\partial u}{\partial n} = 0 & \text{on } \partial \Omega.
\end{cases}
\]

This contradicts and therefore

\[ \mu_N^s(\Omega) < |\Omega| \left( \int_{\Omega} |x|^{-s} \right)^{-\frac{2}{2^*(s)}}. \]

Combining this estimate and Lemma 3 (I), Theorem 1 (I) holds true.
Proof (Proof of Theorem 1 (II)) Since Lemma 2, We can define a constant $m$ by

$$m := \inf \{ C > 0 \mid (2) \text{ holds} \}. \quad (9)$$

$M$ is defined by $M := \sqrt{m}$. In inequality (2), $C$ is replaced by $M^2$ and hence we have

$$\mu_s \leq \frac{\int_{\Omega} (|\nabla u|^2 + M^2 u^2) \, dx}{\left( \int_{\Omega} \frac{|u|^{2^*(s)}}{|x|^s} \, dx \right)^{\frac{2}{2^*(s)}}} \tag{9}$$

for all $u \in H^1(\Omega)$. Therefore by Lemma 1 we obtain

$$\mu_s \leq \inf_{u \in H^1(\Omega) \setminus \{0\}} \frac{\int_{\Omega} (|\nabla u|^2 + M^2 u^2) \, dx}{\left( \int_{\Omega} \frac{|u|^{2^*(s)}}{|x|^s} \, dx \right)^{\frac{2}{2^*(s)}}}$$

$$= \mu^N_{s,M}(\Omega)$$

$$= \mu^N_{s}(M \Omega).$$

Recall that $\mu^N_{s}(\Omega) \leq \mu_s$ holds for all bounded domain $\Omega$ and thus $\mu^N_{s}(M \Omega) = \mu_s$. Consequently we obtain the result of Theorem 1 (II) by Lemma 3 (II).

### 4 Singularity on the boundary

Throughout this section, assume that $0 \in \partial \Omega$. If the mean curvature of $\partial \Omega$ at $0$ is positive, we have obtained the results in Section 1. However, if the mean curvature of $\partial \Omega$ at $0$ vanishes, we don’t obtain results so far, even if the attainability of $\mu^N_{s}(\Omega)$. In this section, we show the following results by using the strategy in Section 2 and Section 3.

**Theorem 3** Let $\Omega \subset \mathbb{R}^N$ is bounded domain with smooth boundary, $0 \in \partial \Omega$ and $\partial \Omega$ is flat near the origin. Then the following statements hold;

(I) If $\Omega$ is sufficiently small, then $\mu^N_{s}(\Omega)$ is attained. Especially, if $\Omega$ satisfies the following:

$$|\Omega| \left( \int_{\Omega} |x|^{-s} \, dx \right)^{-\frac{2}{2^*(s)}} \leq \frac{\mu_s}{2 \pi^{\frac{s}{2}}},$$

then $\mu^N_{s}(\Omega)$ is attained.

(II) There is a positive constant $M$ which depends on only $\Omega$ such that $\mu^N_{s}(r \Omega)$ is never attained if $r > M$.

This condition of the boundary in this theorem is a special case of vanishing of the mean curvature of $\partial \Omega$ at $0$.

We prove the theorem in the same way as in Section 2 and Section 3. Different from the proof of Theorem 1, we need the following lemma instead of Lemma 2.

**Lemma 4** There is a positive constant $C$ depends on only $\Omega$ such that

$$\frac{\mu_s}{2 \pi^{\frac{s}{2}}} \left( \int_{\Omega} \frac{|u|^{2^*(s)}}{|x|^s} \, dx \right)^{\frac{2}{2^*(s)}} \leq \int_{\Omega} |\nabla u|^2 \, dx \int_{\Omega} u^2 \, dx \quad (u \in H^1(\Omega)). \tag{10}$$
Proof We introduce some notation. \( B_r(0) \) is an open ball which center is origin and radius is \( R \). \( \mathbb{R}^N_+ \) is a half space which is defined by \( \mathbb{R}^N_+ := \{(x', x_N) \in \mathbb{R}^N | x_N > 0 \} \) where \( x' := (x_1, \ldots, x_{N-1}) \in \mathbb{R}^{N-1} \).

Since \( \partial \Omega \) is flat near the origin, by rotating coordinate there is a constant \( r > 0 \) such that \( B_r(0) \cap \Omega = B_r^+(0) := B_r(0) \cap \mathbb{R}^N_+ \). For \( u \in H^1(\Omega) \) we have

\[
\left( \int_{\Omega} \frac{|u|^{2^*(s)}}{|x|^s} dx \right)^{\frac{2}{2^*(s)}} = \left( \int_{B_r^+(0)} \frac{|u|^{2^*(s)}}{|x|^s} dx + \int_{\Omega \setminus B_r^+(0)} \frac{|u|^{2^*(s)}}{|x|^s} dx \right)^{\frac{2}{2^*(s)}} \\
\leq \left( \int_{B_r^+(0)} \frac{|u|^{2^*(s)}}{|x|^s} dx \right)^{\frac{2}{2^*(s)}} + \left( \int_{\Omega \setminus B_r^+(0)} \frac{|u|^{2^*(s)}}{|x|^s} dx \right)^{\frac{2}{2^*(s)}} = J_1 + J_2.
\]

For \( u \in H^1(B_r^+(0)), \tilde{u} \in H^1(B_r(0)) \) is defined by the even reflection for the direction \( x_N \), that is,

\[
\tilde{u}(x', x_N) := \begin{cases} 
  u(x', x_N) & \text{if } 0 \leq x_N < 1 \\
  u(x', x_N) & \text{if } -1 < x_N < 0.
\end{cases}
\]

Concerning \( J_1 \), by Lemma 2 we have

\[
J_1 = \left( \int_{B_r^+(0)} \frac{|\tilde{u}|^{2^*(s)}}{|x|^s} dx \right)^{\frac{2}{2^*(s)}} \\
= \left( \frac{1}{2} \right)^{\frac{2}{2^*(s)}} \left( \int_{B_r(0)} \frac{\tilde{u}^{2^*(s)}}{|x|^s} dx \right)^{\frac{2}{2^*(s)}} \\
\leq \left( \frac{1}{2} \right)^{\frac{2}{2^*(s)}} \mu_s^{-1} \left( \int_{B_r(0)} |\nabla \tilde{u}|^2 dx + C_1 \int_{B_r(0)} \tilde{u}^2 dx \right) \\
= \left( \frac{1}{2} \right)^{\frac{2}{2^*(s)}} \mu_s^{-1} \cdot 2 \left( \int_{B_r^+(0)} |\nabla u|^2 dx + C_1 \int_{B_r^+(0)} u^2 dx \right) \\
= \left( \frac{\mu_s}{2} \right)^{-1} \left( \int_{B_r^+(0)} |\nabla u|^2 dx + C_1 \int_{B_r^+(0)} u^2 dx \right)
\]

for some positive constant \( C_1 \) depends on only \( B_r(0) \).

Next, we estimate \( J_2 \). Let \( \delta > 0 \) for sufficiently small. We consider \( \{\phi_i\}_{i=1}^m \) a partition of unity on \( \Omega \setminus B_r^+(0) \) such that \( \phi_i^2 \in C^1 \) and \( |\text{supp } \phi_i| \leq \delta \) for all \( i \). Since \( |x|^{-s} \leq r^{-s} \) for \( x \in \Omega \setminus B_r^+(0) \) we have

\[
J_2 = \left( \int_{\Omega \setminus B_r^+(0)} \frac{|u|^{2^*(s)}}{|x|^s} dx \right)^{\frac{2}{2^*(s)}} \leq m \left( \int_{\Omega \setminus B_r^+(0)} \frac{|\phi_i^2 u|^{2^*(s)}}{|x|^s} dx \right)^{\frac{2}{2^*(s)}} \\
= r^{-\frac{2s}{2^*(s)}} \sum_{i=1}^m \left( \int_{\Omega \setminus B_r^+(0)} \frac{|\phi_i^2 u|^{2^*(s)}}{|x|^s} dx \right)^{\frac{2}{2^*(s)}}.
\]
By Hölder inequalities it follows that
\[
\left( \int_{\Omega \setminus B^+_r(0)} |\phi_i^{1/2} u|^{2^*(s)} \, dx \right)^{\frac{2}{2^*(s)}} \leq |\text{supp} \phi_i|^{\frac{2}{2^*(s)}} \frac{s}{r} \|\phi_i^{1/2} u\|_{L^{2^*(s)}(\Omega \setminus B^+_r(0))}^2 \leq \delta^{\frac{2}{2^*(s)}} \|\phi_i^{1/2} u\|_{L^{2^*(s)}(\Omega \setminus B^+_r(0))}^2
\]
for each \( i \in \mathbb{N} \). Since \( \delta \) is sufficiently small, by using the Sobolev inequalities (If necessary we use the Sobolev inequalities of mixed boundary condition version.) we have
\[
J_2 \leq \left( \frac{\mu_s}{2^{\frac{n-1}{s}}} \right)^{-1} \cdot \frac{1}{2} \sum_{i=1}^{m} \int_{\Omega \setminus B^+_r(0)} |\nabla (\phi_i^{1/2} u)|^2 \, dx.
\]
Consequently we have
\[
J_2 \leq \left( \frac{\mu_s}{2^{\frac{n-1}{s}}} \right)^{-1} \left( \int_{\Omega \setminus B^+_r(0)} |\nabla u|^2 \, dx + C_2 \int_{\Omega \setminus B^+_r(0)} u^2 \, dx \right).
\]
for some positive constant \( C_2 \) depends on only \( \Omega \setminus B^+_r(\Omega) \). Combining the estimates of \( J_1 \) and \( J_2 \) we obtain
\[
\left( \int_{\Omega} \frac{|u|^{2^*(s)}}{|x|^s} \right)^{\frac{2}{2^*(s)}} \leq J_1 + J_2 \leq \left( \frac{\mu_s}{2^{\frac{n-1}{s}}} \right)^{-1} \left( \int_{\Omega} |\nabla u|^2 \, dx + C \int_{\Omega} u^2 \, dx \right)
\]
for some positive constant \( C \) depends on \( \Omega \).

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**References**