On the Heegaard genus, the bridge genus and the braid genus of a three-manifold

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Introduction

Theorem

Every closed orientable connected 3-manifold is obtained by the 0-surgery on S^3 along a link L.

Theorem

$$g_{\mathsf{H}}(M) \leq g_{\mathsf{bridge}}(M) \leq g_{\mathsf{braid}}(M).$$

We shows these invariants are mutually independent.

Definition (Heegaard splitting)

M : a closed connected orientable 3-manifold.

$$M = H_1 \cup_h H_2,$$

H_1, H_2 : handlebodies of genus g,

$h : \partial H_2 \to \partial H_1$: a homeomorphism.

(H_1, H_2, h) : a genus g Heegaard splitting of M.

Definition

The Heegaard genus

$g_{\mathsf{H}}(M)$

$$= \min\{g \mid \exists \text{a genus g Heegaard splitting of } M\}.$$

$L = K_1 \cup K_2 \cup \cdots \cup K_n$: an n-component link in S^3.

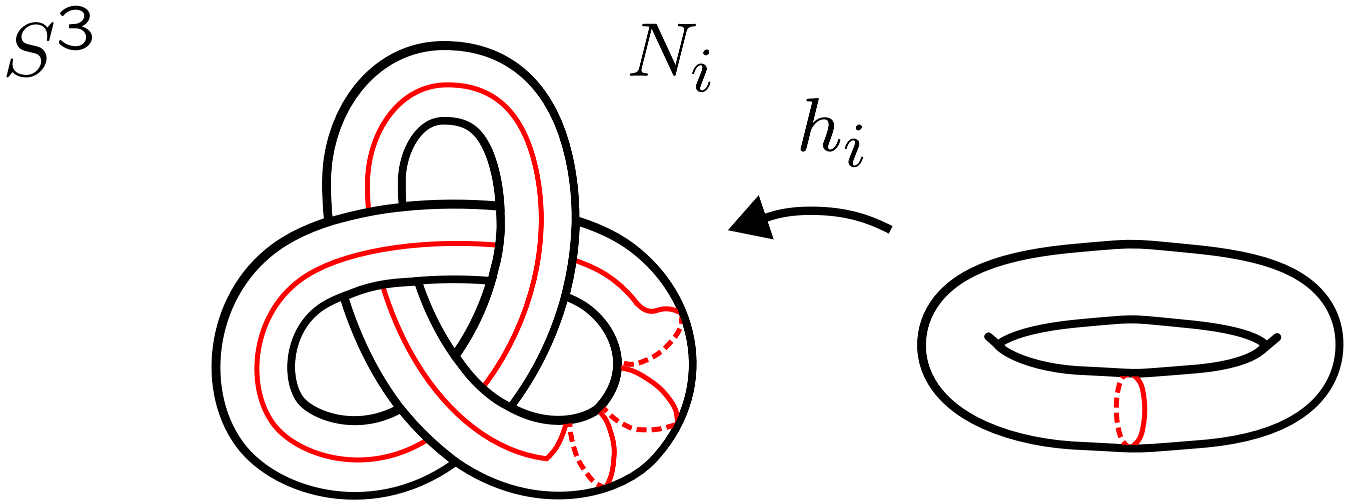
N_i : a tubular neighborhood of K_i in S^3.

Definition (0-surgery on S^3 along L)

$$\chi(L,0) \overset{\text{def}}{=} \left(S^3 - \left(\bigcup_{i=1}^{n} \mathring{N}_i\right)\right) \cup_h \left(\bigcup_{i=1}^{n} N_i\right).$$

h : a union of homeomorphisms

$h_i : \partial N_i \to \partial N_i$ taking a meridian of N_i onto a preferred longitude of N_i.



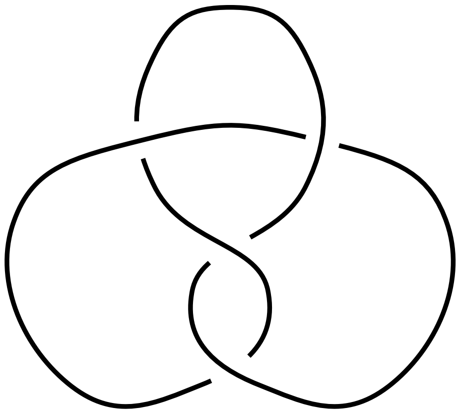
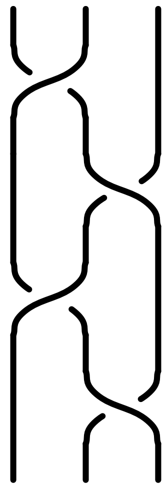
Theorem

Every closed orientable connected 3-manifold is obtained by the 0-surgery on S^3 along a link L.

bridge(L): the bridge index of L.

braid(L): the braid index of L.

Example

4_1 :  $\longrightarrow$  : braid(4_1)$=3$.

Definition

The bridge genus

$$g_{\text{bridge}}(M) = \min\{\text{bridge}(L) \mid \chi(L, 0) = M\}.$$

The braid genus

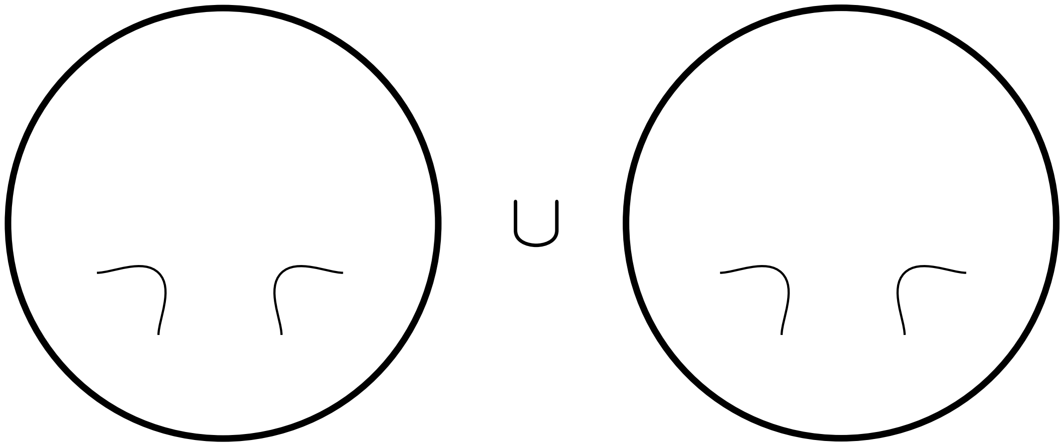
$$g_{\text{braid}}(M) = \min\{\text{braid}(L) \mid \chi(L, 0) = M\}.$$

Theorem

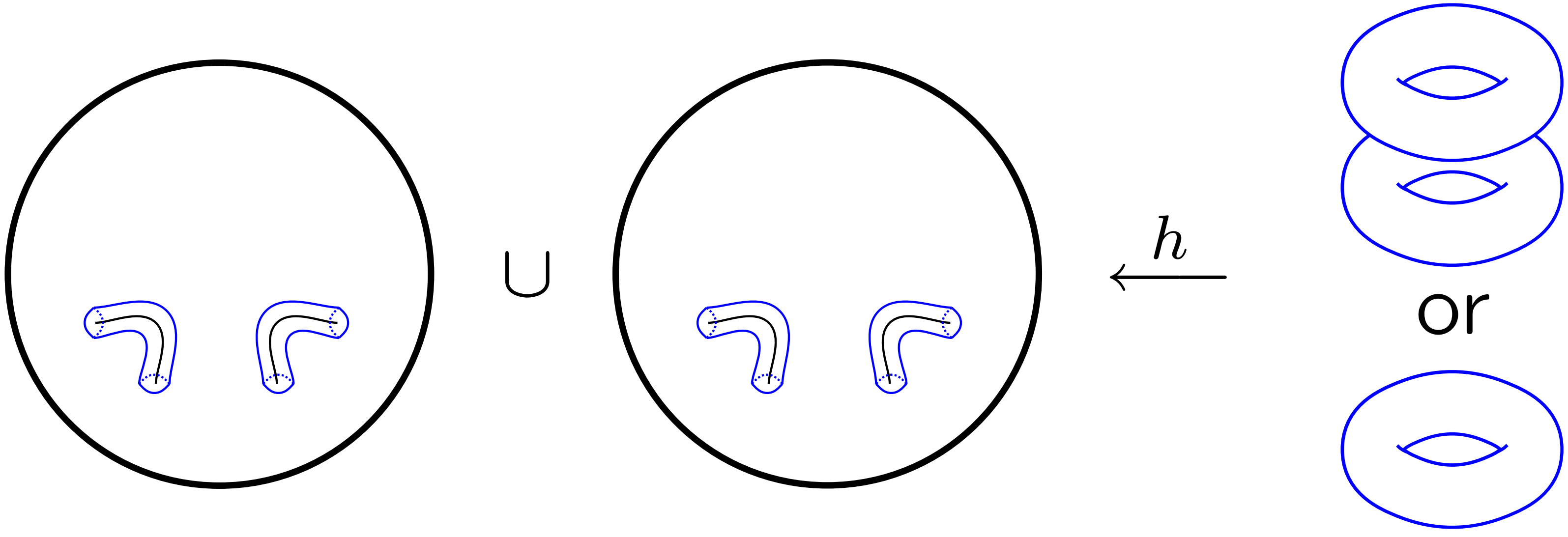
$$g_{\mathsf{H}}(M) \leq g_{\mathsf{bridge}}(M) \leq g_{\mathsf{braid}}(M).$$

Outline of the proof $g_{\mathsf{H}}(M) \leq g_{\mathsf{bridge}}(M)$

Let $g_{\mathsf{bridge}}(M) = 2$.



B_1^3 $\qquad$ B_2^3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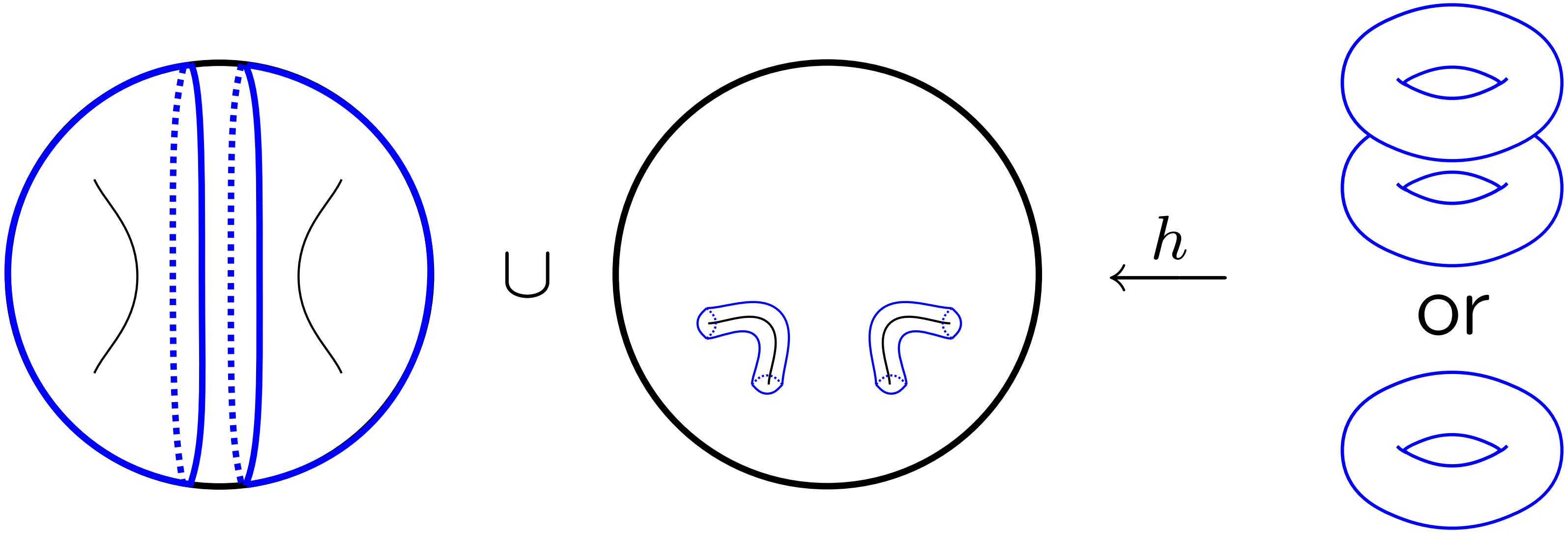
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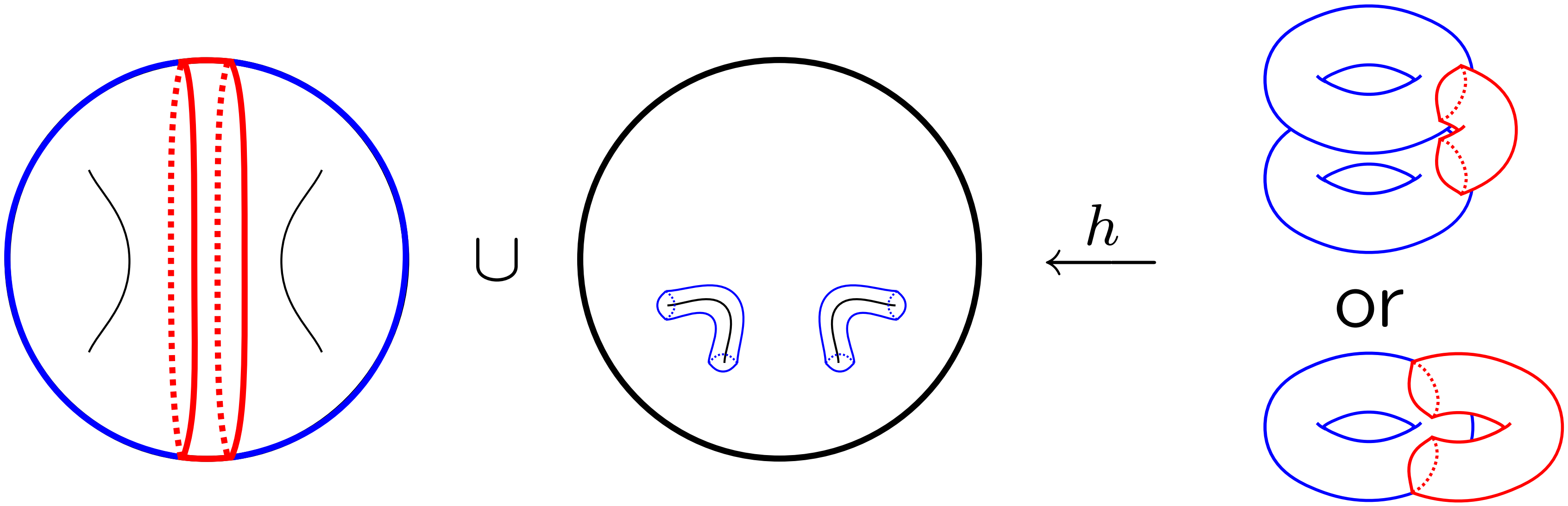

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Fact

$$g_{\mathsf{H}}(M) = 0 \;\Leftrightarrow\; M = S^3 \qquad \Leftrightarrow\; \pi_1(M) = 1.$$

$$g_{\mathsf{H}}(M) = 1 \;\Leftrightarrow\; M = L(p,q) \Leftrightarrow\; \pi_1(M) = \mathbb{Z}_p,$$

or

$$M = S^2 \times S^1 \Rightarrow\; \Delta_K(t) = 1,$$

where p and q are coprime integers s.t. $0 < q < p$.

Fact

K : a knot.

$H_1(\chi(K,0)) = \mathbb{Z}$.

$L = K_1 \cup K_2$: a 2-component link.

$lk(K_1, K_2) = n$.

$H_1(\chi(L,0)) = \mathbb{Z}_n \oplus \mathbb{Z}_n$.

Theorem [Kawauchi]

K, K': knots.

$\Delta_K(t), \Delta_{K'}(t)$: Alexander polynomials of K, K'.

$\chi(K, 0) \approx \chi(K', 0) \Rightarrow \Delta_K(t) \doteq \Delta_{K'}(t)$

Example ($M = \underset{n}{\sharp}\, S^2 \times S^1$)

$$g_{\mathsf{H}}(M) = g_{\mathsf{bridge}}(M) = g_{\mathsf{braid}}(M) = n.$$

($\because$) It is known that $g_{\mathsf{H}}(\underset{n}{\sharp}\, S^2 \times S^1) = n$.

Let L be the n-component trivial link.

Then we have

$$\chi(L, 0) = \underset{n}{\sharp}\, S^2 \times S^1.$$

$\therefore$ $g_{\mathsf{bridge}}(M) = \mathrm{bridge}(L) = n$,

$g_{\mathsf{braid}}(M) = \mathrm{braid}(L) = n$.

Example $(M = S^3)$

$$0 = g_{\mathsf{H}}(M) < g_{\mathsf{bridge}}(M) = g_{\mathsf{braid}}(M) = 2.$$

($\because$) It is known that $g_{\mathsf{H}}(M) = 0$.

Let L be the Hopf link. Then we have

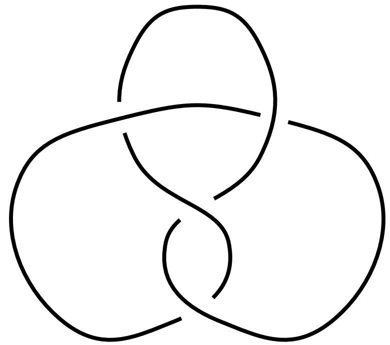
$$\chi(L, 0) = M.$$

$\therefore$ $g_{\mathsf{bridge}}(M) = \mathsf{bridge}(L) = 2$,

$g_{\mathsf{braid}}(M) = \mathsf{braid}(L) = 2$.

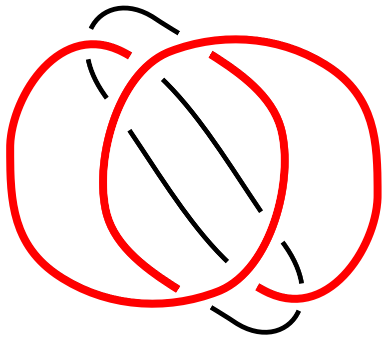
Example $(M = \chi(4_1, 0))$

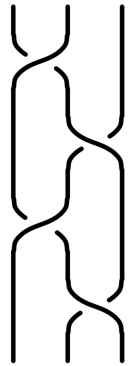
4_1 : the figure-eight knot.



$$2 = g_{\mathrm{H}}(M) = g_{\mathrm{bridge}}(M) < g_{\mathrm{braid}}(M) = 3.$$

$(\because)$





bridge(4_1)$= 2$, braid(4_1)$= 3$.

$\therefore\ g_{\mathrm{bridge}}(M) \leq 2,\ g_{\mathrm{braid}}(M) \leq 3.$

We show that $g_{\mathsf{H}}(M) = g_{\mathsf{bridge}}(M) = 2$.

Since $H_1(M) = \mathbb{Z}$, $M \neq L(p,q)$, S^3.

Since $\Delta_{4_1}(t) = t^2 - 3t + 1$, $M \neq S^2 \times S^1$.

that is, $g_{\mathsf{H}}(M) \geq 2$.

$\therefore\ 2 \leq g_{\mathsf{H}}(M) = g_{\mathsf{bridge}}(M) \leq 2$.

$$\therefore g_{\mathsf{H}}(M) = g_{\mathsf{bridge}}(M) = 2.$$

Next, we show that $g_{\text{braid}}(M) \geq 3$.

If $g_{\text{braid}}(M) = 2$,

then $\exists$a torus knot $K = T(2n+1, 2)$

s.t. $M = \chi(K, 0)$.

$\Delta_{4_1}(t) = t^2 - 3t + 1$.

$\Delta_K(t) = t^n - t^{n-1} + \cdots + t^2 - t + 1$.

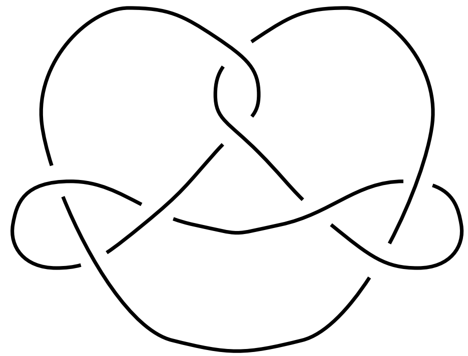
$\therefore$ $\Delta_{4_1}(t) \neq \Delta_K(t)$, that is, $g_{\text{braid}}(M) \neq 2$.

$\therefore$ $g_{\text{braid}}(M) \geq 3$.

Then we have

$$2 = g_{\text{H}}(M) = g_{\text{bridge}}(M) < g_{\text{braid}}(M) = 3.$$

Example $(M = \chi(8_{15}, 0))$

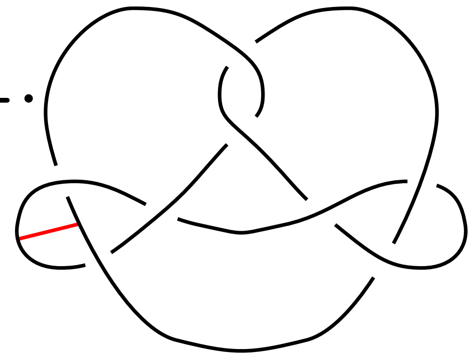


$g_{\mathsf{H}}(M) = 2,\ g_{\mathsf{bridge}}(M) = 3,\ g_{\mathsf{braid}}(M) \overset{?}{=} 4.$

$$\therefore \quad g_{\mathsf{H}}(M) < g_{\mathsf{bridge}}(M) \overset{?}{<} g_{\mathsf{braid}}(M).$$

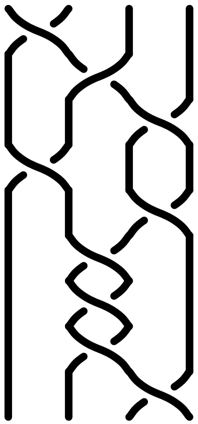
$(\because)$ The tunnel number of K is 1.

$\therefore\ g_{\mathsf{H}}(M) \leq 2.$



$:\text{bridge}(8_{15}) = 3.$

$\therefore\ g_{\text{bridge}}(M) \leq 3.$



$:\text{braid}(8_{15}) = 4.$

$\therefore\ g_{\text{braid}}(M) \leq 4.$

We show that $g_{\mathsf{H}}(M) \geq 2$.

Since $H_1(M) \cong \mathbb{Z}$, $\pi_1(M) \neq \mathbb{Z}_p$.

$\therefore\ M \neq L(p,q),\ S^3$.

Since $\Delta_{8_{15}}(t) = 3t^4 - 8t^3 + 11t^2 - 8t + 3$,

$M \neq S^2 \times S^1$.

$\therefore\ g_{\mathsf{H}}(M) = 2$.

Next, we show that $g_{\text{bridge}}(M) \geq 3$.

Since $H_1(M) \cong \mathbb{Z}$, M is not obtained by 0-surgery along any 2-component 2-bridge link.

Theorem [Murasugi]

If K is a 2-bridge knot, then

$$\Delta_K(t) \equiv \frac{1 - t^\lambda}{1 - t} \pmod 2.$$

λ : some odd integer.

Since
$$\begin{aligned}\Delta_{8_{15}}(t) &= 3t^4 - 8t^3 + 11t^2 - 8t + 3 \\ &\equiv t^4 + t^2 + 1 \pmod 2,\end{aligned}$$

M is not obtained by the 0-surgery along any 2-bridge knot.

$\therefore\ g_{\text{bridge}}(M) = 3.$

Next we show that $g_{\text{braid}}(M) \geq 4$.

<u>Theorem</u> [Jones]

K : a knot.

If $|\, \Delta_K(i)\, | > 3$, then braid$(K) \neq 3$.

Since $|\, \Delta_{8_{15}}(i)\, | = 5$, M is not obtained by 0-surgery along any 3-braid knot.

Then we have

$$g_{\text{H}}(M) = 2 < g_{\text{bridge}}(M) = 3 \overset{?}{<} g_{\text{braid}}(M) \overset{?}{=} 4.$$

Example An infinite family

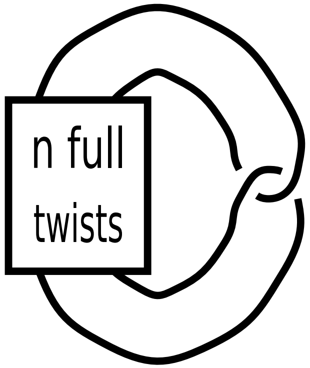
$$M = \chi(T(2n, 2), 0) \qquad (n = 2, 3, 4, \ldots)$$

$$g_{\mathsf{H}}(M) = g_{\mathsf{bridge}}(M) = g_{\mathsf{braid}}(M) = 2.$$

$$M = \chi(T(3n + 1, 3), 0) \qquad (n = 1, 2, 3, \ldots)$$

$$2 = g_{\mathsf{H}}(M) < g_{\mathsf{bridge}}(M) = g_{\mathsf{braid}}(M) = 3.$$

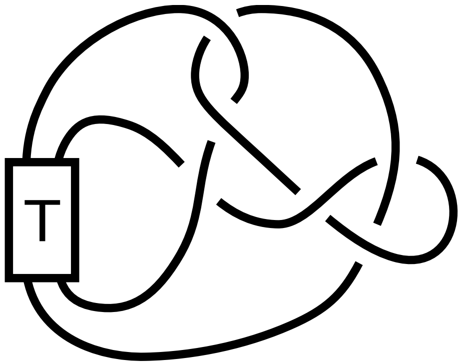
K_n :



$$M = \chi(K_n, 0) \qquad\qquad (n = 1, 2, 3, \ldots)$$

$$2 = g_{\mathsf{H}}(M) = g_{\mathsf{bridge}}(M) < g_{\mathsf{braid}}(M).$$

K_n :



T : 2n+1 full twists.

$$M = \chi(K_n, 0) \qquad\qquad (n = 1, 2, 3, \ldots)$$

$$g_{\mathsf{H}}(M) = 2 < g_{\mathsf{bridge}}(M) = 3 \overset{?}{<} g_{\mathsf{braid}}(M) \overset{?}{=} 4.$$