TANGLE SURGERIES ON THE DOUBLE OF A TANGLE AND LINK POLYNOMIALS

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Dedicated to Professor Shin'ichi Suzuki on his 60th birthday

Abstract. We introduce some tangle surgeries on the double of a tangle. If the tangle satisfies certain conditions, then the resulting link has the same polynomial invariant as the original one. We give some pairs of links sharing the same polynomial invariants making use of our tangle surgeries, which also show that our results are the best possible.

1. Introduction

There are several methods for producing different knots or links with the same polynomial invariant such as the Alexander, Jones, HOMFLY, or Kauffman polynomials. Conway's mutation [4] and its generalizations, satellites of mutants, rotants and the theory of spectral parameter tangles of Jones provide such examples [1, 7, 8, 10, 11, 12, 13, 18, 20, 23, 24, 26, 27, 30]. They are performed by removing a tangle from somewhere in the diagram and then sewing that same tangle back in differently by rotating it or flipping it over. In this paper, we introduce other three types of tangle surgeries that do not alter some polynomial invariants. The first one is as follows: Given a tangle T containing a tangle R, we construct the double of T. Then it contains R and its mirror image. Instead of rotating a tangle $180°$, we rotate these two tangles $90°$ simultaneously and take mirror images. If the tangle T satisfies certain conditions, then the resulting link has the same Kauffman bracket polynomial (Theorem 2.1(i)) or the HOMFLY polynomial (Theorem 2.2(i)) as the original one. This tangle surgery provides

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a pair of links of more than one component having the same Kauffman bracket or HOMFLY polynomials. The second and third tangle surgeries are done similarly. The second one provides a pair of links of more than one component having the same Kauffman bracket or HOMFLY polynomials and also the Q polynomial. The third one can provide a pair of knots having the same Kauffman bracket or HOMFLY polynomials.

This paper is organized as follows: In Sect. 2, we introduce our tangle surgeries and state the main theorems (Theorems 2.1–2.3). In Sect. 3, we give the definitions of the polynomial invariants such as the Kauffman bracket, Jones, Λ, Kauffman, Q, HOMFLY, and Conway polynomials. In Sect. 4, we prove the main theorems. We use a linear skein theory essentially due to Conway [4]. In Sect. 5, we give some pairs of links sharing the same polynomial invariants making use of Theorems 2.1–2.3, which also show that our theorems are the best possible. We also give another tangle surgery that does not alter the Λ polynomial, and hence the Kauffman's F polynomial (Theorem 5.1).

In his master thesis, Hirofusa Saito [28] classified 2-string tangles of 6 crossings or less up to freely equivalence (see Sect. 4), where he uses that the double of a tangle is an invariant link, and thus its polynomial invariant is an invariant for a tangle; see Lemma 4.1. Furthermore, he and Satoh [29] found several pairs of tangles that cannot be classified by the polynomial invariants of their doubles. In this paper, the author generalized their examples.

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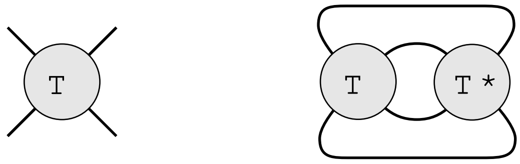
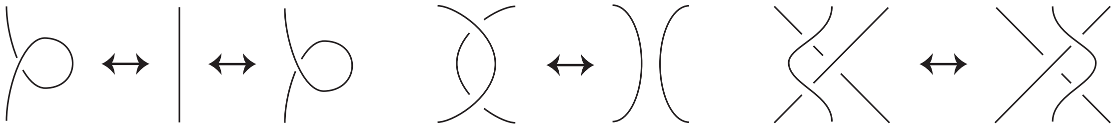
2. Tangle Surgeries

A *tangle* is a pair (B^3, τ), where τ is a 1-manifold properly embedded in a 3-ball B^3 with 4 boundary components. We express a tangle T by a diagram as in Fig. 1(a), where we use the projection $(x, y, z) \mapsto (x, y)$. Two tangles are *equivalent* if there is an isotopy of the 3-ball that takes one tangle to the other while fixing each point of the boundary, that is, their diagrams are related by a finite sequence of Reidemeister moves (Fig. 2) inside the circle defining the tangle while the endpoints of the strings remain fixed; cf. [4, p. 331].



(a) (b)

FIGURE 1. (a) A tangle T. (b) The double of T, DT.



C_+ C C_-

I II III

FIGURE 2. Reidemeister moves.

We define the *double* of T, DT, by the link diagram or the link given by Fig. 1(b), where T^* denotes the image of T under the reflection with regard to the yz-plane; $T^* = \rho_{yz}T$ with $\rho_{yz}(x, y, z) = (-x, y, z)$. If T is an oriented tangle, then its double DT is oriented so that the strings of T keep the original orientation and that the strings of T^* reverse the orientation that is induced from that of T.

Suppose that a tangle T contains a tangle R in its interior. We denote by $T(R')$ the new tangle obtained from T by replacing R with another tangle R'. Thus, in particular, $T = T(R)$. Then the double DT contains the two tangles R and R^*. We denote by $D(T; R_1, R_2)$ the new link obtained from DT by replacing R and R^* with other tangles R_1 and R_2, respectively. Thus, in particular, $DT = D(T; R, R^*)$.

Two tangles $T = (B^3, \tau)$ and $T' = (B^3, \tau')$ are *freely equivalent* if there is an isotopy of B^3 taking τ to τ' (without the restriction that the endpoints stay fixed). When we consider oriented tangles, we require the isotopy should preserve the orientation of the strings.

We define the integral tangle or n tangle, $n \in \mathbb{Z}$, and the ∞ tangle as in Fig. 3, where $n > 0$. Further, we define the $1/m$ tangle, $m \in \mathbb{Z} \setminus \{0\}$, as in Fig. 4, where $m > 0$.

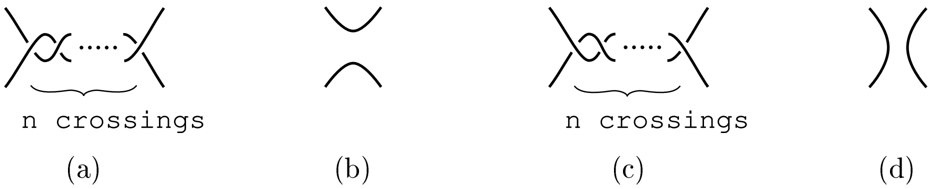


FIGURE 3. (a) The n tangle. (b) The 0 tangle. (c) The $-n$ tangle. (d) The ∞ tangle.

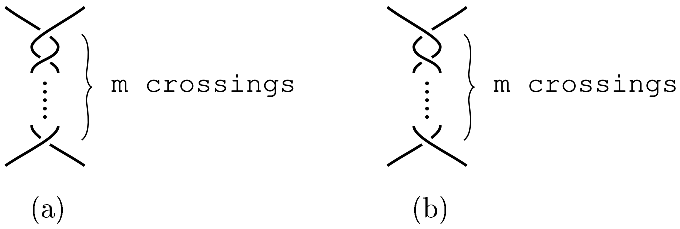


FIGURE 4. (a) The $\frac{1}{m}$ tangle. (b) The $-\frac{1}{m}$ tangle.

For a tangle R, we denote by $R_\perp$ the image of R under the rotation by angle 90° given by $\nu(x, y, z) = (-y, x, z)$, $R_\perp = \nu R$, and by $R^*_\perp$ the tangle $\nu\rho_{yz}R$. Thus if R is the n tangle, $n \in \mathbb{Z} \setminus \{0\}$, then $R_\perp$ and $R^*_\perp$ are the the $-1/n$ and $1/n$ tangles, respectively.

When R is the n tangle, $n \in \mathbb{Z} \cup \{\infty\}$ (resp. $1/m$ tangle, $m \in \mathbb{Z} \setminus \{0\}$), we denote a tangle $T(R)$ by $T(n)$ (resp. $T(1/m)$). In the following, we shall use a similar notation. We consider a tangle $T(R)$ satisfying the condition:

($\star$) The two tangles $T(0)$ and $T(\infty)$ are freely equivalent.

In Sect. 5, we give some tangles satisfying the condition ($\star$).

Now we state our main theorems. The definitions of the Kauffman bracket, HOMFLY, Q, and Conway polynomials will be given in the next section.

Theorem 2.1. *Suppose that a tangle $T(R)$ satisfies the condition ($\star$). Then each of the following pairs of diagrams share the same Kauffman bracket polynomial.*

(i) $D(T; R, R^*)$ ($= DT(R)$) *and* $D(T; R^*_\perp, R_\perp)$ ($= DT(R^*_\perp)$).
(ii) $D(T; R, R)$ *and* $D(T; R_\perp, R_\perp)$.
(iii) $D(T; R, R^*_\perp)$ *and* $D(T; R^*, R_\perp)$.

Theorem 2.2. *Suppose that an oriented tangle $T(R)$ satisfies the condition* $(\star)$ *and R is oriented as in Fig.* 5. *Then each of the pairs of the oriented links* (i)–(iii) *in Theorem* 2.1 *share the same HOMFLY polynomial.*

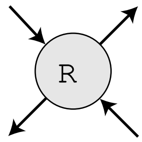


FIGURE 5. An oriented tangle R.

Theorem 2.3. *Suppose that a tangle $T(R)$ satisfies the condition* $(\star)$. *Then the pair of the link* (ii) *in Theorem* 2.1 *share the same Q polynomial.*

Remark 2.4. For any tangle T, it is easy to see that its double DT is a *slice link in the strong sense*, that is, DT is cobordant to a trivial link with the same number of components. Then the multivariable Alexander polynomial of DT is zero, and hence the Conway polynomial is zero; see [15, 21, 22], cf. [16, Sect. 12.3].

3. Link Polynomials

The *Kauffman bracket polynomial* $\langle G\rangle \in \mathbb{Z}[A^{\pm 1}]$ [14] of a link diagram G is defined by the following formulas:

$$\langle O\rangle = 1; \tag{3.1}$$

$$\langle G_1\rangle = A\langle G_\infty\rangle + A^{-1}\langle G_0\rangle; \tag{3.2}$$

$$\langle G_{-1}\rangle = A\langle G_0\rangle + A^{-1}\langle G_\infty\rangle; \tag{3.3}$$

$$\langle G \sqcup O\rangle = \left(-A^2 - A^{-2}\right)\langle G\rangle, \tag{3.4}$$

where O is a diagram of the unknot with no crossing and G_n, $n \in \mathbb{Z}\cup\{\infty\}$, are link diagrams that are identical except near one point where they are the n tangles. Then $\langle G\rangle$ is invariant under *regular isotopy* of G, that is, invariant under the Reidemeister moves II and III; see Fig. 2. The writhe $w(G)$ of a diagram of an oriented link is the sum of the signs of the crossings of G; the convention is shown in Fig. 6. The *Jones polynomial* $V(L;t) \in \mathbb{Z}[t^{\pm 1/2}]$ [9] is an invariant of an oriented link L given by

$$V(L;t) = \left[(-A)^{-3w(G)}\langle G\rangle\right]_{A=t^{-1/4}}, \tag{3.5}$$

where G is a diagram of L.

+1  −1

FIGURE 6. Crossing signs.

The Λ-*polynomial* $\Lambda(G) \in \mathbb{Z}[a^{\pm 1}, x^{\pm 1}]$ [14] of a link diagram G is defined by the following formulas:

$$\Lambda(O) = 1; \tag{3.6}$$

$$\Lambda(C_+) = a\Lambda(C), \qquad \Lambda(C_-) = a^{-1}\Lambda(C); \tag{3.7}$$

$$\Lambda(G_1) + \Lambda(G_{-1}) = x\left(\Lambda(G_0) + \Lambda(G_\infty)\right); \tag{3.8}$$

where O and G_n, $n \in \mathbb{Z} \cup \{\infty\}$, are link diagrams as above and C_+, C, C_- are link diagrams that are identical except near one point where they are as shown in Fig. 2. Then $\Lambda(G)$ is invariant under regular isotopy of G and can be modified to the *Kauffman polynomial* $F(L;a,x) \in \mathbb{Z}[a^{\pm 1}, x^{\pm 1}]$ of an oriented link L by

$$F(L;a,x) = a^{-w(G)}\Lambda(G), \tag{3.9}$$

where G is a diagram of L.

The Q *polynomial* $Q(L;x) \in \mathbb{Z}[x^{\pm 1}]$ [2, 6] is an invariant of an unoriented link L, which is defined by the following formulas:

$$Q(U;x) = 1; \tag{3.10}$$

$$Q(L_1;x) + Q(L_{-1};x) = x\left(Q(L_0;x) + Q(L_\infty;x)\right); \tag{3.11}$$

where U is a trivial knot and L_n, $n \in \mathbb{Z} \cup \{\infty\}$, are links having diagrams that are identical except near one point where they are the n tangles. The Q polynomial is obtained from the

Kauffman polynomial:

$$Q(L;x) = F(L,1,x). \tag{3.12}$$

The *HOMFLY polynomial* $P(L;t,z) \in \mathbb{Z}[t^{\pm 1}, z^{\pm 1}]$ [5, 25] is an invariant of an oriented link L, which is defined, as in [9], by the following formulas:

$$P(U;t,z) = 1; \tag{3.13}$$

$$t^{-1}P(L_+;t,z) - tP(L_-;t,z) = zP(L_0;t,z), \tag{3.14}$$

where U is a trivial knot and L_+, L_-, L_0 are three links that are identical except near one point where they are as in Fig. 7. The *Conway polynomial* $\nabla(L;z) \in \mathbb{Z}[z]$ [4] of an oriented link L is given by

$$\nabla(L;z) = P(L;1,z). \tag{3.15}$$

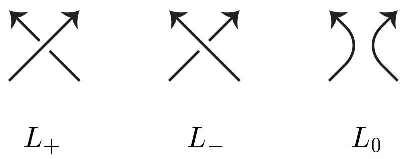


FIGURE 7. Skein triple.

Note that the Jones polynomial is obtained from both the HOMFLY and Kauffman polynomials; cf. [17]:

$$V(L;t) = P(L;t,t^{1/2} - t^{-1/2}); \tag{3.16}$$

$$= F(L;t^{-3/4}, -t^{1/4} - t^{-1/4}). \tag{3.17}$$

4. Proofs of Theorems

Two link diagrams G and G' are said to be *balanced isotopic* [1, p. 240] if they are related by a finite sequence of the Reidemeister moves II and III and the move BI as shown in Fig. 8, which introduces or deletes an opposite pair of curls. Then the Kauffman bracket and Λ polynomials are invariants of balanced isotopy.

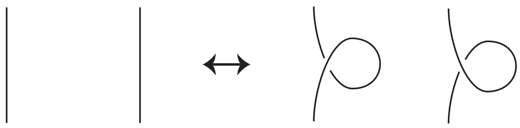


FIGURE 8. Balanced Reidemeister move BI.

Lemma 4.1. *Suppose that T_0 and T_1 are freely equivalent tangles. Then their doubles DT_0 and DT_1 are balanced isotopic.*

Proof. Let $T_1 = (B^3, \tau)$. Then T_0 is equivalent to $T_1' = (B^3, \tau) \cup_h (S^2 \times [0,1], \sigma_0)$ for some 4-string braid σ_0 on the 2-sphere S^2, where $h : S^2 \times \{0\} \to \partial B^3$ is a homeomorphism sending the endpoints of σ_0 in $S^2 \times \{0\}$ to those of τ; cf. [3, Lemma 2.3]. Then T_1' can be expressed as in Fig. 9, where σ is a 4-string braid. Thus DT_1 and DT_1' are related by a finite sequence of the Reidemeister move II. On the other hand, since T_0 and T_1' are equivalent, they are related by a finite sequence of the Reidemeister moves keeping the endpoints fixed, and thus DT_0 and DT_1' are balanced isotopic. This completes the proof. □

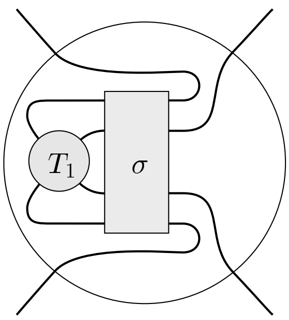


FIGURE 9. The tangle T_1'.

Proof of Theorem 2.1. Let L_R be a link diagram that contains a tangle R. When R is the n tangle, $n \in \mathbb{Z} \cup \{\infty\}$, we denote L_R by L_n. Applying the axioms (3.2)–(3.4), we may express the bracket polynomial $\langle L_R \rangle$ in terms of $\langle L_0 \rangle$ and $\langle L_\infty \rangle$:

$$\langle L_R \rangle = \alpha_R \langle L_0 \rangle + \beta_R \langle L_\infty \rangle, \tag{4.1}$$

where $\alpha_R = \alpha_R(A)$, $\beta_R = \beta_R(A) \in \mathbb{Z}[A^{\pm 1}]$. In the terminology of the linear skein theory [19], the 0 and ∞ tangles skein-generate the room inhabited by R, and the link diagrams obtained by substituting these tangles for R generate L_R.

From (4.1), we have

$$\begin{aligned}\langle D(T;R_1,R_2)\rangle &= \alpha_{R_1}\alpha_{R_2}\langle D(T;0,0)\rangle + \alpha_{R_1}\beta_{R_2}\langle D(T;0,\infty)\rangle \\ &\quad + \beta_{R_1}\alpha_{R_2}\langle D(T;\infty,0)\rangle + \beta_{R_1}\beta_{R_2}\langle D(T;\infty,\infty)\rangle.\end{aligned} \tag{4.2}$$

Using Lemma 4.1, the condition $(\star)$ implies

$$\langle D(T;0,0)\rangle = \langle D(T;\infty,\infty)\rangle, \tag{4.3}$$

which we denote by γ. Then we have

$$\begin{aligned}\langle D(T;R_1,R_2)\rangle &= (\alpha_{R_1}\alpha_{R_2} + \beta_{R_1}\beta_{R_2})\,\gamma \\ &\quad + \alpha_{R_1}\beta_{R_2}\langle D(T;0,\infty)\rangle + \beta_{R_1}\alpha_{R_2}\langle D(T;\infty,0)\rangle.\end{aligned} \tag{4.4}$$

Replacing R with R^*, $R_\perp$, $R_\perp^*$ in (4.1), we have:

$$\langle L(R^*)\rangle = \bar{\alpha}\langle L(0)\rangle + \bar{\beta}\langle L(\infty)\rangle; \tag{4.5}$$

$$\langle L(R_\perp)\rangle = \beta\langle L(0)\rangle + \alpha\langle L(\infty)\rangle; \tag{4.6}$$

$$\langle L(R_\perp^*)\rangle = \bar{\beta}\langle L(0)\rangle + \bar{\alpha}\langle L(\infty)\rangle, \tag{4.7}$$

where $\bar{\alpha} = \alpha_{R^*}(A) = \alpha_R(A^{-1})$, $\bar{\beta} = \beta_{R^*}(A) = \beta_R(A^{-1})$, $\beta = \beta_R$ and $\alpha = \alpha_R$. Thus from (4.4), we obtain

$$\begin{aligned}\langle D(T;R,R^*)\rangle &= \langle D(T;R_\perp,R_\perp^*)\rangle \\ &= (\alpha\bar{\alpha} + \beta\bar{\beta})\,\gamma + \alpha\bar{\beta}\langle D(T;0,\infty)\rangle + \bar{\alpha}\beta\langle D(T;\infty,0)\rangle;\end{aligned} \tag{4.8}$$

$$\begin{aligned}\langle D(T;R,R)\rangle &= \langle D(T;R_\perp,R_\perp)\rangle \\ &= \left(\alpha^2 + \beta^2\right)\gamma + \alpha\beta\left(\langle D(T;0,\infty)\rangle + \langle D(T;\infty,0)\rangle\right);\end{aligned} \tag{4.9}$$

$$\begin{aligned}\langle D(T;R,R_\perp^*)\rangle &= \langle D(T;R^*,R_\perp)\rangle \\ &= (\alpha\bar{\beta} + \bar{\alpha}\beta)\,\gamma + \alpha\bar{\alpha}\langle D(T;0,\infty)\rangle + \beta\bar{\beta}\langle D(T;\infty,0)\rangle,\end{aligned} \tag{4.10}$$

completing the proof. □

Proof of Theorem 2.2. We use a similar equation to (4.1). Let L_R be an oriented link diagram that contains a tangle R, which is oriented as in Fig. 5. Applying the axioms (3.13) and (3.14), we may express the HOMFLY polynomial $P(L_R)$ of L_R in terms of $P(L_0)$ and $P(L_\infty)$:

$$P(L_R) = \varphi P(L_0) + \psi P(L_\infty), \tag{4.11}$$

where $\varphi = \varphi(t, z)$, $\psi = \psi(t, z) \in \mathbb{Z}[t^{\pm 1}, z^{\pm 1}]$. Then we have

$$P(L_{R^*}) = \bar{\varphi} P(L_0) + \bar{\psi} P(L_\infty); \tag{4.12}$$

$$P(L_{R_\perp}) = \psi P(L_0) + \varphi P(L_\infty); \tag{4.13}$$

$$P(L_{R_\perp^*}) = \bar{\psi} P(L_0) + \bar{\varphi} P(L_\infty), \tag{4.14}$$

where $\bar{\varphi} = \bar{\varphi}(t, z) = \varphi(-t^{-1}, z)$ and $\bar{\psi} = \bar{\psi}(t, z) = \psi(-t^{-1}, z)$. Using these formulas, we may prove in a similar way to Theorem 2.1. □

Proof of Theorem 2.3. Let L_R be an unoriented link diagram that contains a tangle R. Applying the axioms (3.10) and (3.11), we may express the Q polynomial $Q(L_R)$ of L_R in terms of $Q(L_0)$ and $Q(L_\infty)$:

$$Q(L_R) = fQ(L_0) + gQ(L_\infty) + hQ(L_1), \tag{4.15}$$

where $f = f(x)$, $g = g(x)$, $h = h(x) \in \mathbb{Z}[x^{\pm 1}]$. Then we have

$$Q(L_{R_\perp}) = gQ(L_0) + fQ(L_\infty) + hQ(L_{-1}). \tag{4.16}$$

Let $Q(R_1, R_2)$ be the Q polynomial of the link $D(T; R_1, R_2)$. Then using (4.15), we have

$$\begin{aligned} Q(R, R) &= f^2 Q(0,0) + fgQ(0,\infty) + fhQ(0,1) \\ &\quad + fgQ(\infty,0) + g^2 Q(\infty,\infty) + ghQ(\infty,1) \\ &\quad + fhQ(1,0) + ghQ(1,\infty) + h^2 Q(1,1) \\ &= \left(f^2 + g^2\right) k + fg\left(Q(0,\infty) + Q(\infty,0)\right) \\ &\quad + fh\left(Q(0,1) + Q(1,0)\right) \\ &\quad + gh\left(Q(\infty,1) + Q(1,\infty)\right) + h^2 Q(1,1), \end{aligned} \tag{4.17}$$

where

$$k = Q(0,0) = Q(\infty,\infty), \tag{4.18}$$

which follows from the condition ($\star$). Similarly, using (4.16), we have

$$Q(R_\perp, R_\perp) = \left(f^2+g^2\right)k + fg\left(Q(0,\infty)+Q(\infty,0)\right) + gh\left(Q(0,-1)+Q(-1,0)\right)$$
$$+ fh\left(Q(\infty,-1)+Q(-1,\infty)\right) + h^2 Q(-1,-1). \tag{4.19}$$

If we flip over the link $D(T;R_1,R_2)$ by μ_y, we obtain $D(\mu_y T^*;\mu_y R_2,\mu_y R_1)$, where $\mu_y T^*$ is the image of T under the reflection with regard to the xy-plane; $\mu_y T^* = \rho_{xy}T$ with $\rho_{xy}(x,y,z) = (x,y,-z)$. Thus $D(\mu_y T^*;\mu_y R_2,\mu_y R_1) = D(\rho_{xy}T;\rho_{xy}R_2^*,\rho_{xy}R_1^*)$, which is the mirror image of the link $D(T;R_2^*,R_1^*)$, and so by [2, Property 1(c)]

$$Q(R_1,R_2) = Q(R_2^*,R_1^*). \tag{4.20}$$

By (3.11),

$$Q(0,1)+Q(0,-1) = x\left(Q(0,0)+Q(0,\infty)\right). \tag{4.21}$$

From (4.20), $Q(0,-1)=Q(1,0)$ and $Q(0,1)=Q(-1,0)$, and so this becomes

$$Q(0,1)+Q(1,0) = Q(-1,0)+Q(0,-1) = x\left(k+Q(0,\infty)\right). \tag{4.22}$$

Similarly, using $Q(-1,\infty) = Q(\infty,1)$ and $Q(1,\infty) = Q(\infty,-1)$, $Q(1,\infty)+Q(-1,\infty) = x\left(Q(\infty,\infty)+Q(0,\infty)\right)$ becomes

$$Q(1,\infty)+Q(\infty,1) = Q(\infty,-1)+Q(-1,\infty) = x\left(k+Q(0,\infty)\right). \tag{4.23}$$

Substituting (4.22) and (4.23) into (4.17) and (4.19) respectively, and using $Q(1,1)=Q(-1,-1)$ and $Q(0,\infty)=Q(\infty,0)$, which follow from (4.20), we obtain

$$\begin{aligned} Q(R,R) &= Q(R_\perp,R_\perp) \\ &= \left(f^2+g^2\right)k + 2fgQ(0,\infty) + x(f+g)h\left(k+Q(0,\infty)\right) + h^2Q(1,1), \end{aligned} \tag{4.24}$$

completing the proof. $\square$

5. Examples

Let $T(R_0,R_1,R_2,R_3,R_4)$ be the tangle as shown in Fig. 10. Then the following tangles satisfy the condition ($\star$) with respect to the sub-tangle R:

- $T(m,\epsilon,R,n,\epsilon)$ for any integers m, n.
- $T(R,\epsilon,R_2,\epsilon,R_4)$ for any tangles R_2, R_4.

Here $\epsilon = \pm 1$. In fact, $T(m, \epsilon, 0, n, \epsilon)$ and $T(m, \epsilon, \infty, n, \epsilon)$ are rational tangles [4], which are freely equivalent to the trivial tangles, and $T(0, \epsilon, R_2, \epsilon, R_4)$ and $T(\infty, \epsilon, R_2, \epsilon, R_4)$ are easily seen to be freely equivalent each other.

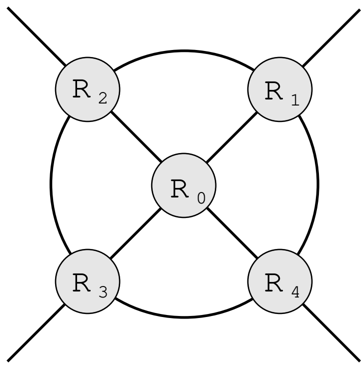


FIGURE 10. $T(R_0, R_1, R_2, R_3, R_4)$.

5.1. $T(3, -1, R, -2, -1)$. For the tangle $T(R) = T(3, -1, R, -2, -1)$, let us consider the oriented link diagram or the oriented link $DT(R_1, R_2)$ as shown in Fig. 11, which we denote by $L(R_1, R_2)$. When $L(R_1, R_2)$ is a 2-component oriented link, we denote by $\bar{L}(R_1, R_2)$ the oriented link obtained from $L(R_1, R_2)$ by changing the orientation of one component.

From Theorems 2.1 and 2.2, each of the following pairs of link diagrams or oriented links have the same Kauffman bracket and HOMFLY polynomials.

(i) $L(-2, 2)$ and $L\left(-\frac{1}{2}, \frac{1}{2}\right)$.
(ii) $L(-2, -2)$ and $L\left(\frac{1}{2}, \frac{1}{2}\right)$.
(iii) $L\left(-2, -\frac{1}{2}\right)$ and $L\left(2, \frac{1}{2}\right)$.

The pairs (i), (ii) are of two components and the pair (iii) are knots. Let us examine each pair.

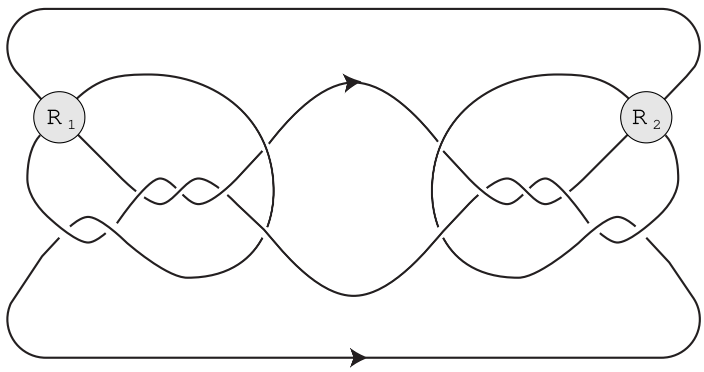


FIGURE 11. $L(R_1, R_2)$.

(i) $L(-2,2)$ and $L\left(-\frac{1}{2}, \frac{1}{2}\right)$. This pair have different Q polynomials, and thus have different Kauffman polynomials:

$$\begin{aligned}
Q(L(-2,2)) &= 2x^{-1} - 1 + 64x^2 + 96x^3 - 432x^4 - 120x^5 + 936x^6 + 56x^7 - 1088x^8 \\
&\quad - 104x^9 + 764x^{10} + 200x^{11} - 280x^{12} - 140x^{13} + 20x^{14} + 24x^{15} + 4x^{16}; \\
Q\left(L\left(-\tfrac{1}{2}, \tfrac{1}{2}\right)\right) &= 2x^{-1} - 1 - 32x + 48x^2 + 232x^3 - 396x^4 - 424x^5 + 832x^6 + 396x^7 - 904x^8 \\
&\quad - 252x^9 + 644x^{10} + 204x^{11} - 260x^{12} - 136x^{13} + 20x^{14} + 24x^{15} + 4x^{16}.
\end{aligned}$$

From Remark 2.4, the Conway polynomials of the oriented links $L(-2,2)$, $L(-1/2,1/2)$, $\bar{L}(-2,2)$, $\bar{L}(-1/2,1/2)$ are zeros. Further, since the writhes of these oriented diagrams are zeros and their linking numbers are zeros, they have the same Jones polynomials. However, the pairs $\bar{L}(-2,2)$, $\bar{L}(-1/2,1/2)$, which do not satisfy the condition for orientations of the sub-tangles in Theorem 2.2, have different HOMFLY polynomials:

$$\begin{aligned}
P(\bar{L}(-2,2)) &= (t^{-1} - t)z + (-t^{-5} + t^{-3} + 2t^{-1} - 2t - t^3 + t^5)z \\
&\quad + (-5t^{-5} + 6t^{-3} + 11t^{-1} - 11t - 6t^3 + 5t^5)z^3 \\
&\quad + (-4t^{-5} + 9t^{-3} + 13t^{-1} - 13t - 9t^3 + 4t^5)z^5 \\
&\quad + (-t^{-5} + 5t^{-3} + 6t^{-1} - 6t - 5t^3 + t^5)z^7 + (t^{-3} + t^{-1} - t - t^3)z^9;
\end{aligned}$$

$$
\begin{aligned}
P\left(\bar{L}\left(-\tfrac{1}{2},\tfrac{1}{2}\right)\right) = {} & (t^{-1}-t)z + (-4t^{-3}+12t^{-1}-12t+4t^3)z \\
& + (-24t^{-3}+76t^{-1}-76t+24t^3)z^3 + (-37t^{-3}+135t^{-1}-135t+37t^3)z^5 \\
& + (-25t^{-3}+112t^{-1}-112t+25t^3)z^7 + (-8t^{-3}+49t^{-1}-49t+8t^3)z^9 \\
& + (-t^{-3}+11t^{-1}-11t+t^3)z^{11} + (t^{-1}-t)z^{13}.
\end{aligned}
$$

(ii) $L(-2,-2)$ and $L\left(\frac{1}{2},\frac{1}{2}\right)$. From Theorem 2.3, this pair have the same Q polynomials. As mentioned above, they have the same HOMFLY polynomials, and thus have the same Jones and Conway polynomials. Note that their linking numbers are -2. However, this pair have different Kauffman polynomials:

$$
\begin{aligned}
F(L(-2,-2)) = {} & (a^5+a^3)x^{-1} - a^4 \\
& + (-2a^9-5a^7-4a^5+4a^3+5a^1-a^{-1}-a^{-3})x + O(x^2); \\
F\left(L\left(\tfrac{1}{2},\tfrac{1}{2}\right)\right) = {} & (a^5+a^3)x^{-1} - a^4 \\
& + (-a^9-2a^7-2a^5+2a^3+2a^1-2a^{-1}-a^{-3})x + O(x^2).
\end{aligned}
$$

Similarly, the pair $\bar{L}(-2,-2)$, $\bar{L}(1/2,1/2)$ have the same Jones and Q polynomials, but have different Kauffman polynomials. Further they have different Conway polynomials, and thus have different HOMFLY polynomials:

$$
\begin{aligned}
\nabla(\bar{L}(-2,-2)) &= 2z + z^3 - 4z^5 - 10z^7 - 4z^9; \\
\nabla\left(\bar{L}\left(\tfrac{1}{2},\tfrac{1}{2}\right)\right) &= 2z + z^3 - 12z^5 - 24z^7 - 19z^9 - 7z^{11} - z^{13}.
\end{aligned}
$$

(iii) $L\left(-2,-\frac{1}{2}\right)$ and $L\left(2,\frac{1}{2}\right)$. As mentioned above, these are knots and have the same HOMFLY, Jones and Conway polynomials. They have different Q polynomials, and thus have different Kauffman polynomials:

$$
\begin{aligned}
Q\left(L\left(-2,-\tfrac{1}{2}\right)\right) = {} & -3 - 26x - 12x^2 + 36x^3 + 114x^4 - 24x^5 - 300x^6 - 60x^7 + 404x^8 \\
& + 212x^9 - 266x^{10} - 252x^{11} + 24x^{12} + 104x^{13} + 44x^{14} + 6x^{15}; \\
Q\left(L\left(2,\tfrac{1}{2}\right)\right) = {} & -3 + 6x + 4x^2 - 100x^3 + 78x^4 + 280x^5 - 196x^6 - 400x^7 + 220x^8 \\
& + 360x^9 - 146x^{10} - 256x^{11} + 4x^{12} + 100x^{13} + 44x^{14} + 6x^{15}.
\end{aligned}
$$

5.2. $T(2,-1,R,-3,-1)$. For the tangle $T(R)=T(2,-1,R,-3,-1)$, let us consider the oriented link diagram or the oriented link $DT(R_1,R_2)$ as shown in Fig. 12, which we denote by $M(R_1,R_2)$.

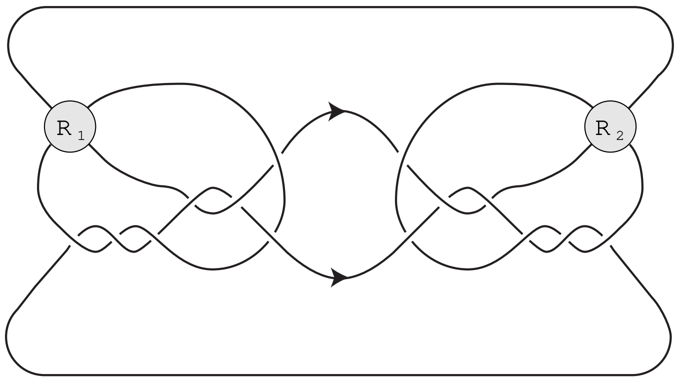


FIGURE 12. $M(R_1,R_2)$.

From Theorem 2.1 (iii), the following pair of 2-component link diagrams have the same Kauffman bracket polynomials:

- $M\left(-2,-\frac{1}{2}\right)$ and $M\left(2,\frac{1}{2}\right)$.

Since the writhes of these oriented diagrams are 10, they have the same Jones polynomials. They have different Conway polynomials, and thus have different HOMFLY polynomials:

$$\nabla\left(M\left(-2,-\tfrac{1}{2}\right)\right)=4z+5z^3-7z^5+3z^9;$$
$$\nabla\left(M\left(2,\tfrac{1}{2}\right)\right)=4z+5z^3-15z^5-6z^7+2z^9.$$

Further, they have different Q polynomials and thus have different Kauffman polynomials:

$$\begin{aligned}Q\left(M\left(-2,-\tfrac{1}{2}\right)\right)=&-6x^{-1}+7-6x-74x^2+74x^3+314x^4-74x^5-594x^6-90x^7\\&+602x^8+234x^9-374x^{10}-268x^{11}+72x^{12}+130x^{13}+48x^{14}+6x^{15};\\Q\left(M\left(2,\tfrac{1}{2}\right)\right)=&-6x^{-1}+7+58x-42x^2-230x^3+98x^4+414x^5-158x^6-402x^7\\&+234x^8+286x^9-242x^{10}-252x^{11}+56x^{12}+126x^{13}+48x^{14}+6x^{15}.\end{aligned}$$

5.3. $T(R,1,R_2,1,R_4)$. In addition to Theorems 2.1–2.3, the following holds in this class.

Theorem 5.1. *Let $U = U(R) = T(R, 1, R_2, 1, R_4)$. Then the pair of the link diagrams $D(U; R, R^*)$ ($= DU(R)$) and $D(U; R_\perp^*, R_\perp)$ ($= DU(R_\perp^*)$) share the same Λ polynomial for any tangles R_2 and R_4.*

Proof. Let G_R be a link diagram that contains a tangle R. Applying the axioms (3.6)–(3.8), we may express the Λ polynomial $\Lambda(G_R)$ of G_R in terms of $\Lambda(G_0)$, $\Lambda(G_\infty)$ and $\Lambda(G_1)$:

$$\Lambda(G_R) = f\Lambda(G_0) + g\Lambda(G_\infty) + h\Lambda(G_1), \tag{5.1}$$

where $f = f(a, x)$, $g = g(a, x)$, $h = h(a, x) \in \mathbb{Z}[a^{\pm 1}, x^{\pm 1}]$. Then we have

$$\Lambda(G_{R^*}) = \bar{f}\Lambda(G_0) + \bar{g}\Lambda(G_\infty) + \bar{h}\Lambda(G_{-1}); \tag{5.2}$$

$$\Lambda(G_{R_\perp}) = g\Lambda(G_0) + f\Lambda(G_\infty) + h\Lambda(G_{-1}); \tag{5.3}$$

$$\Lambda(G_{R_\perp^*}) = \bar{g}\Lambda(G_0) + \bar{f}\Lambda(G_\infty) + \bar{h}\Lambda(G_1), \tag{5.4}$$

where $\bar{f} = \bar{f}(a, x) = f(a^{-1}, x)$, $\bar{g} = \bar{g}(a, x) = g(a^{-1}, x)$, $\bar{h} = \bar{h}(a, x) = h(a^{-1}, x)$. Let $\Lambda(R_1, R_2)$ be the Λ polynomial of the link diagram $D(U; R_1, R_2)$. Then using (5.1), we have

$$\begin{aligned} \Lambda(R, R^*) = {} & f\bar{f}\Lambda(0,0) + f\bar{g}\Lambda(0,\infty) + f\bar{h}\Lambda(0,-1) \\ & + \bar{f}g\Lambda(\infty,0) + g\bar{g}\Lambda(\infty,\infty) + g\bar{h}\Lambda(\infty,-1) \\ & + \bar{f}h\Lambda(1,0) + \bar{g}h\Lambda(1,\infty) + h\bar{h}\Lambda(1,-1). \end{aligned} \tag{5.5}$$

It is easy to see that the following pairs of link diagrams are regular isotopic:

$D(U; 0, 0)$ and $D(U; \infty, \infty)$; $D(U; 0, -1)$ and $D(U; 1, \infty)$; $D(U; \infty, -1)$ and $D(U; 1, 0)$.

Thus we have

$$\begin{aligned} \Lambda(R, R^*) = {} & (f\bar{f} + g\bar{g})\Lambda(0,0) + (f\bar{h} + \bar{g}h)\Lambda(0,-1) + (g\bar{h} + \bar{f}h)\Lambda(\infty,-1) \\ & + f\bar{g}\Lambda(0,\infty) + \bar{f}g\Lambda(\infty,0) + h\bar{h}\Lambda(1,-1). \end{aligned} \tag{5.6}$$

From (5.1)–(5.4), we see that $\Lambda(R_\perp^*, R_\perp)$ is obtained from (5.6) by exchanging f, g, h for $\bar{g}$, $\bar{f}$, $\bar{h}$, respectively. Therefore we obtain:

$$\Lambda(R, R^*) = \Lambda(R_\perp^*, R_\perp). \tag{5.7}$$

This completes the proof. □

For the tangle $U(R) = T(R, 1, 1, 1, 2)$, let us consider the oriented link diagram $D(U; R, R^*)$ as shown in Fig. 13, which we denote by $N(R)$. Then the pair $N(R)$ and $N(R_\perp^*)$ share the same Kauffman bracket (Theorem 2.1 (i)), HOMFLY (Theorem 2.2 (i)), and Λ (Theorem 5.1) polynomials. In general, $N(R)$ and $N(R^*_\perp)$ are distinct. Take $N(-2)$ and $N(-1/2)$ for example; each of them is a 2-component link consisting of a trivial knot and a square knot. Let $N(-2)'$ and $N(-1/2)'$ be the 4-component links obtained from $N(-2)$ and $N(-1/2)$ by taking 3-cables about the unknotted components, respectively. Then by the computer calculation, we see that the Jones polynomials of $N(-2)'$ and $N(-1/2)'$ are distinct. We omit the detail, which will be found in [29].

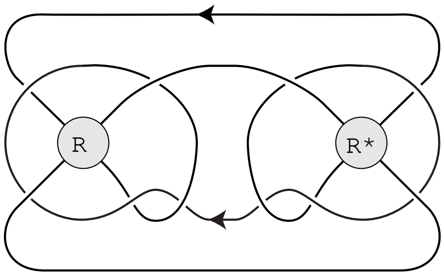


FIGURE 13. $N(R)$.

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