The warping degree of a knot diagram

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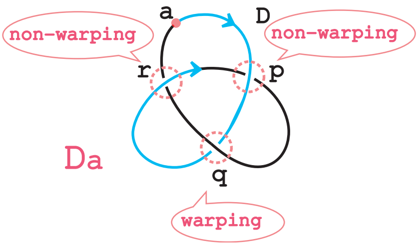
Warping Crossing Point

D : an oriented knot diagram
a : a point on D (*base point*)



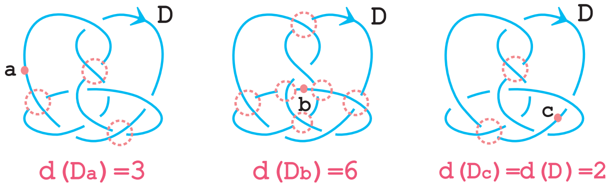
Definition (warping crossing point)

A crossing point of D_a is *warping* if we meet the point first at the under-crossing when we go along the oriented diagram D by starting from a.

The Warping Degree

the warping degree of D_a

$d(D_a) = \#\{\text{warping crossing points of } D_a\}$

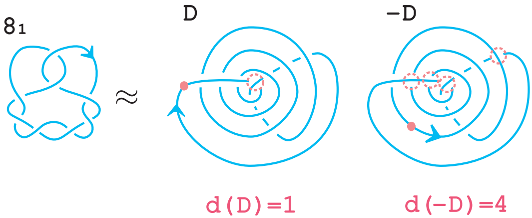


the warping degree of D

$d(D) = \min\{d(D_a) | \, a : \text{a base point of } D\}$

Remark.

The warping degree depends on the orientation. Let $-D$ be the inverse of D.

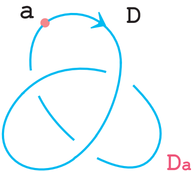


Theorem (M.Ozawa, T.S.Fung)

A knot K which has an oriented diagram D with $d(D) = 1$ is a twist knot.

Monotone Diagram

D_a is *monotone* if $d(D_a) = 0$.



- D_a: monotone $\Rightarrow$ D: a diagram of the trivial knot
- $u(D) \leq d(D)$ ($u(D)$: the unknotting number of D)

Warping Degrees of Knot Diagrams in Rolfsen's Book

D	$d(D)$	$d(-D)$	$d(D)+d(-D)$	$c(D)$
3_1	1	1	2	3
4_1	1	2	3	4
$5_1, 5_2$	2	2	4	5
$6_1, \ldots, 6_3$	2	3	5	6
$7_1, \ldots, 7_6$	3	3	6	7
7_7	2	4	6	7
$8_1, \ldots, 8_{17}$	3	4	7	8
8_{18}	2	5	7	8
8_{19} (non-alt)	3	3	6	8
8_{20} (non-alt)	2	3	5	8
8_{21} (non-alt)	2	2	4	8

Main Theorem

Main Theorem

For an oriented knot diagram D which has at least one crossing point, we have

$$d(D) + d(-D) + 1 \leq c(D).$$

Further, the equality holds if and only if D is an alternating diagram.

Corollary

Corollary

Let $u(D)$ be the unknotting number of D. Then we have

$$u(D) \leq \min\{d(D), d(-D)\} \leq \frac{c(D) - 1}{2}.$$

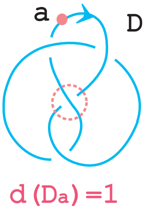
Further the equality holds if and only if D is a reduced alternating diagram of some $(2, p)$-torus knot, or D is a diagram with $c(D) = 1$ (by Taniyama).

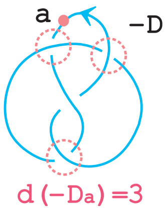
Properties of The Warping Degree

p: a crossing point of D
p is warping of D_a $\Leftrightarrow$ p is non-warping of $-D_a$.

Lemma 1

Let $c(D)$ be the crossing number of D. Then we have
$d(D_a) + d(-D_a) = c(D)$.

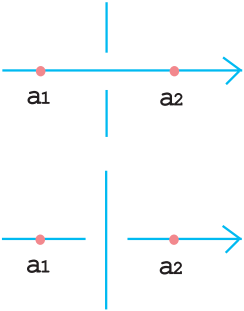


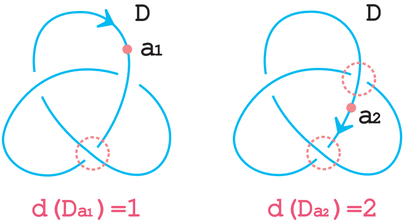


Properties of The Warping Degree

Lemma 2

For base points a_1, a_2 which are put across an over-crossing (resp. under-crossing), we have
$d(D_{a_2}) = d(D_{a_1}) + 1$ (resp. -1).

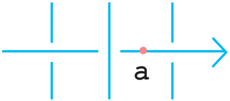




Properties of The Warping Degree

Lemma 3

Let D be an oriented alternating knot diagram. If a base point a is just before an over-crossing, then we have $d(D_a) = d(D)$.



Main Theorem

Main Theorem

For an oriented knot diagram D which has at least one crossing point, we have

$$d(D) + d(-D) + 1 \leq c(D).$$

Further, the equality holds if and only if D is an alternating diagram.

Outline of Proof

(i) $d(D) + d(-D) + 1 \leq c(D)$
(ii) D: alternating $\Rightarrow d(D) + d(-D) + 1 = c(D)$
(iii) $d(D) + d(-D) + 1 = c(D) \Rightarrow D$: alternating

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Proof of Main Theorem

(ii) D : alternating $\Rightarrow d(D) + d(-D) + 1 = c(D)$

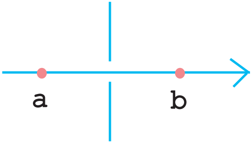
$d(D_a) = d(D)$ (by Lemma 3),

$d(D_a) + d(-D_a) = c(D)$ (by Lemma 1).

$d(-D_a) = d(-D_b) + 1$ (by Lemma 2)

$= d(-D) + 1$ (by Lemma 3).

$\therefore d(D) + d(-D) + 1 = c(D)$

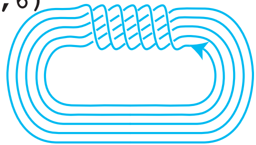


Torus Knot

Let $T(p,q)$ be a (p,q)-torus knot ($0 < p < q$, p and q are coprime) and $D(p,q)$ the standard diagram of $T(p,q)$. Then we have

$$d(D(p,q)) = d(-D(p,q)) = \frac{(p-1)(q-1)}{2}.$$

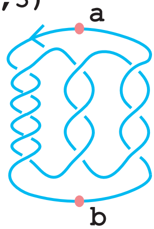
D(5,6)



Pretzel Knot

Let $D = P(\varepsilon_1 n_1, \varepsilon_2 n_2, \ldots, \varepsilon_m n_m)$ be a pretzel knot diagram of odd type
($\varepsilon_i \in \{+1, -1\}, n_i, m : odd > 0$). Then we have

P(5,3,3)



$$d(D_a) = d(-D_b) = \frac{c(D)}{2} + \sum_i \frac{(-1)^{i+1} \varepsilon_i}{2},$$

$$\sharp \lfloor D_a \rfloor = \sharp \lfloor -D_b \rfloor,$$

$$\therefore\ d(D) = d(-D).$$

D: alternating i.e. $\varepsilon_1 = \varepsilon_2 = \cdots = \varepsilon_m = \pm 1 \Rightarrow d(D) = \frac{c(D)}{2} - \frac{1}{2}$,

$$d(D) + d(-D) + 1 = c(D). \quad \text{(Main Theorem)}$$

To a Knot Invariant

K: a nontrivial knot

$e(K) := \min\{d(D) + d(-D) \mid D$: an oriented diagram of K with c(D)=c(K) $\}$

Theorem

(1) We have $e(K) + 1 \leq c(K)$.
Further, the equality holds if and only if K is a prime alternating knot.

(2) For any positive integer n, there exists a prime knot K s.t. $c(K) - e(K) = n$.

Corollary

Corollary

Let $u(K)$ be the unknotting number of K. Then we have

$$u(K) \leq \frac{e(K)}{2} \leq \frac{c(K)-1}{2}.$$

Further the equality holds if and only if K is a $(2, p)$-torus knot (p:odd,$\neq \pm 1$), (by Taniyama).

Thank you.