SOME IMPROVEMENTS FOR A CLASS OF THE 
CAFFARELLI-KOHN-NIRENBERG INEQUALITIES

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Abstract. In this paper, we concern a weighted version of the 
Hardy inequality, which is a special case of the more general Caffarelli-
Kohn-Nirenberg inequalities. We improve the inequality on the 
whole space or on a bounded domain by adding various remainder 
terms. On the whole space, we show the existence of a remain-
der term which has the form of ratio of two weighted integrals. 
Also we give a simple derivation of the remainder term involving 
a distance from the manifold of the “virtual extremals”. Finally 
on a bounded domain, we prove the existence of remainder terms 
involving the gradient of functions.

1. Introduction

In this paper, we are concerned with the weighted version of the 
Hardy inequality:

\[
\int_{\Omega} |\nabla u|^p |x|^{-pa} \, dx \geq \left( \frac{N - p - pa}{p} \right)^p \int_{\Omega} \frac{|u|^p}{|x|^{p(a+1)}} \, dx
\]

for all \( u \in C_0^\infty(\Omega) \), where \( \Omega \) is a smooth bounded domain in \( \mathbb{R}^N \) \( (N \geq 3) \) 
with \( 0 \in \Omega \), or \( \Omega = \mathbb{R}^N \), \( 1 < p < N \) and \( -\infty < a < \frac{N-p}{p} \). Actually, 
much more general weighted type inequalities are shown by Caffarelli, 
Kohn and Nirenberg [4] and (1.1) is one of the special cases. Let 
\( D_{0,a}^{1,p}(\Omega) \) and \( W_{0,a}^{1,p}(\Omega) \) be the completion of \( C_0^\infty(\Omega) \) with respect to each 
norm

\[
\|u\|_{D_{0,a}^{1,p}(\Omega)} = \left( \int_{\Omega} |\nabla u|^p |x|^{-pa} \, dx \right)^{1/p},
\]

\[
\|u\|_{W_{0,a}^{1,p}(\Omega)} = \left( \int_{\Omega} (|\nabla u|^p + |u|^p) |x|^{-pa} \, dx \right)^{1/p},
\]

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respectively. Then (1.1) holds true for all \( u \in D^{1,p}_{0,\alpha}(\Omega) \). If \( \Omega \) is bounded, the Poincaré type inequality implies that \( D^{1,p}_{0,\alpha}(\Omega) = W^{1,p}_{0,\alpha}(\Omega) \). Also the constant \( \left( \frac{N-p-\alpha p}{p} \right)^p \) in (1.1) is known optimal and never attained in \( D^{1,p}_{0,\alpha}(\Omega) \).

When \( \alpha = 0 \), (1.1) becomes the classical Hardy inequality

\[
\int_{\Omega} |\nabla u|^p dx \geq \left( \frac{N-p}{2} \right)^2 \int_{\Omega} \frac{|u|^2}{|x|^2} dx,
\]

again the equality in (1.2) is never achieved by any function in \( D^{1,p}_{0,0}(\Omega) \).

There are many papers up to now that treat the improvement of (1.2) when \( \Omega \) is a smooth bounded domain (see [2], [3], [6], [8], [10], [11], [12], [17], [18], and references therein). On the other hand, when \( \Omega = \mathbb{R}^N \), Ghoussoub and Moradifam [11] show that there is no strictly positive \( V \in C^1((0,1)) \) such that the inequality

\[
\int_{\mathbb{R}^N} |\nabla u|^2 dx \geq \left( \frac{N-2}{2} \right)^2 \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} dx + \int_{\mathbb{R}^N} V(|x|)|u|^2 dx
\]

holds for all \( u \in W^{1,2}(\mathbb{R}^N) \). Therefore we cannot expect the same type of remainder terms as in the bounded domain case, one of the reasons of what is a lack of the Poincaré inequality in the whole space. Instead, Cianchi and Ferone [7] provided the following “non-standard” remainder term: Let \( p^* = \frac{Np}{N-p} \) be the critical Sobolev exponent, \( u_0(x) = |x|^{\frac{N-p}{p}} \) for \( x \in \mathbb{R}^N \), and define

\[
d_p(u) = \inf_{c \in \mathbb{R}} \frac{\|u - cu_0\|_{L^{p^*}((\mathbb{R}^N))}}{\|u\|_{L^{p^*}((\mathbb{R}^N))}} \quad (1 < p < N).
\]

Here \( L^{\tau,\sigma}(\mathbb{R}^N) \) \((0 < \tau \leq \infty, 1 \leq \sigma \leq \infty)\) is the Lorentz space with the norm

\[
\|u\|_{L^{\tau,\sigma}(\mathbb{R}^N)} = \|s^{\frac{1}{\tau} - \frac{1}{\sigma}} u^*(\cdot)\|_{L^\sigma((0,\infty))},
\]

where \( u^* \) denotes the (one-dimensional) decreasing rearrangement of \( u \). Then in [7] it is shown that for any \( 1 < p < N \) there exists a constant \( C = C(p, N) \) such that

\[
\int_{\mathbb{R}^N} |\nabla u|^p dx \geq \left( \frac{N-p}{p} \right)^p \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^p} dx \left( 1 + C d_p(u)^{2p^*} \right)
\]

holds for every real-valued weakly differentiable function \( u \) in \( \mathbb{R}^N \) decaying to zero at infinity with \( |\nabla u| \in L^p(\mathbb{R}^N) \). Recently, the authors of this paper have succeeded to obtain a remainder term for the classical Hardy inequality (1.2) on the whole space [16]. Our method consists
Concerning the inequality (1.1), on the other hand, Wang and Willem [19] obtained the following improved version of (1.1) on a bounded domain $\Omega \subset \mathbb{R}^N$: Assume $\Omega \subset \subset B_R(0)$ for some $R > 0$. Then

$$\int_\Omega |\nabla u|^2 |x|^{-2a} \, dx - \left( \frac{N - 2 - 2a}{2} \right)^2 \int_\Omega \frac{|u|^2}{|x|^{2(a+1)}} \, dx \geq C \int_\Omega \left( \log \left( \frac{R}{|x|} \right) \right)^{-2} |x|^{-2a} |\nabla u|^2 \, dx$$

holds for all $u \in D^{1,2}_{0,a}(\Omega)$, where $C = C(\Omega, a)$ is a positive constant. Their method consists of the use of a conformal transformation introduced by Catrina and Wang [5], which transforms the problem on a bounded domain of $\mathbb{R}^N$ to that of a cylinder $C = \mathbb{R} \times S^{N-1}$; see §4. Later, Abdellaoui, Colorado and Peral [1] obtained the following improvement of (1.1): Let $|\Omega|$ denote the volume of $\Omega$. Then for all $1 < q < p$, there exists a positive constant $C = C(N, p, q, a, |\Omega|)$ such that the inequality

$$\int_\Omega |\nabla u|^p |x|^{-pa} \, dx - \left( \frac{N - p - pa}{p} \right)^p \int_\Omega \frac{|u|^p}{|x|^{p(a+1)}} \, dx \geq C \left( \int_\Omega |x|^{-ra} |\nabla u|^q \, dx \right)^{p/q}$$

holds true for all $u \in D^{1,p}_{0,a}(\Omega)$, where $r$ is any number such that $q \leq r < \infty$ if $a \leq 0$ and $1 \leq r < p + \rho$ for some positive constant $\rho$ when $a > 0$. Their method is based on the Picone type inequality for the operator $\text{div}(|x|^{-pa} \nabla |u|^{p-2} \nabla u)$. Note that from the proof of [1], the constant $C(N, p, q, a, |\Omega|) \to 0$ as $|\Omega| \to \infty$.

In this paper, firstly we improve the inequality (1.1) when $\Omega = \mathbb{R}^N$ by adding a remainder term of the form of ratio of two weighted integrals to (1.1).

**Theorem 1.** (A remainder term of the form of ratio on the whole space) Let $N \geq 3$, $2 \leq p < N$ and $-\infty < a < -\frac{N-p}{p}$. For given $n \in \mathbb{N}$, $t \in (0, 1)$ and $\gamma < \min\{1-t, \frac{p-n}{p}\}$, set $\delta = n - N + \frac{n}{1-t-\gamma} \left( \gamma + \frac{N-p-pa}{p} \right)$. 


Then there exists a constant $C > 0$ such that the inequality
\[
\int_{\mathbb{R}^N} |\nabla u|^p |x|^{-pa} \, dx - \left( \frac{N - p - pa}{p} \right) \int_{\mathbb{R}^N} |u|^p |x|^{(a+1)p} \, dx
\]
holds for all radially symmetric function $u \in W^{1,p}_{0,\lambda}(\mathbb{R}^N)$, $u \neq 0$.

**Remark 1.** If we put $F_{A,B}(u) = \int_{\mathbb{R}^N} |u|^A |x|^B \, dx$ for constants $A, B$, and put $u_\lambda(x) = \lambda^C u(\lambda x)$ for $x \in \mathbb{R}^N$ and $C \in \mathbb{R}$, then a simple computation shows that $F_{A,B}(u_\lambda) = \lambda^{AC-B-N} F_{A,B}(u)$. The remainder term in Theorem 1 can be written as
\[
R(u) = \frac{\int_{\mathbb{R}^N} |u|^{\frac{n}{1-t}} |x|^{\frac{N}{1-t}} \, dx}{\left( \int_{\mathbb{R}^N} |u|^p |x|^{-pa} \, dx \right)^{\frac{1}{1-t}}}
\]
and
\[
R(u_\lambda) = \left\{ \frac{\lambda^{p(\frac{n}{1-t}) \left( \frac{N-p-pa}{p} - \delta - N \right)} \frac{(1-t-\gamma)}{\alpha}}{\left\{ \lambda^{p(\frac{n}{1-t}) \left( \frac{N-p-pa}{p} + pa - N \right)} \frac{(1-t-\gamma)}{\alpha} \right\} \frac{(1-t-\gamma)}{\alpha}} R(u) = \left( \frac{\lambda^{-p(\frac{n}{1-t})}}{\lambda^{-p(\frac{n}{1-t})}} \right) R(u)
\]
\[
= R(u).
\]
Thus the remainder term is invariant under the scaling $u(x) \mapsto u_\lambda(x) = \lambda^\frac{N-p-pa}{p} u(\lambda x)$.

**Remark 2.** If we choose $n, t, \gamma$ satisfying $\frac{n}{1-t-\gamma} = p$, then (1.3) in Theorem 1 also holds even for non-radial functions. Indeed, for a non-radial function $u$, let us consider the radial function
\[
U(r) = \left( \omega_N^{\frac{1}{p}} \int_{S_{N-1}} |u(r\omega)|^p \, dS_\omega \right)^{\frac{1}{p}}.
\]
Then Hölder’s inequality implies that
\[
U'(r) \leq \left( \omega_N^{\frac{1}{p}} \int_{S_{N-1}} |\partial_r u(r\omega)|^p \, dS_\omega \right)^{\frac{1}{p}}
\]
so we obtain
\begin{equation}
\omega_N \int_0^\infty |U'(r)|^p r^{N-1-ps} dr \leq \int_{\mathbb{R}^N} \left| \nabla u - \frac{x}{|x|} \right|^p |x|^{-ps} dx.
\end{equation}
Also we have
\begin{equation}
\int_{\mathbb{R}^N} |U(|x|)|^p |x|^A dx = \int_{\mathbb{R}^N} |u|^p |x|^A dx
\end{equation}
for any $A \in \mathbb{R}$. Thus when $\frac{n}{1-t-\gamma} = p$, (1.3) for $U$, (1.4) and (1.5) imply that the same inequality holds for all non-radial functions.

Note that the standard rearrangement argument, see for example [13], is not applicable because of the presence of weights, since $a$ may be negative and $|x|^{-p(a+1)}$ may be an increasing function. Thus, differently from [16], here we use another type of pointwise estimate for the expansion of $|a - b|^p$, see Lemma 1. The use of full version of the Caffarelli-Kohn-Nirenberg inequality, see Proposition 1, is another key point. Since the technique used here is different from that of [16], the remainder term obtained in Theorem 1 is also different from that in [16] even when $a = 0$.

In §3, we improve (1.1) on the whole space by adding a remainder term which involves a distance from “the manifold of the virtual extremals” $\{cu \mid c \in \mathbb{R}\}$, where $u_a(x) = |x|^{-\frac{n-2a}{p-2}}$. For the proof, differently from that of Cianchi-Ferone [7], we use a new inequality recently obtained by Machihara, Ozawa and Wadade [14]. It is surprising for the authors that a direct use of the Machihara-Ozawa-Wadade inequality leads to the existence of “non-standard” type remainder terms for (1.1) very simply, at least in the radially symmetric case.

In §4, we will show some improvements of (1.1) when $p = 2$:
\[\int_\Omega |\nabla u|^2 |x|^{-2a} dx \geq \left( \frac{N - 2 - 2a}{2} \right)^2 \int_\Omega \frac{|u|^2}{|x|^{2(a+1)}} dx,\]
where $\Omega$ is a bounded domain in $\mathbb{R}^N$ ($N \geq 3$). In this part, the method of the proof is to combine two ideas: One is to transform the problem to the one on a cylinder, which was initiated by Catrina and Wang [5] in this context, and the another is to improve the one-dimensional Hardy (weighted Poincaré) inequality by the method of Picone’s identity see [11], [12] for $p = 2$ case and [1] for $p \neq 2$ case.

2. Proof of Theorem 1.

In this section, we prove Theorem 1. For the proof, we need the following lemma.
Lemma 1 ([9]). Let $p \geq 2$ and $a, b$ be real numbers. Then there exists $c_p > 0$ such that

$$|a - b|^p \geq |a|^p - p|a|^{p-2}ab + c_p|b|^p$$

holds true. $c_p$ is explicitly given as $c_p = \min_{0 < t \leq 1/2} ((1 - t)^p - t^p + pt^{p-1})$ and sharp in this inequality.

We recall here the following inequality obtained by Caffarelli, Kohn and Nirenberg [4] in its full version:

Proposition 1. (Caffarelli-Kohn-Nirenberg [4]) Let $n \in \mathbb{N}$ and let $p, q, r, \alpha, \beta, \sigma$ and $t$ be real constants such that $p, q \geq 1$, $r > 0$, $0 \leq t \leq 1$, and

$$\frac{1}{p} + \frac{\alpha}{n} = \frac{1}{q} + \frac{\beta}{n} = \frac{1}{r} + \frac{\gamma}{n} > 0$$

where $\gamma = t\sigma + (1 - t)\beta$. Then there exists a positive constant $C$ such that

$$\left\| |x|^\gamma u \right\|_{L^r(\mathbb{R}^n)} \leq C \left\| |x|^\alpha |\nabla u| \right\|_{L^p(\mathbb{R}^n)} t \left\| |x|^\beta u \right\|_{L^q(\mathbb{R}^n)}^{1-t}$$

holds for all $u \in C_0^\infty(\mathbb{R}^n)$, if and only if the following conditions hold:

1. (balance of dimension)

$$\frac{1}{r} + \frac{\gamma}{n} = t \left( \frac{1}{p} + \frac{\alpha - 1}{n} \right) + (1 - t) \left( \frac{1}{q} + \frac{\beta}{n} \right),$$

2. $0 \leq \alpha - \sigma$ if $t > 0$,
3. $1 \geq \alpha - \sigma$ if $t > 0$ and $\frac{1}{r} + \frac{\gamma}{n} = \frac{1}{p} + \frac{\alpha - 1}{n}$.

Proof of Theorem 1. We show Theorem 1 for a radial function $u \in C_0^\infty(\mathbb{R}^N)$. Then a density argument implies the desired result. Since $u$ is radial, $u$ can be written as $u(x) = \tilde{u}(|x|)$ for some function $\tilde{u} \in C_0^\infty([0, +\infty))$. Appealing to Brezis-Vázquez’s idea, we put

$$\tilde{v}(r) = r \frac{N-p-pn}{r} \tilde{u}(r).$$
We see \( \tilde{v}(0) = 0 \) since \( a < \frac{N-p}{p} \) and \( \tilde{v}(+\infty) = 0 \), since \( \tilde{u} \equiv 0 \) near \( r = +\infty \). Put \( v(y) = \tilde{v}(|y|) \) for \( y \in \mathbb{R}^n, n \in \mathbb{N} \). Calculation shows that

\[
J := \int_{\mathbb{R}^n} |\nabla u|^p |x|^{-pa} \, dx - \left( \frac{N - p - pa}{p} \right)^p \int_{\mathbb{R}^n} \frac{|u|^p}{|x|^{p(a+1)}} \, dx
\]

\[
= \omega_N \int_0^\infty |\tilde{u}'(r)|^{p - pa} r^{-1} \, dr - \omega_N \left( \frac{N - p - pa}{p} \right)^p \int_0^\infty |\tilde{u}(r)|^{p - p(a+1)} r^{-1} \, dr
\]

\[
= \omega_N \left( \frac{N - p - pa}{p} \right)^p \int_0^\infty |\tilde{u}(r)|^{p - 1} \, dr.
\]

We apply Lemma 1 to the integrand of the first term:

\[
\left| \left( \frac{N - p - pa}{p} \right)^p r^{-\frac{N-pa}{p}} \tilde{v}(r) - r^{-\frac{N-pa}{p}} \tilde{v}'(r) \right|^p r^{N-1-pa}
\]

\[
\geq \left[ \left( \frac{N - p - pa}{p} \right)^p r^{-N+pa} |\tilde{v}(r)|^p
\right]^{p-1}
\]

\[
- \frac{p}{p} \left( \frac{N - p - pa}{p} \right)^{p-1} |\tilde{v}(r)|^{p-2} \tilde{v}(r) \tilde{v}'(r) r^{-\left( \frac{N-pa}{p} \right)(p-1)} r^{-\left( \frac{N-pa}{p} \right)}
\]

\[
+ c_p |\tilde{v}'(r)|^{p-1} r^{N-1-pa}
\]

\[
= \left( \frac{N - p - pa}{p} \right)^p r^{-1} |\tilde{v}(r)|^p - \frac{p}{p} \left( \frac{N - p - pa}{p} \right)^{p-1} |\tilde{v}(r)|^{p-2} \tilde{v}(r) \tilde{v}'(r)
\]

\[
+ c_p |\tilde{v}'(r)|^{p-1}.
\]

By using the fact \( \tilde{v}(0) = \tilde{v}(+\infty) = 0 \) and \( p \geq 2 \), we see

\[
p \int_0^\infty |\tilde{v}(r)|^{p-2} \tilde{v}(r) \tilde{v}'(r) \, dr = \int_0^\infty \frac{d}{dr} (|\tilde{v}(r)|^p) \, dr = 0.
\]

Finally, we note that the terms involving \( \int_0^\infty |\tilde{v}(r)|^{p-1} \, dr \) cancel out by subtracting each other. Thus we obtain

\[
(2.3) \quad J \geq c_p \omega_N \int_0^\infty |\tilde{v}'(r)|^p r^{p-1} \, dr = c_p \frac{\omega_N}{\omega_n} \int_{\mathbb{R}^n} |\nabla v(y)|^p |y|^{p-n} \, dy.
\]

From now on, we estimate the right hand side of (2.3) by using the Caffarelli-Kohn-Nirenberg inequality (2.1) (Proposition 1) for \( v \) on \( \mathbb{R}^n \).

We take

\[ q = p \quad \text{and} \quad \alpha = \beta = \frac{p-n}{p}. \]
By these choices, we see
\[ \frac{1}{p} + \frac{\alpha}{n} = \frac{1}{q} + \frac{\beta}{n} = \frac{1}{n} > 0. \]

The first condition of the Proposition 1 (balance of dimension) reduces to
\[ \frac{1}{r} + \frac{n}{n} = \frac{1}{n}, \quad \text{i.e.,} \quad r = \frac{n}{1 - t - \gamma}. \]

Thus \( \frac{1}{r} + \frac{n}{n} > 0 \) if \( t < 1 \), and by the assumption \( \gamma < \min\{1 - t, \frac{p - n}{p}\} \), \( r \)

is positive and the condition \( \alpha \geq \sigma = \frac{\gamma}{t} \) when \( t > 0 \) is fulfilled.

Also under the choice \( 0 < t < 1 \), we see \( 0 = \frac{p}{n} + \frac{1}{n} \neq \frac{r}{n} + \frac{n}{n} = \frac{1 - t}{n} \),

thus we do not need to consider the third condition. In conclusion, we assure that the following inequality holds true for \( v \):
\[ \left\| y^{\gamma} v \right\|_{L^{\frac{n}{1 - t - \gamma}}(\mathbb{R}^n)} \leq C \left\| y^{\frac{n - \gamma}{n - t}} \left| \nabla v \right| \right\|_{L^p(\mathbb{R}^n)} \left\| y^{\frac{p - \gamma}{p}} v \right\|_{L^p(\mathbb{R}^n)}^{1 - t}, \]

that is,
\[ \left( \int_{\mathbb{R}^n} |y|^{\frac{n - \gamma}{n - t}} |v|^{\frac{p - \gamma}{p}} dy \right)^{\frac{n(t - 1)}{n(t - 1 - \gamma)}} \leq C \left( \int_{\mathbb{R}^n} |y|^{p - n} |\nabla v|^{p} dy \right) \left( \int_{\mathbb{R}^n} |y|^{p - n} |v|^{p} dy \right)^{\frac{1 - t}{t}}. \]

Combining this to (2.3), we have
\[ (2.4) \quad J \geq C' \left( \frac{\int_{\mathbb{R}^n} |y|^{\frac{n\gamma}{n - t}} |v|^{\frac{n}{t - \gamma}} dy}{\frac{n(t - 1)}{n(t - 1 - \gamma)}} \right)^{\frac{n(t - 1)}{n(t - 1 - \gamma)}} \left( \frac{\int_{\mathbb{R}^n} |y|^{p - n} |v|^{p} dy}{\frac{n(t - 1)}{n(t - 1 - \gamma)}} \right)^{\frac{1 - t}{t}} \]

where \( C' = C^{-1} C_r \omega_N^N \). By the definition \( v(y) = \hat{v}(|y|), y \in \mathbb{R}^n \) and the assumption that \( u \in W^{1,p}_0(\mathbb{R}^N) \), we have
\[ \int_{\mathbb{R}^n} |y|^{p - n} |v|^{p} dy = \omega_n \int_{0}^{\infty} r^{N - p - p\alpha} |\hat{v}(r)|^{p} r^{n - n\alpha - 1} dr \]
\[ = \frac{\omega_n}{\omega_N} \int_{\mathbb{R}^N} |u|^p |x|^{-p\alpha} dx < \infty. \]

On the other hand,
\[ \int_{\mathbb{R}^n} |y|^{\frac{n\gamma}{n - t}} |v|^{\frac{n}{n - t}} dy = \omega_n \int_{0}^{\infty} \left( r^{N - p - p\alpha} |\hat{u}(r)| \right)^{\frac{n}{n - t}} r^{\frac{n\gamma}{n - t}} r^{n - n\alpha - 1} dr \]
\[ = \frac{\omega_n}{\omega_N} \int_{\mathbb{R}^N} |u|^\frac{n\gamma}{n - t} |x|^{\gamma} dx. \]

Inserting these into (2.4), we obtain the desired conclusion. \( \Box \)
Remark 3. If we assume that the function \( u \in W^{1,p}_{0,a}(\mathbb{R}^N) \) is nonnegative, radially symmetric and radially decreasing, then we can apply the same argument in [16] also in our situation. In this case we obtain the following theorem, the proof of it is exactly the same as in [16].

**Theorem 2.** For given \( N \geq 3, \ 2 \leq p < N \) and \( q > 2 \), set \( \alpha = \alpha(p,q,N) = 2 - N + \frac{q(N-p-pa)}{2} \). Then there exists \( D = D(p,q,N) > 0 \) such that the inequality

\[
\int_{\mathbb{R}^N} |\nabla u|^p |x|^{-pa} dx - \left( \frac{N-p-pa}{p} \right)^p \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^{p(a+1)}} dx \\
\geq D \left( \frac{\int_{\mathbb{R}^N} |u|^\frac{p}{2} |x|^\alpha dx}{\int_{\mathbb{R}^N} |u|^p |x|^{2-p-pa} dx} \right)^{\frac{2}{p-2}}
\]

holds for all nonnegative, radially symmetric and radially decreasing function \( u \in W^{1,p}_{0,a}(\mathbb{R}^N), \ u \neq 0 \).

3. A REMAINDER TERM INVOLVING A DISTANCE FROM THE VIRTUAL EXTREMALS

For \(-\infty < a < \frac{N-p}{p}\), let \( u_a(x) = |x|^{-\frac{N-p-pa}{p}} \). Note that \( u_a \) is a solution to the Euler-Lagrange equation associated with the best constant of the inequality (1.1)

\[
-\text{div} \left( |x|^{-pa} |\nabla u|^{p-2} \nabla u \right) = \left( \frac{N-p-pa}{p} \right)^p |x|^{-p(a+1)} u^{p-1},
\]

\( u \geq 0 \) in \( \mathbb{R}^N \),

however, \( u_a \notin D^{1,p}_{0,a}(B_R(0)) \) for any \( R > 0 \). Thus \( u_a \) is not a genuine minimizer for the best constant of (1.1) in the admissible class \( D^{1,p}_{0,a}(\mathbb{R}^N) \), but just approximates the non-existing extremals on the whole space.

In this section, we prove an improved version of (1.1), which involves a sort of the “distance” of the associated function \( u \) from the one-dimensional space of “virtual extremals” \( \{ cu_a | c \in \mathbb{R} \} \). For the (sub-critical, also the critical) Hardy, or higher order Hardy-Rellich inequalities, see [15].

For \( R > 0 \), let us define

\[
d_R(f,g) = \left( \int_{\mathbb{R}^N} \frac{|f(x) - g(x)|^p}{\log \frac{R}{|x|}} |x|^{p(a+1)} dx \right)^{1/p}
\]

for functions \( f, g \), for which the right hand side is finite.
Theorem 3. Let $N \geq 3$, $2 \leq p < N$ and assume $-\infty < a < \frac{N-p}{p}$. Then
\[
\int_{\mathbb{R}^N} |\nabla u|^p |x|^{-pa} dx - \left( \frac{N-p-pa}{p} \right)^p \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^{p(a+1)}} dx
\geq c_p \left( \frac{p-1}{p} \right)^p \sup_{R>0} \inf_{c \in \mathbb{R}} d_R(u, cu)^p
\]
holds for any radially symmetric function $u \in D^{1,p}_0(\mathbb{R}^N)$. Here $c_p$ is a constant in Lemma 1 and $d_R(\cdot, \cdot)$ is defined in (3.1).

For the proof, we need the following result.

Proposition 2. (Machihara-Ozawa-Wadade [14]: Theorem 1.1) Let $N \in \mathbb{N}$, $1 < \alpha < \infty$ and $\max\{1, \alpha - 1\} < \beta < \infty$. Then for any $R > 0$, the inequality
\[
\int_{\mathbb{R}^N} \frac{|f(x) - f(R \frac{x}{|x|})|^\beta}{\log \frac{R}{|x|}} \frac{|x|^\alpha}{|x|^N} dx \leq \left( \frac{\beta}{\alpha-1} \right)^\beta \int_{\mathbb{R}^N} \frac{|x|^N \cdot \nabla f(x)|^\beta}{|x|^N \log \frac{R}{|x|}} \frac{|x|^\alpha}{|x|^N} dx
\]
holds for all $f \in W^{1,L}_{N,\beta,\beta-\alpha}(\mathbb{R}^N)$. Also the constant $\left( \frac{\beta}{\alpha-1} \right)^\beta$ is best possible in (3.3).

Remark 4. Here, $W^{1,L}_{p,q,\lambda}(\mathbb{R}^N)$ denotes the Sobolev-Lorentz-Zygmund spaces. For the precise definition of these spaces, we refer to [14]. However, we note that if $\int_{\mathbb{R}^N} (|\nabla u|^p + |u|^p) |x|^{-N} dx < \infty$, then the function $u \in W^{1,L}_{N,p,0}(\mathbb{R}^N)$.

Proof of Theorem 3. First, we prove Theorem for a radial function $u \in C_0^\infty(\mathbb{R}^N)$. Let $u \in C_0^\infty(\mathbb{R}^N)$, $u(x) = \tilde{u}(r)$, $r = |x|$ be a radial function. Define $v(x) = \tilde{v}(|x|)$ for $x \in \mathbb{R}^N$ where $\tilde{v}(r)$ is defined in (2.2). As in the proof of Theorem 1, we obtain
\[
J(u) = \int_{\mathbb{R}^N} |\nabla u|^p |x|^{-pa} dx - \left( \frac{N-p-pa}{p} \right)^p \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^{p(a+1)}} dx
\geq c_p \omega_N \int_0^\infty |\tilde{v}'|^p r^{p-1} dr = c_p \int_{\mathbb{R}^N} |\nabla v|^p |x|^{-N} dx,
\]
since we assume $p \geq 2$, see (2.3). Here we claim that if $u \in C_0^\infty(\mathbb{R}^N) \subset W^{1,p}_{0,a}(\mathbb{R}^N)$, $v$ satisfies
\[
\int_{\mathbb{R}^N} (|\nabla v|^p + |v|^p) |x|^{-N} dx < \infty.
\]
In particular, \( v \in W^1L_{N,p,0}(\mathbb{R}^N) \) by the above remark. Indeed, from (3.4), we have \( \int_{\mathbb{R}^N} |\nabla v|^p |x|^{N-p} dx \leq J(u)/c_p < \infty \). Also by the definition of \( v \), we see \( \int_{\mathbb{R}^N} |x|^{N-p} |v|^p dx = \int_{\mathbb{R}^N} |x|^{-pa} |u|^p dx < \infty \). Thus we have obtained the claim.

By the claim, we can apply Proposition 2 to \( v \in W^1L_{N,p,0}(\mathbb{R}^N) \). Then we derive

\[
J(u) \geq c_p \int_{\mathbb{R}^N} |\nabla v|^p |x|^{N-p} dx \geq c_p \left( \frac{p-1}{p} \right)^p \int_{\mathbb{R}^N} \frac{|x|^{N-p-pa} u(x) - R^{N-p-pa} u(R|x|^{p})|^p}{\log \frac{R}{|x|}} |x|^N dx
\]

\[
= c_p \left( \frac{p-1}{p} \right)^p \int_{\mathbb{R}^N} \frac{|u(x) - R^{N-p-pa} \tilde{u}(R)|x|^{N-p-pa}|^p}{\log \frac{R}{|x|}} |x|^p |x|^{p(\alpha+1)} dx
\]

(3.5) \[
\geq c_p \left( \frac{p-1}{p} \right)^p \inf_{c \in \mathbb{R}} \int_{\mathbb{R}^N} \frac{|u(x) - c|x|^{N-p-pa}|^p}{\log \frac{R}{|x|}} |x|^{p(\alpha+1)} dx
\]

for any \( R > 0 \), here we have used (3.4) in the first inequality, (3.3) with \( f = v \) and \( \alpha = \beta = p \) in the second inequality, the definition of \( v \) in the first equality, and the fact that \( u \) is radially symmetric in the second equality. This proves Theorem for a radial function \( u \in C_0^\infty(\mathbb{R}^N) \).

Next, we prove Theorem for a radial function \( u \in D_{0,a}^{1,p}(\mathbb{R}^N) \). Here we follow an argument by Machihara, Ozawa and Wadade [14]. Let \( \{u_m\}_{m=1}^\infty \subset C_0^\infty(\mathbb{R}^N) \) be a sequence of radially symmetric functions such that \( u_m \to u \) in \( D_{0,a}^{1,p}(\mathbb{R}^N) \) as \( m \to \infty \). Then there exists a subsequence \( \{u_{m_j}\}_{j=1}^\infty \) such that

\[
\frac{u_m}{|x|^{\alpha+1}} \to \frac{u}{|x|^{\alpha+1}} \quad \text{in} \quad L^p(\mathbb{R}^N),
\]

\[
u_{m_j} \to u \quad \text{a.e. in} \quad \mathbb{R}^N
\]

by (1.1). Now, define

\[
w_R(x) = \frac{w(x) - R^{N-p-pa} \tilde{w}(R)|x|^{N-p-pa}}{|x|^{\alpha+1} \log \frac{R}{|x|}}
\]

for a radial function \( w \in L^1_{loc}(\mathbb{R}^N) \), \( w(x) = \tilde{w}(|x|) \), and \( R > 0 \). Since the inequality (3.5) holds for \( u_{m_j} \), we observe that \( \{u_{m_j}\}_{j=1}^\infty \) is a Cauchy sequence in \( L^p(\mathbb{R}^N) \). Since \( (u_{m_j})_R \to u_R \) a.e.
in \( \mathbb{R}^N \), we easily see that \((u_m)_R \to u_R \) in \( L^p(\mathbb{R}^N) \) as \( j \to \infty \). Therefore, for any \( R > 0 \), we have

\[
J(u) = \lim_{j \to \infty} J(u_m)_R
\]

\[
\geq c_p \left( \frac{p - 1}{p} \right)^p \lim_{j \to \infty} \int_{\mathbb{R}^N} \frac{|u_m(x) - R^{-\frac{N-p-pa}{p}} \bar{u}_m(R)x|^{\frac{N-p-pa}{p}}}{|\log \frac{R}{|x|}|^{p} |x|^{p(a+1)}} \, dx
\]

\[
= c_p \left( \frac{p - 1}{p} \right)^p \int_{\mathbb{R}^N} \frac{|u(x) - R^{-\frac{N-p-pa}{p}} \bar{u}(R)x|^{\frac{N-p-pa}{p}}}{|\log \frac{R}{|x|}|^{p} |x|^{p(a+1)}} \, dx
\]

for all radial functions \( u \in D^{1,p}_a(\mathbb{R}^N) \), here we have used (3.5) for \( u_m \in C_0^\infty(\mathbb{R}^N) \). As before, this ends the proof.

## 4. Improved Caffarelli-Kohn-Nirenberg Type Inequalities on a Bounded Domain

In this section, we revisit the idea by Catrina and Wang [5], Wang-Willem [19] to improve the inequality

\[
\int_{\Omega} |\nabla u|^2 |x|^{-2a} \, dx \geq \left( \frac{N - 2 - 2a}{2} \right)^2 \int_{\Omega} \frac{|u|^2}{|x|^{2(a+1)}} \, dx
\]

on a smooth bounded domain \( \Omega \) in \( \mathbb{R}^N (N \geq 3) \) with \( 0 \in \Omega \). Here \( u \) is a function in \( C_0^\infty(\Omega \setminus \{0\}) \) and \(-\infty < a < \frac{N-2}{2}\) is always assumed. We can check that under the assumption \( a < \frac{N-2}{2} \), \( D^{1,2}_0(\Omega \setminus \{0\}) \) is identical to the completion of \( C_0^\infty(\Omega \setminus \{0\}) \) with respect to the norm \( \| \cdot \|_{D^{1,2}_0(\Omega \setminus \{0\})} \). The idea of Catrina and Wang consists of the use of a conformal transformation which converts a problem to an equivalent one in a domain on a cylinder \( \mathcal{C} = \mathbb{R} \times S^{N-1} \). More precisely, for a function \( u \in C_0^\infty(\Omega \setminus \{0\}) \), let us associate \( v \in C_0^\infty(\tilde{\Omega}) \) by the transformation

\[
u(x) = |x|^{\frac{N-2-2a}{2}} v \left( -\log |x|, \frac{x}{|x|} \right),
\]

where \( \tilde{\Omega} \) is a domain on the cylinder \( \mathcal{C} \) defined as

\[
\tilde{\Omega} = \left\{(t, \theta) \in \mathbb{R} \times S^{N-1} : t = -\log |x|, \theta = \frac{x}{|x|}, x \in \Omega \right\}.
\]

Catrina and Wang proved in [5] that when \( \Omega = \mathbb{R}^N \) and \( a < \frac{N-2}{2} \), the transformation (4.2) provides an isomorphism between two Hilbert
spaces $D_{0,a}^{1,2}(\mathbb{R}^N)$ and $H^1(C)$, the inner product of the latter is given by

$$(v, w)_{H^1(C)} = \int_C \left\{ \nabla v \cdot \nabla w + \left( \frac{N - 2 - 2a}{2} \right)^2 vw \right\} d\mu.$$ 

Here $|\nabla v|^2 = v_t^2 + |\nabla \theta v|^2$ and $d\mu = dt d\theta = |x|^{-N} dx$ are the length of the gradient vector and the volume element on $C$. By a direct computation, we see

$$\nabla u(x) = -|x|^{-N+2a} \left[ \left\{ v_t(t, \theta) + \left( \frac{N - 2 - 2a}{2} \right) v \right\} \theta + \nabla \theta v(t, \theta) \right],$$

and since $\langle \theta, \nabla \theta v \rangle \equiv 0$, it holds

(4.3)

$$|\nabla u(x)|^2 = |x|^{-N+2a} \left[ \left\{ v_t(t, \theta) + \left( \frac{N - 2 - 2a}{2} \right) v \right\}^2 + |\nabla \theta v(t, \theta)|^2 \right].$$

Furthermore, we have

(4.4)

$$\int_{\tilde{\Omega}} |\nabla v|^2 d\mu = \int_{\Omega} |\nabla u|^2 |x|^{-2a} dx - \left( \frac{N - 2 - 2a}{2} \right)^2 \int_{\Omega} \frac{u^2}{|x|^{2(a+1)}} dx,$$

(4.5)

$$\int_{\tilde{\Omega}} v^2 d\mu = \int_{\Omega} \frac{u^2}{|x|^{2(a+1)}} dx.$$

On the other hand, Wang and Willem [19] proved the weighted Poincaré inequality of the form

$$\int_{\tilde{\Omega}} |\nabla v|^2 d\mu \geq \frac{1}{4} \int_{\Omega} \frac{v^2}{t^2} d\mu$$

for $v \in C_0^\infty(\tilde{\Omega})$. Using this inequality and the transformation (4.2), they obtained the improved inequality mentioned in the Introduction, See [5], [19].

Following their arguments, we prove the next theorem.

**Theorem 4.** Let $V = V(t)$ is nonnegative, monotone decreasing function on $t \in (0, +\infty)$ and assume that there exists a strictly positive function $\phi = \phi(t) \in C^2((0, +\infty))$ such that $-\phi''(t) \geq V(t) \phi(t)$ holds on $(0, +\infty)$. Let $\Omega$ be a domain in $\mathbb{R}^N$ such that $\Omega \subset \subset B_R(0)$ for some
For example, \( V(t) = \frac{1}{4t^2} \) satisfies the assumption of Theorem 4 with \( \phi(t) = t^{1/2} \), \( t \in (0, +\infty) \).

As a corollary, we obtain the following improved inequalities.

**Corollary 1.** Let \( \Omega \) be a domain in \( \mathbb{R}^N \) such that \( \Omega \subset B_R(0) \) for some \( R > 0 \). Assume \( -\infty < a < \frac{N-2}{2} \). Then for any integer \( k \in \mathbb{N} \), there exists a constant \( C = C(a, \Omega, k) > 0 \) such that the followings hold.

(i) Define the functions \( X_i = X_i(s) \) for \( s \in (0, 1] \) iteratively as \( X_1(s) = (1 - \log s)^{-1} \), \( X_i(s) = X_i(X_{i-1}(s)) \) for \( i \geq 2 \). Then

\[
\int_{\Omega} |\nabla u|^2 |x|^{-2a} dx - \left( \frac{N - 2 - 2a}{2} \right)^2 \int_{\Omega} \frac{|u|^2}{|x|^{2(a+1)}} dx \\
\geq C \sum_{i=1}^{k} \int_{\Omega} \frac{1}{|x|^{2a_i}} X_i^2 \left( \frac{|x|}{R} \right) X_1^2 \left( \frac{|x|}{R} \right) \cdots X_i^2 \left( \frac{|x|}{R} \right) |\nabla u|^2 dx
\]

holds for all \( u \in D_{0,a}^{1,2}(\Omega) \).

(ii) Let \( \epsilon_{-1} = 0 \), \( \epsilon_0 = 1 \), \( \epsilon_i = e^{\epsilon_{i-1}} \) for \( i \geq 1 \), and define functions iteratively as \( \log^{(1)} s = \log s \), \( \log^{(i)}(s) = \log^{(1)} \left( \log^{(i-1)} s \right) \) for \( i \geq 2 \), those are well defined when \( s > \epsilon_{i-2} \) and positive when \( s > \epsilon_{i-1} \). For any integer \( k \in \mathbb{N} \), take \( \rho \geq R \epsilon_{k-1} \). Then there exists a constant \( C = C(a, \Omega, k) > 0 \) such that

\[
\int_{\Omega} |\nabla u|^2 |x|^{-2a} dx - \left( \frac{N - 2 - 2a}{2} \right)^2 \int_{\Omega} \frac{|u|^2}{|x|^{2(a+1)}} dx \\
\geq C \sum_{i=1}^{k} \int_{\Omega} \frac{1}{|x|^{2a_i}} \left( \log^{(1)} \frac{\rho}{|x|} \right)^{-2} \left( \log^{(2)} \frac{\rho}{|x|} \right)^{-2} \cdots \left( \log^{(i)} \frac{\rho}{|x|} \right)^{-2} |\nabla u|^2 dx
\]

holds for all \( u \in D_{0,a}^{1,2}(\Omega) \).

For the proof of Theorem 4, first we show a lemma.

**Lemma 2.** (The weighted Poincaré inequality on a cylinder) Let \( \Omega \subset C_+ \) be a bounded domain where \( C_+ = \{(t, \theta) \in \mathbb{R} \times S^{N-1} : t > 0 \} \). Let
$V = V(t)$ satisfy the assumptions in Theorem 4. Then it holds

\begin{equation}
\int_{\Omega} |\nabla v|^2 d\mu \geq \int_{\Omega} V(t)v^2 d\mu
\end{equation}

for all $v \in C_0^\infty(\Omega)$.

**Proof.** Put $v(t, \theta) = v(t, \theta)/\phi(t)$ where $\phi(t) > 0$, $\phi = \phi(t) \in C^2((0, +\infty))$ such that $-\phi''(t) \geq V(t)\phi(t)$ on $(0, +\infty)$. Then $\psi \equiv 0$ near $t = 0$ and $t = \infty$, since $v \in C_0^\infty(\Omega)$. Thus

\[
\int_0^\infty v_t^2(t, \theta) dt = \int_0^\infty (\phi_t v + \phi v_t)^2 dt = \int_0^\infty (\phi_t^2 v^2 + \phi^2 v_t^2 + \phi\phi_t(v^2)_t) dt
\]

\[
= \int_0^\infty (\phi_t^2 v^2 + \phi^2 v_t^2) dt + \left[\psi^2\phi\right]_0^\infty - \int_0^\infty \psi^2(\phi\phi_t) dt
\]

\[
= \int_0^\infty \phi_t^2 v_t^2 dt - \int_0^\infty \psi^2 \phi \phi_t dt \geq - \int_0^\infty \psi^2 \phi \phi_t dt
\]

\[
= \int_0^\infty v^2 \left( -\frac{\phi''(t)}{\phi(t)} \right) dt \geq \int_0^\infty V(t)v^2 dt.
\]

Integrating both sides on $S^{N-1}$ with respect to $dS_\theta$ and adding $\int_\Omega |\nabla v|^2 dt dS_\theta$ to the right hand side, we obtain (4.7). \[\square\]

The inequality (4.7) also holds for $v \in H_0^1(\Omega)$ by a density argument.

**Proof of Theorem 4.** By using a scaling $x \mapsto x/R$, it is enough to prove Theorem when $R = 1$. In this case, $\Omega \subset C_+$ where $C_+ = \{(t, \theta) \in \mathbb{R} \times S^{N-1} : t > 0\}$. Now, we will prove

\begin{equation}
\int_{\Omega} |\nabla v|^2 d\mu \geq C \int_{\Omega} V(t) \left[ |\nabla v|^2 + \left\{ v_t(t, \theta) + \left( \frac{N - 2 - 2a}{2} \right) v \right\}^2 \right] d\mu
\end{equation}

for some $C > 0$ independent of $v$. Indeed, since $r_0 = \sup_{x \in \Omega} |x| < 1$, we have $t = -\log |x| \geq -\log r_0 > 0$ on $\tilde{\Omega}$. Then the positivity and the
decreasing property of $V$ imply that
\[
\int_{\Omega} V(t) \left[ |\nabla v|^2 + \left\{ v_t(t, \theta) + \left( \frac{N-2-2a}{2} \right) v \right\}^2 \right] d\mu \\
\leq 2 \int_{\Omega} V(t) |\nabla v|^2 d\mu + 2 \left( \frac{N-2-2a}{2} \right)^2 \int_{\Omega} V(t) v^2 d\mu \\
\leq 2V(-\log r_0) \int_{\Omega} |\nabla v|^2 d\mu + 2 \left( \frac{N-2-2a}{2} \right)^2 \int_{\Omega} V(t) v^2 d\mu \\
\leq \left( 2V(-\log r_0) + 2 \left( \frac{N-2-2a}{2} \right)^2 \right) \int_{\Omega} |\nabla v|^2 d\mu,
\]
where we have used the weighted Poincaré inequality (4.7) in the last inequality. Thus we get (4.8). Simple computation using (4.3), (4.4), (4.5) in (4.8) yields the desired inequality (4.6).

\[\square\]

Remark 5. The constant $C$ in Theorem 4 can be chosen that
\[C = \left( 2V(-\log r_0) + 2 \left( \frac{N-2-2a}{2} \right)^2 \right)^{-1}.\]

Proof of Corollary 1. We follow the computation in [17].

(i) We may assume $R = 1$. Note that $X_i$ is well-defined and $X_i(0) = 0$, $X_i(1) = 1$, $0 < X_i(s) < 1$ for $s \in (0, 1)$. We compute
\[
X'_i(s) = \frac{1}{s} X_i(s)^2, \quad X'_i(s) = \frac{X_1(s) \cdots X_i(s)}{s} X_i(s)^2, \\
(X_1 \cdots X_k)'(s) = \frac{X_1(s) \cdots X_k(s)}{s} \{X_1(s) + (X_1 X_2)(s) + \cdots + (X_1 \cdots X_k)(s)\}. \\
\]
Define $Y_i(t) = X_i(e^{-t})$ for $t \in [0, +\infty)$. Then we obtain
\[
Y'_i(t) = -Y_i(t)^2, \quad Y'_i(t) = -(Y_1 \cdots Y_i)(t) Y_i(t), \\
(Y_1 \cdots Y_k)'(t) = (Y_1 \cdots Y_k)(t) \{Y_1(t) + (Y_1 Y_2)(t) + \cdots + (Y_1 \cdots Y_k)(t)\}. \\
\]
Now, define
\[
\phi_k(t) = (Y_1 \cdots Y_k)^{-1/2}(t) > 0.
\]
Then by differentiating $\log \phi_k(t) = -\frac{1}{2} \sum_{i=1}^k \log Y_i(t)$, we check that
\[
\phi'_k(t) = \frac{\phi_k(t)}{2} \sum_{i=1}^k (Y_1 \cdots Y_i)(t), \quad \phi''_k(t) = \frac{\phi_k(t)}{4} \sum_{i=1}^k (Y_1 \cdots Y_i)^2(t),
\]
thus $\phi_k$ is a solution of $-\frac{\phi_k''(t)}{\phi_k(t)} = \frac{1}{4} \sum_{i=1}^{k} (Y_1 \cdots Y_i)^2(t)$. It is easy to check that $V_k(t) = \frac{1}{4} \sum_{i=1}^{k} (Y_1 \cdots Y_i)^2(t)$ is nonnegative and decreasing on $(0, +\infty)$. Applying Theorem 4, we obtain the result.

(ii) We compute
\[
\left( \log^{(i)} s \right)' = s^{-1} \left( \log^{(1)} s \right)^{-1} \left( \log^{(2)} s \right)^{-1} \cdots \left( \log^{(i-1)} s \right)^{-1}
\]
for $i = 1, 2, \cdots, k$. Put $L_i(t) = \log^{(i)}(pe^t)$ and $W_i(t) = (L_1 \cdots L_i)^{-1}(t)$ for $i = 1, 2, \cdots, k$. We see $L_i(t) > 0$ if $\rho > e^{-t e_{i-1}}$ and obtain
\[
L'_i(t) = 1, \quad L'_i(t) = (L_1 \cdots L_{i-1})^{-1}(t) \quad \text{for } i = 2, 3, \ldots, k,
\]
\[
W'_i(t) = -W_i(t) \sum_{j=1}^{i} (L_1 \cdots L_j)^{-1}(t).
\]
Define
\[
\psi_k(t) = (L_1 \cdots L_k)^{1/2}(t) > 0.
\]
Then we check that
\[
\psi'_k(t) = \frac{\psi_k(t)}{2} \sum_{i=1}^{k} (L_1 \cdots L_i)^{-1}(t) = \frac{\psi_k(t)}{2} \sum_{i=1}^{k} W_i(t),
\]
\[
\psi''_k(t) = \frac{\psi_k(t)}{4} \left( \sum_{i=1}^{k} W_i(t) \right)^2 - \frac{\psi_k(t)}{2} \sum_{i=1}^{k} W_i(t) \sum_{j=1}^{i} W_j(t),
\]
\[- \frac{\psi''_k(t)}{\psi_k(t)} = \frac{1}{4} \sum_{i=1}^{k} W_i(t)^2.
\]
On the way of computation, we have used the identity
\[
2 \sum_{i=1}^{k} a_i \sum_{j=1}^{i} a_j - \left( \sum_{i=1}^{k} a_i \right) \left( \sum_{j=1}^{k} a_j \right) = \sum_{i=1}^{k} a_i^2
\]
for any $a_1, \cdots, a_k \in \mathbb{R}$. Again $V_k(t) = \frac{1}{4} \sum_{i=1}^{k} (L_1 \cdots L_i)^{-2}(t)$ is nonnegative and decreasing on $(0, +\infty)$, so we may apply Theorem 4 to obtain the result.

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