ELLIPTIC GENERA, TORUS MANIFOLDS AND MULTI-FANS

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1. Introduction

The rigidity theorem of Witten-Bott-Taubes-Hirzebruch [W, BT, H] tells us that, if the circle group acts on a closed almost complex (or more generally unitary) manifold whose first Chern class is divisible by a positive integer N greater than 1, then its equivariant elliptic genus of level N is rigid. Applying this to a non-singular compact toric variety we see that its elliptic genus of level N is rigid if its first Chern class is divisible by N. But, using a vanishing theorem of Hirzebruch [H], we can show moreover that the genus actually vanishes. In this note we shall extend this result to torus manifolds [M, HM]. A torus manifold is an oriented closed manifold of even dimension which admits an action of a torus of half the dimenstion of the manifold with some orientation data concerning codimension two fixed point set components of circle subgroups. Though there is not an almost complex nor unitary structure on a torus manifold in general, one can still define elliptic genus of level N on torus manifolds. In the almost complex or unitary case the elliptic genus is defined as the index of a twisted Dirac operator, so that the equivariant index is a virtual character of the torus acting on the manifold. For general torus manifolds the equivariant elliptic genus of level N is defined in terms of the multi-fan associated with the torus manifold. Thus the genus is in fact defined for multi-fans (precisely speaking for complete simplicial multi-fans). The fact that it is a character of a torus is proved by using the multiplicity formula for Duistermaat-Heckman function for multi-polytopes given in [HM]. The Chern class is also defined for multi-fans, and rigidity and vanishing of elliptic genus of level N can be formulated and proved. One of the main results is Corollary 4.4 which states that, if the first Chern class of a complete non-singular multi-fan is divisible by N, then the elliptic genus of level N vanishes. The proof of rigidity and vanishing of level N elliptic genus follows the idea of the proof given in [H]. The formula for Duistermaat-Heckman function is also used to modify topological terms into combinatorial terms so as to be applicable to multi-fans. When $N = 2$, the torus manifold is a spin manifold. The corresponding multi-fan might be called a spin multi-fan. As a corollary we see that its signature vanishes in this case.

The T_y-genus can be considered as a special value of equivariant elliptic genus. Thus, if the first Chern class is divisible by N, then T_y-genus vanishes for $(-y)^N = 1$. One can derive some applications from this fact. For example if Δ is a complete non-singular multi-fan of dimension n with first Chern class $c_1(\Delta)$ divisible by N and with non-vanishing Todd genus, then N must be equal or less than $n + 1$ (Proposition 5.2). In the extremal case $N = n + 1$, if Δ is assumed to be a complete non-singular ordinary fan, then Δ must be isomorphic to the fan of projective space $\mathbb{P}^n$. Hence a complete non-singular toric variety M of dimension n with $c_1(M)$ divisible by $n + 1$ must be isomorphic to $\mathbb{P}^n$ as toric varity (Corollary 5.4). We show furthermore that, in case $c_1(M)$ is divisible by n, M is isomorphic to a certain projective space bundle over $\mathbb{P}^1$ (Corollary 5.8).

The paper is organizd as follows. In Section 2 we recall some basic facts about multi-fans from [HM]. In Section 3 we define equivariant first Chern class of multi-fans and discuss properties concerning its divisibility. Section 4 is the main part of the paper. The equivariant elliptic genus of complete non-singular multi-fans is introduced and the main results are stated and proved here. The main results are Theorem 4.3 and Corollary 4.4. Section 5 is devoted to applications. Finally in Section 6 we discuss phenomena which arises when the multi-fan is not assumed to be non-singular.

2. Multi-fans

We refer to [HM] for notions and notations concerning multi-fans and torus manifolds. We shall summarize some of them. Let $\Delta = (\Sigma, C, w^{\pm})$ be an n-dimensional complete non-singular multi-fan. Here Σ is an augmented simplicial set which means that Σ is a simplicial set with empty set $* = \emptyset$ added as (-1)-dimenional simplex. $\Sigma^{(k)}$ denotes the $k-1$ skeleton of Σ so that $* \in \Sigma^{(0)}$. We assume that $\Sigma = \sum_{k=0}^{n} \Sigma^{(k)}$. There is associated with Δ an n-dimensional lattice L (the notation N was used in [HM]). C is a map from $\Sigma^{(k)}$ into the set of k-dimensional strongly convex rational polyhedral cones in the vector space $L_{\mathbb{R}} = L \otimes \mathbb{R}$ for each k such that, if J is a face of I, then $C(J)$ is a face of $C(I)$. $w^{\pm}$ are maps $\Sigma^{(n)} \to \mathbb{Z}$ which, when Σ is complete, satisfy certain compatibility conditions. We set $w(I) = w^+(I) - w^-(I)$. T will denote an n-dimensinal torus. We identify $\mathrm{Hom}(S^1, T) = H_2(BT)$ with the lattice L. Thus there is a unique primitive vector $v_i \in H_2(BT)$ which generates the cone $C(i)$ for each $i \in \Sigma^{(1)}$. Non-singularity of Σ means that $\{v_i \mid i \in I\}$ is a basis of the lattice $L = H_2(BT)$ for each $I \in \Sigma^{(n)}$. Let $\{u_i^I\}$ be the dual basis of $\{v_i \mid i \in I\}$ in the dual lattice $L^* = H^2(BT)$. $H^2(BT)$ is canonically identified with $\mathrm{Hom}(T, S^1)$. The latter is embedded in the character ring $R(T)$. In fact $R(T)$ can be considered as the group ring of the group $\mathrm{Hom}(T, S^1)$. It is convenient to write the element in $R(T)$ corresponding to $u \in H^2(BT)$ by t^u. The homomorphism $v^* : R(T) \to R(S^1)$ induced by an element $v \in H_2(BT) = \mathrm{Hom}(S^1, T)$ can be written in the form

$$v^*(t^u) = t^{\langle u, v \rangle},$$

where $\langle u, v \rangle$ is the dual pairing and $t^m \in \mathrm{Hom}(S^1, S^1) \subset R(S^1)$, $m \in \mathbb{Z}$, is defined by $t^m(\lambda) = \lambda^m$.

We define the equivariant cohomology $H_T^*(\Delta)$ of a complete non-singular multi-fan Δ as the face ring of the simplicial complex Σ. Namely let $\{x_i\}$ be indeterminates indexed by $\Sigma^{(1)}$, and let R be the the polynomial ring over the integers generated by $\{x_i\}$. We denote by $\mathcal{I}$ the ideal in R generated by monomials $\prod_{i \in J} x_i$ such that $J \notin \Sigma$. $H_T^*(\Delta)$ is by definition the quotient $R/\mathcal{I}$. We regard $H^2(BT)$ as a submodule of $H_T^2(\Delta)$ by the formula

$$u = \sum_{i \in \Sigma^{(1)}} \langle u, v_i \rangle x_i. \tag{1}$$

This determines an $H^*(BT)$-module structure of $H_T^*(\Delta)$. For each $I \in \Sigma^{(n)}$ we define the restriction homomorphism $i_I^* : H_T^*(\Delta) \to H^*(BT)$ by

$$i_I^*(x_i) = \begin{cases} u_i^I & \text{for } i \in I \\ 0 & \text{for } i \notin I. \end{cases} \tag{2}$$

It follows from (1) that $i_I^*|H^2(BT)$ is the identity map for any I, and $\sum_{I \in \Sigma^{(n)}} i_I^*$ is injective.

The following lemma is a consequence of Corollary 7.4 in [HM].

Lemma 2.1. *For any $x = \sum_{i\in\Sigma^{(1)}} c_i x_i \in H^2_T(\Delta)$, $c_i \in \mathbb{Z}$, the element*

$$\sum_{I\in\Sigma^{(n)}} w(I)\frac{t^{i_I^*(x)}}{\prod_{i\in I}(1-t^{-u_i^I})}$$

in the quotient ring of $R(T)$ actually belongs to $R(T)$.

We also use an extended version of Corollary 7.4 in [HM]. Given $K \in \Sigma^{(k)}$ an $(n-k)$-dimensional multi-fan $\Delta_K = (\Sigma_K, C_K, w_K^{\pm})$, which we called projected multi-fan, was defined in [HM]. Σ_K consists of such $I \in \Sigma$ that $K \subset I$. If I is in $\Sigma^{(l)}$ then I is considered as lying in $\Sigma_K^{(l-k)}$. An element of $\Sigma_K^{(1)}$ is of the form $K \cup \{i\}$ which we identify with i. When $K = \emptyset$ then $\Delta_K = \Delta$. The lattice L_K is the quotient of L by the susbmodule generated by $\{v_i \mid i \in K\}$, and $C_K(I) \in L_K \otimes \mathbb{R}$ is the projection image of $C(I) \in L_{\mathbb{R}} = L \otimes \mathbb{R}$. The torus T_K corresponding to L_K is a quotient of T. We consider the polynomial ring R_K generated by $\{x_i \mid i \in K \cup \Sigma_K^{(1)}\}$ and the ideal $\mathcal{I}_K$ generated by monomials $\prod_{i\in J} x_i$ such that $J \notin \Sigma_K$. We define the equivariant cohomology $H^2_T(\Delta_K)$ of Δ_K with respect to the torus T as the quotient ring $R_K/\mathcal{I}_K$. $H^2(BT)$ is regarded as a submodule of $H^2_T(\Delta_K)$ by a formula similar to (1). The projection $H^2_T(\Delta) \to H^2_T(\Delta_K)$ is defined by putting $x_i = 0$ for $i \notin K \cup \Sigma_K^{(1)}$, which is an $H^2(BT)$-module homomorphism. The restriction homomorphism $i_I^* : H^2_T(\Delta_K) \to H^2(BT)$ is also defined for $I \in \Sigma_K^{(n-k)}$. Given $x = \sum_i c_i x_i \in H^2_T(\Delta_K; \mathbb{R})$, $c_i \in \mathbb{R}$, let A^* be the affine subspace in the dual space $L^*_{\mathbb{R}}$ defined by $\langle u, v_i\rangle = c_i$ for $i \in K$. Then we introduce a collection $\mathcal{F}_K = \{F_i \mid i \in \Sigma_K^{(1)}\}$ of affine hyperplanes in A^* by setting

$$F_i = \{u \mid u \in A^*,\ \langle u, v_i\rangle = c_i\}.$$

The pair $\mathcal{P}_K = (\Delta_K, \mathcal{F}_K)$ will be called a multi-polytope associated with x; see [HM] for the case $K = \emptyset$. For $I \in \Sigma_K^{(n-k)}$, i.e. $I \in \Sigma^{(n)}$ with $I \supset K$, we put $u_I = \cap_{i\in I} F_i \in A^*$. Note that u_I is equal to $i_I^*(x)$. The dual vector space $L_K^* \otimes \mathbb{R}$ of $L_K \otimes \mathbb{R}$ is canonically idenitified with the subspace $\{u \mid \langle u, v_i\rangle = 0,\ i \in K\}$ of $L^* \otimes \mathbb{R} = H^2(BT; \mathbb{R})$. It is parallel to A^*, and u_i^I lies in $L_K^* \otimes \mathbb{R}$ for $I \in \Sigma_K^{(n-k)}$ and $i \in I \setminus K$. A vector $v \in L_K \otimes \mathbb{R}$ is called generic if $\langle u_i^I, v\rangle \neq 0$ for any $I \in \Sigma_K^{(n-k)}$ and $i \in I \setminus K$. We take a generic vector $v \in L_K \otimes \mathbb{R}$, and define, for $I \in \Sigma_K^{(n-k)}$ and $i \in I \setminus K$,

$$(-1)^I := (-1)^{\#\{j\in I\setminus K \mid \langle u_j^I, v\rangle > 0\}} \quad \text{and} \quad (u_i^I)^+ := \begin{cases} u_i^I & \text{if } \langle u_i^I, v\rangle > 0 \\ -u_i^I & \text{if } \langle u_i^I, v\rangle < 0. \end{cases}$$

We denote by $C_K^*(I)^+$ the cone in A^* spanned by the $(u_i^I)^+$, $i \in I \setminus K$, with apex at u_I, and by ϕ_I its characteristic function. With these understood, we define a function $\mathrm{DH}_{\mathcal{P}_K}$ on $A^* \setminus \cup_i F_i$ by

$$\mathrm{DH}_{\mathcal{P}_K} := \sum_{I\in\Sigma_K^{(n-k)}} (-1)^I w(I)\phi_I.$$

As in [HM] we call this function the Duistermaat-Heckman function associated with $\mathcal{P}_K$.

Lemma 2.2. *The support of the function $\mathrm{DH}_{\mathcal{P}_K}$ is bounded, and the function is independent of the choice of generic vector v.*

The proof is similar to that of Lemma 5.4 in [HM]. We shall denote by $\mathcal{P}_{K+}$ the multi-polytope associated with $x_+ = \sum_{i\in K} c_i x_i + \sum_{i\in\Sigma_K^{(1)}}(c_i+\epsilon)x_i$ where $0<\epsilon<1$. The following theorem is a generalization of Corollary 7.4 in [HM].

Theorem 2.3. *Let x and x_+ be as above with all c_i integers. Then*

$$\sum_{u\in A^*\cap L^*} \mathrm{DH}_{\mathcal{P}_{K+}}(u)t^u = \sum_{I\in\Sigma_K^{(n-k)}} w(I)\frac{t^{i_I^*(x)}}{\prod_{i\in I\setminus K}(1-t^{-u_i^I})}.$$

In particular the right hand side belongs to $R(T)$.

The proof is similar to that of Corollary 7.4 in [HM].

Take $I\in\Sigma_K^{(n-k)}$ and let G_I be the subgroup of the permutation group of I consisting of those elements which are identity on K. Let $\mathcal{L}_I$ be the set of all linear forms $\sum_{i\in I} m_i u_i^I$ with integer coefficients m_i. The group G_I acts on $\mathcal{L}_I$. Let $\mathcal{O}_I$ denote the set of orbits of that action. If I' is also in $\Sigma_K^{(n-k)}$, take a bijection $f: I\to I'$ which is the identity on K. It induces a bijection $f_*:\mathcal{O}_I\to\mathcal{O}_{I'}$. It is easy to see that f_* does not depend on the choice of particular f. Thus we can write them simply $\mathcal{O}$. For $\alpha\in\mathcal{O}$ we define t_I^α by

$$t_I^\alpha = \sum_{l\in\alpha} t^l,$$

where α is regarded as contained in $\mathcal{O}_I$ in the sum at the right hand side.

Corollary 2.4. *For any $\alpha\in\mathcal{O}$ the expression*

$$\sum_{I\in\Sigma_K^{(n-k)}} w(I)\frac{t_I^\alpha}{\prod_{i\in I\setminus K}(1-t^{-u_i^I})}$$

belongs to $R(T)$.

Proof. Let $\bar{G}$ denote the subgroup of permutation group of $K\cup\Sigma_K^{(1)}$ consisting of those elements which are identity on K. It acts on the set $\bar{\mathcal{L}}$ of linear forms $l=\sum_{i\in K\cup\Sigma_K^{(1)}} m_i x_i$. Let $\bar{\mathcal{O}}$ denote the set of orbits of that action on $\bar{\mathcal{L}}$. There is an obvious map of $\mathcal{O}$ in $\bar{\mathcal{O}}$. We denote the image of $\alpha\in\mathcal{O}$ in $\bar{\mathcal{O}}$ by $\bar\alpha$. We define the length of a linear form l as the maximal number of i's with $i\in\Sigma_K^{(1)}$ and $m_i\neq 0$. The length is invariant under the action of $\bar{G}$, so that the length $|\bar\alpha|$ of $\bar\alpha\in\bar{\mathcal{O}}$ is defined as that of a linear form contained in $\bar\alpha$. The length is also defined for $\alpha\in\mathcal{O}$ independently of I and we have $|\alpha|=|\bar\alpha|$. The proof will proceed by induction on the length of α. If $|\alpha|=0$, then it is clear that α consists of a single linear form $l=\sum_{i\in K} m_i x_i$. Applying Theorem 2.3 to $x=l$ we see that the statement of Corollary is true in this case. Suppose that $|\alpha|>0$. Then it is not difficult to see that

$$\sum_{l\in\bar\alpha} t^{i_I^*(l)} = t_I^\alpha + \sum_{|\beta|<|\alpha|} a_\beta t_I^\beta$$

where $a_\beta\in\mathbb{Z}$ does not depend on I. We apply Theorem 2.3 and use induction assumption to conclude that

$$\sum_{I\in\Sigma_K^{(n-k)}} w(I)\frac{t_I^\alpha}{\prod_{i\in I\setminus K}(1-t^{-u_i^I})}$$

belongs to $R(T)$. ∎

If we take K to be the empty set, the G_I is the permutation group of I, and t_I^α is a symmetric character with respect to G_I. We obtain

Corollary 2.5. *For any $\alpha \in \mathcal{O}$ the expression*

$$\sum_{I \in \Sigma^{(n)}} w(I) \frac{t_I^\alpha}{\prod_{i \in I}(1 - t^{-u_i^I})}$$

belongs to $R(T)$.

3. Equivariant first Chern class

Let Δ be a complete non-singular multi-fan as in the previous section. The class

$$\sum_{i \in \Sigma^{(1)}} x_i$$

will be called equivariant first Chern class of Δ, and will be denoted by $c_1^T(\Delta)$. Its image $c_1(\Delta)$ in $H^2(\Delta) = H_T^2(\Delta)/H^2(BT)$ is called first Chern of Δ. Let $N > 1$ be an integer. The first Chern class $c_1(\Delta)$ is divisible by N if and only if $c_1^T(\Delta)$ is of the form

$$c_1^T(\Delta) = Nx + u, \ x \in H_T^2(\Delta), \ u \in H^2(BT).$$

We set $u^I = i_I^*(c_1^T(\Delta)) = \sum_{i \in I} u_i^I \in H^2(BT)$.

Lemma 3.1. *The following three conditions are equivalent:*

1. *the first Chern class $c_1(\Delta)$ is divisible by N,*
2. $u^I \bmod N$ *is independent of $I \in \Sigma^{(n)}$,*
3. *there is an element $u \in H^2(BT)$ such that $\langle u, v_i \rangle = 1 \bmod N$ for all $i \in \Sigma^{(1)}$.*

Proof. Suppose $c_1^T(\Delta)$ is of the form

$$c_1^T(\Delta) = Nx + u, \ x \in H_T^2(\Delta), \ u \in H^2(BT).$$

Then $u^I = i_I^*(c_1^T(\Delta)) \bmod N$ is equal to $i_I^*(u) = u$, and hence is independent of I. Thus (1) implies (2).

Suppose that $u \bmod N$ is equal to $u^I \bmod N$ for any $I \in \Sigma^{(n)}$. Then $\langle u, v_i \rangle = \langle u^I, v_i \rangle \bmod N$, and hence $\langle u, v_i \rangle = \sum_{j \in I} \langle u_j^I, v_i \rangle = 1 \bmod N$ for $i \in I$. Thus (2) implies (3).

Suppose $\langle u, v_i \rangle = 1 \bmod N$ for any $i \in \Sigma^{(1)}$. Then, by (1),

$$c_1^T(\Delta) - u = \sum_{i \in \Sigma^{(1)}} (1 - \langle u, v_i \rangle) x_i = 0 \bmod N.$$

Hence $c_1^T(\Delta)$ is of the form $c_1^T(\Delta) = Nx + u$ because $H_T^2(\Delta)$ is a free module. Thus (3) implies (1). □

Remark. Let $\hat{H}_T^2(M) = H_T^2(M)/H^*(BT)$-torsion. In [M] it was shown that, if M is a torus manifold and $\Delta(M)$ is its associated multi-fan, then there is a canonical identification of $H_T^2(\Delta(M))$ with $\hat{H}_T^2(M)$, and, in case M is a unitary torus manifold, $c_1^T(M) \in H_T^2(M)$ descends to $c_1^T(\Delta)$. It follows that, if M is a unitary torus manifold and $c_1(M)$ is divisible by N, then $c_1(\Delta(M))$ is also divisible by N.

Let $v \in H^2(BT)$ be a generic integral vector. If we fix I and write

$$v = \sum_{i \in I} m_i v_i,$$

then $m_i = \langle u_i^I, v\rangle$. Since $u^I = \sum_{i\in I} u_i^I$, we have $\langle u^I, v\rangle = \sum_i m_i$. Let $m \geq 1$ be an integer. Fixing a generic integral vector v and m, we put

$$I_{(m)} := \{i \in I \mid m \text{ does not divide } m_i\}$$

for $I \in \Sigma^{(n)}$. It will be called mod$m$ face of I.

Lemma 3.2. *Let v, m and I be as above, and let $K = I_{(m)}$ be the* mod$m$ *face of I. If $I' \in \Sigma^{(n)}$ contains K, then K is also the* mod$m$ *face of I'. Moreover, $\langle u_i^{I'}, v\rangle = \langle u_i^I, v\rangle \bmod m$ for $i \in K$.*

Proof. We put $m_i = \langle u_i^I, v\rangle$ and $m'_{i'} = \langle u_{i'}^{I'}, v\rangle$. Then we have

$$v = \sum_{i\in I} m_i v_i = \sum_{i' \in I'} m'_{i'} v_{i'}.$$

By assumption $m_i = 0 \bmod m$ for $i \notin K$. Hence

$$\sum_{i\in K} m_i v_i = \sum_{i'\in I'} m'_{i'} v_{i'} \mod m.$$

or

$$\sum_{i\in K} (m'_i - m_i) v_i + \sum_{i' \in I',\ i' \notin K} m'_{i'} v_{i'} = 0 \mod m.$$

Since $\{v_{i'} \mid i' \in I'\}$ is a basis of the free module $H_2(BT)$, we see that $m'_{i'} = 0 \bmod m$ for $i' \in I'$, $i' \notin K$ and $m'_i = m_i \bmod m$ for $i \in K$. □

We shall say that I and I' are (v, m)-equivalent and write $I \sim I'$ if $I_{(m)} = I'_{(m)}$. This defines an equivalence relation $\sim$ in $\Sigma^{(n)}$. Lemma 3.2 implies that its equivalence class X forms a projected multi-fan $\Delta_K = (\Sigma_K, C_K, w_K^{\pm})$ where K is the common mod m face of the members of X. We shall call this K the core of the equivalence class X.

Lemma 3.3. *Let X be an equivalence class of (v, m)-equivalence relation. For $x \in H_T^2(\Delta)$ the value $\langle i_I^*(x), v\rangle$* mod m *does not depend on the choice of I in X.*

Proof. Write $x = \sum_{i\in\Sigma^{(1)}} a_i x_i$. Let K denote the common mod m face of $I \in X$. Then

$$\langle i_I^*(x), v\rangle = \sum_{i\in K} a_i \langle u_i^I, v\rangle + \sum_{i \in I, \notin K} a_i \langle u_i^I, v\rangle.$$

Since $\langle u_i^I, v\rangle = 0 \mod m$ for $i \notin K$ and $\langle u_i^I, v\rangle \mod m$ does not depend on I in X by Lemma 3.2, $\langle i_I^*(x), v\rangle \mod m$ does not depend on the choice of I in X. □

Corollary 3.4. *Assume that $c_1(\Delta)$ is divisible by N, and write $c_1^T(\Delta) = Nx + u, u \in H^2(BT)$. Let v and m be as above. If we write $\langle u_i^I, v\rangle$ in the form*

$$\langle u_i^I, v\rangle = mh_i + r_i \text{ with } 0 \leq r_i < m,$$

then the sum $\sum_{i\in I} h_i$ mod N *depends only on the (v, m)-equivalence class X of I.*

Proof. We put $h_I = \sum_{i\in I} h_i$ and $r_I = \sum_{i\in I} r_i$. Then $\langle u^I, v\rangle = mh_I + r_I$. By Lemma 3.3 $\langle i_I^*(x), v\rangle$ is of the form

$$\langle i_I^*(x), v\rangle = mh'_I + r',$$

for $I \in X$, where r' is independent of I. Therefore, if we write $\langle u, v\rangle = r''$, then

$$\langle u^I, v\rangle = Nmh'_I + Nr' + r''.$$

If we compare this with

$$\langle u^I, v\rangle = mh_I + r_I,$$

we see that $Nr' + r''$ is of the form $Nr' + r'' = mh' + r_I$ and $h_I = Nh'_I + h'$. This shows that h_I mod N depends only on X. □

Under the situation of Corollary 3.4, the mod N value of $\sum_{i\in I} h_i$ will be called (v, m)-type of X and will be denoted by $h(v, m, X)$. Similarly the $\bmod N$ value of $\langle u^I, v\rangle$ (which is independent of $I \in \Sigma^{(n)}$ by Lemma 3.1) will be called v-type and will be denoted by $h(v)$.

Lemma 3.5. *Assume $c_1(\Delta)$ is divisible by N. Any non-zero $b \in \mathbb{Z}/N$ can occur as v-type when v varies over generic vectors in $H_2(BT)$.*

Proof. This follows readily from the fact that $\{u_i^I \mid i \in I\}$ is a basis of $H^2(BT)$. □

4. Elliptic genus of level N

We define the equivariant elliptic genus of a multi-fan. For that purpose we first consider the theta function

$$\theta_y(\lambda) = (1 + y\lambda)\prod_{n=1}^{\infty}(1 + yq^n\lambda)(1 + y^{-1}q^n\lambda^{-1}).$$

It is defined for $y \in \mathbb{C}^*$ and $q \in \mathbb{C}$ with $|q| < 1$. For $q \neq 0$ we have

$$\theta_y(q\lambda) = y^{-1}\lambda^{-1}\theta_y(\lambda). \tag{3}$$

We define

$$\phi_y(\lambda) = \frac{\theta_y(\lambda)}{\theta_{-1}(\lambda)}.$$

It transforms by

$$\phi_y(q\lambda) = -y^{-1}\phi_y(\lambda) \quad \text{or} \quad \phi_y(q^{-1}\lambda) = -y\phi_y(\lambda), \tag{4}$$

by virtue of (3). We exclude the case $y = -1$. It follows that, if $(-y)^N = 1$, then ϕ_y is the pull-back of a rational function on the torus $\mathbb{C}^*/q^N$. In this case we shall regard ϕ_y as the function on that torus.

Let $\Delta = (\Sigma, C, w^{\pm})$ be a complete non-singular multi-fan. We set formally

$$\varphi_y(t) = \sum_{I\in\Sigma^{(n)}} w(I)\prod_{i\in I}\phi_y(t^{-u_i^I}),$$

and call it the equivariant elliptic genus of the multi-fan Δ.

Lemma 4.1. *Let $\varphi_y(t) = \sum_{n=0}^{\infty}\varphi_{y,n}(t)q^n$ be the expansion into power series, then $\varphi_{y,n}(t)$ belongs to $R(T)\otimes\mathbb{Z}[y, y^{-1}]$ where $\mathbb{Z}[y, y^{-1}]$ denotes the ring of Laurent polynomials over $\mathbb{Z}$.*

Proof. $\varphi_{y,n}(t)$ is a linear combination with coefficients in $\mathbb{Z}[y, y^{-1}]$ of the expression of the following form:

$$\sum_{I \in \Sigma^{(n)}} w(I) \frac{t_I^\alpha}{\prod_{i \in I}(1 - t^{-u_i^I})},$$

where $\alpha \in \mathcal{O}$ as was introduced in Section 2. Therefore it belongs to $R(T) \otimes \mathbb{Z}[y, y^{-1}]$ by Corollary 2.5. □

Let $v \in H_2(BT)$ be a generic vector. We set

$$\varphi_y^v(t) = \sum_{I \in \Sigma^{(n)}} w(I) \prod_{i \in I} \phi_y(t^{-\langle u_i^I, v \rangle}).$$

If $(-y)^N = 1$, then $\varphi_y^v(t)$ is an elliptic function on $\mathbb{C}^*/q^N$, because $\phi_y(t)$ is such a function. In this case $\varphi_y^v(t)$ is called elliptic genus of level N. We have furthermore

Lemma 4.2. *Suppose that $c_1(\Delta)$ is divisible by N. Let $v \in H_2(BT)$ be a generic vector and $h(v)$ the v-type. Then the elliptic genus $\varphi_y^v(t)$ of level N transfoms by*

$$\varphi_y^v(qt) = \zeta^{h(v)} \varphi_y^v(t),$$

where $\zeta = -y$.

Proof.

$$\prod_{i \in I} \phi_y((qt)^{-\langle u_i^I, v \rangle}) = \zeta^{\sum_{i \in I} \langle u_i^I, v \rangle} \prod_{i \in I} \phi_y(t^{-\langle u_i^I, v \rangle})$$

by (4). But $\sum_{i \in I} \langle u_i^I, v \rangle = h(v) \mod N$ which is independent of I. Hence we obtain

$$\varphi_y^v(qt) = \zeta^{h(v)} \varphi_y^v(t).$$

□

The following theorem and corollary are versions of rigidity theorem and vanishing theorem for multi-fans.

Theorem 4.3. *Let Δ be a non-singular complete multi-fan and $v \in H_2(BT)$ a generic vector. Assume that $c_1(\Delta)$ is divisible by an integer $N > 1$. Then the equivariant elliptic genus $\varphi_y^v(t)$ $((-y)^N = 1)$ of level N is rigid, i.e. $\varphi_y^v(t)$ is constant.*

Corollary 4.4. *Under the same situation as in Theorem 4.3 the equivariant elliptic genus $\varphi_y(t)$ $((-y)^N = 1)$ of level N constantly vanishes.*

Proof. We postpone the proof of Theorem 4.3. As to Corollary 4.4, we take a generic vector v such that $\zeta^{h(v)} \neq 1$, which is possible by Lemma 3.5. Since $\varphi_y^v(t)$ is constant by Theorem 4.3 for any v, $\varphi_y(t)$ is also constant, which we denote by φ_y. Since $\varphi_y = \zeta^{h(v)} \varphi_y$ by Lemma 4.2, φ_y must be equal to 0. □

The degree 0 term $\varphi_{y,0}(t)$ in the expansion in Lemma 4.1 reduces to the T_y-genus $T_y[\Delta]$ (cf. [HM]). We obtain

Corollary 4.5. *If $c_1(\Delta)$ is divisible by N, then the T_y-genus $T_y[\Delta]$ vanishes for $(-y)^N = 1$.*

For $N = 2$ and $y = 1$ the T_y-genus equals the signature $\mathrm{Sign}(\Delta)$. Hence

Corollary 4.6. *The signature* $\mathrm{Sign}(\Delta)$ *of a spin multi-fan vanishes.*

Remark. If M is a torus manifold, we define its elliptic genus to be that of its associated multi-fan $\Delta(M)$. If $c_1(M)$ is divisible by N, $c_1(\Delta(M))$ is also divisible by N as was remarked in Section 3, and hence its equivariant elliptic genus of level N vanishes for $(-y)^N = 1$. In case $N = 2$ we have the following conclusion. The equivariant Stiefel-Whitney class $w_2^T(M) \in H_T^2(M;\mathbb{Z}/2)$ is defined and descends to $c_1^T(\Delta(M))$ mod 2. If M is a spin torus manifold, then $w_2^T(M)$ lies in $H^2(BT;\mathbb{Z}/2)$. Therfore $c_1^T(\Delta(M))$ is divisible by 2. It follows from Corollary 4.6 that the signature $\mathrm{Sign}(M)$ of M vanishes. This can be also deduced from Corollary in 1.5 of [HS].

The rest of this section is devoted to the proof of Theorem 4.3. We assume throughout that $c_1(\Delta)$ is divisible by N. It suffices to show that $\varphi_y^v(t)$ has no poles since it is an elliptic function. It is clear that possible poles λ satisfy $\lambda^m q^{-s} = 1$ for some integers $m \geq 1$ and s. Hence it suffices to show that $\varphi_y^v(tq^{\frac{s}{m}})$ has no poles λ with $\lambda^m = 1$. Let

$$\Sigma^{(n)} = \sqcup_\nu X_\nu$$

be the decomposition into (v,m)-equivalence classes. Let K_ν denote the core of X_ν. We fix a class X_ν and write

$$\langle u_i^I, v\rangle = m_i = mh_i + r_i,\ \ 0 \leq r_i < m,$$

for $I \in X_\nu$. From the definition of core it follows that

$$r_i = 0 \ \Leftrightarrow\ i \notin K_\nu.$$

We decompose K_ν into a disjoint union $K_\nu = \sqcup_{r=1}^{m-1} K_\nu^r$ where

$$K_\nu^r = \{i \in K_\nu \mid r_i = r\}.$$

Note that $\sum_{i\in I} h_i = h(v,m,X_\nu) \mod N$. With these understandings a straightforworard calculation yields

Lemma 4.7.

$$\varphi_y^v(tq^{\frac{s}{m}}) = \sum_\nu \zeta^{sh(v,m,X_\nu)}\varphi_y^v(tq^{\frac{s}{m}})_\nu,$$

where

$$\begin{aligned}\varphi_y^v(tq^{\frac{s}{m}})_\nu = &\sum_{I\in X_\nu} \frac{w(I)}{\prod_{i\in I\backslash K_\nu}(1-t^{-\langle u_i^I,v\rangle})}\\ &\cdot \prod_{i\in I\backslash K_\nu}\left((1+yt^{-\langle u_i^I,v\rangle})\prod_{n=1}^\infty \frac{(1+yq^nt^{-\langle u_i^I,v\rangle})(1+y^{-1}q^nt^{\langle u_i^I,v\rangle})}{(1-q^nt^{-\langle u_i^I,v\rangle})(1-q^nt^{\langle u_i^I,v\rangle})}\right)\\ &\cdot \prod_{r=1}^{m-1}\prod_{i\in K_\nu^r}\left(\frac{1+yq^{-\frac{s}{m}r}t^{-\langle u_i^I,v\rangle}}{1-q^{-\frac{s}{m}r}t^{-\langle u_i^I,v\rangle}}\prod_{n=1}^\infty \frac{(1+yq^nq^{-\frac{s}{m}r}t^{-\langle u_i^I,v\rangle})(1+y^{-1}q^nq^{\frac{s}{m}r}t^{\langle u_i^I,v\rangle})}{(1-q^nq^{-\frac{s}{m}r}t^{-\langle u_i^I,v\rangle})(1-q^nq^{\frac{s}{m}r}t^{\langle u_i^I,v\rangle})}.\right)\end{aligned}$$

Lemma 4.8. *Fix ν. Let*

$$\varphi_y^v(tq^{\frac{s}{m}})_\nu = \sum_{n=0}^\infty \bar\varphi_n(t)q^{\frac{n}{m}}$$

be the expansion of $\varphi_y^v(tq^{\frac{s}{m}})_\nu$ into power series in $q^{\frac{1}{m}}$. Then each $\bar\varphi_n(t)$ is in $R(S^1)\otimes\mathbb{C}$.

Proof. The proof is similar to that of Lemma 4.1. $\bar{\varphi}_n(t)$ is a linear combination with complex coefficients of the expression of the form:

$$\sum_{I\in\Sigma_{K_\nu}^{(n-k_\nu)}} w(I)\frac{t_I^{\langle\alpha,v\rangle}}{\prod_{i\in I\setminus K_\nu}(1-t^{-\langle u_i^I,v\rangle})}$$

where k_ν is the dimension of $C(K_\nu)$, $\alpha\in\mathcal{O}$ is as in Section 2, and $t_I^{\langle\alpha,v\rangle}=\sum_{l\in\alpha}t^{\langle l,v\rangle}$. Thus $\bar{\varphi}_n(t)$ is in $R(S^1)\otimes\mathbb{C}$ by Corollary 2.4. $\square$

We shall show that each $\varphi_y^v(tq^{\frac{s}{m}})_\nu$ has no poles at $\lambda^m=1$. Let λ be a possible pole with $\lambda^m=1$. Let $\bar{\varphi}_{n,I}(t)$ be the contribution from I in $\bar{\varphi}_n(t)$. There is an open set U containing λ such that $\bar{\varphi}_{n,I}(t)$ is holomorphic in $U\setminus\{\lambda\}$ for each I. $\sum_{n=0}^\infty\bar{\varphi}_{n,I}(t)q^{\frac{n}{m}}$ converges uniformly in any compact set in $U-\{\lambda\}$. Note that $\sum_I\bar{\varphi}_{n,I}(t)=\bar{\varphi}_n(t)$ is holomorphic in U because it is a finite Laurent series by Lemma 4.8. We now quote a lemma from [H].

Lemma 4.9 (Hirzebruch). *Let $b_{n,j}$ be meromorphic functions on U with j running over some finite set J. Suppose that they satisfy the following properties:*

(1) *$b_{n,j}$ is holomorphic in $U\setminus\{\lambda\}$,*
(2) *$b_n=\sum_j b_{n,j}$ is holomorphic in U,*
(3) *$\sum_{n=0}^\infty b_{n,j}$ converges uniformly in any compact set in $U\setminus\{\lambda\}$.*

Then $\sum_{n=0}^\infty b_n$ converges uniformly in any compact set in U and is a holomophic extension of $\sum_{j\in J}\sum_{n=0}^\infty b_{n,j}\mid U\setminus\{\lambda\}$.

We apply this Lemma to $\bar{\varphi}_{n,I}(t)q^{\frac{n}{m}}$ and $\bar{\varphi}_n(t)q^{\frac{n}{m}}$. It follows that $\varphi_y^v(tq^{\frac{s}{m}})_\nu$ and hence $\varphi_y^v(tq^{\frac{s}{m}})$ has no pole at $t=\lambda$. Hence $\varphi_y^v(t)$ has no poles and it is a constant. This proves Theorem 4.3.

Remark. In case $s=0$, we can use Lemma 4.1 instead of Lemma 4.8,

5. Applications

Let Δ be a complete simplicial multi-fan of dimension n. If I is in $\Sigma^{(n)}$ and v is a generic vector, one can write v in the form $v=\sum_{i\in I}a_iv_i$ with non zero real numbers a_i. Fixing a generic vector v, we set

$$\mu(I)=\#\{i\in I\mid a_i>0,\}$$

and define

$$h_q(\Delta):=\sum_{I\in\Sigma^{(n)},\ \mu(I)=q}w(I),$$

for each integer q with $0\le q\le n$. We also define

$$e_k:=\sum_{K\in\Sigma^{(k)}}T_0[\Delta_K],$$

where Δ_K is the projected multi-fan associated with K. If the Todd genus $T_0[\Delta]$ equals 1 and $w(I)=1$ for all $I\in\Sigma^{(n)}$, then

$$h_q(\Delta)=\#\{I\in\Sigma^{(n)}\mid\mu(I)=q\}, \tag{5}$$

and

$$e_k = \#\Sigma^{(k)}. \tag{6}$$

Remark. Note that these conditions are always satisfied for complete simplicial ordinary fans. In particular, if M is a complete toric variety, the fan $\Delta(M)$ associated with M satisfies these conditions. It is known that $T_{-t}[\Delta(M)]$ is equal to the Poincaré polynomial $P(t)$ of M if M is a non-singular projective toric variety, see e.g. [F].

The following Lemma was proved in [HM].

Lemma 5.1.

$$\begin{aligned} T_y[\Delta] &= \sum_{q=0}^{n} h_q(\Delta)(-y)^q \\ &= \sum_{m=0}^{n} e_{n-m}(\Delta)(-1-y)^m. \end{aligned}$$

Moreover $h_0 = h_n$ and they are the Todd genus of the multi-fan Δ.

It was also proved there that the equivariant T_y-genus of a torus manifold M is always rigid, and hence reduces to the ordinary T_y-genus $T_y[M]$. The proof given there can also be applicable in the case of complete non-singular multi-fan.

Proposition 5.2. *Let Δ be a complete non-singular multi-fan with Todd genus $T_0[\Delta] \neq 0$. If $c_1(\Delta)$ is divisible by a positive integer N, then N is equal to or less than $n+1$. In the cases $N = n+1$ and $N = n$ the T_y- genus must be of the following forms*

$$T_y[\Delta] = T_0[\Delta] \sum_{q=0}^{n} (-y)^q \quad (N = n+1), \tag{7}$$

and

$$T_y[\Delta] = T_0[\Delta](1-y) \sum_{q=0}^{n-1} (-y)^q \quad (N = n). \tag{8}$$

Proof. Suppose that $c_1(\Delta)$ is divisible by N. Then, by Corollary 4.5 $T_y[\Delta]$ considered as a polynomial in $-y$ has roots at all N-th roots of unity other than 1. Hence it must be divisible by $\sum_{q=0}^{N-1}(-y)^q$. On the other hand it is a polynomial of degree n with constant term $T_0[\Delta]$ by Lemma 5.1 since $T_0[\Delta] \neq 0$. Therefore we must have $N + 1 \leq n$.

Suppose that $N = n+1$. Then the same reasoning as above proves (7). If $N = n$, then $T_y[\Delta]$ is divisible by $\sum_{q=0}^{n-1}(-y)^q$. Since the constant term and the coefficient (as a polynomial of $-y$) of the highest term do not vanish by assumption and Lemma 5.1, $T_y[\Delta]$ must be of the form (8). ☐

Lemma 5.3. *Let Δ be a complete non-singular multi-fan with $T_0[\Delta] = 1$ and $w(I) = 1$ for all $I \in \Sigma^{(n)}$. If $T_y[\Delta]$ is of the form (7), then*

$$\#\Sigma^{(1)} = n+1 \text{ and } \#\Sigma^{(n)} = n+1.$$

If $T_y[\Delta]$ is of the form (8), then

$$\#\Sigma^{(1)} = n+2 \text{ and } \#\Sigma^{(n)} = 2n.$$

Moreover, in case $n \geq 3$, $\#\Sigma^{(2)} = \frac{1}{2}n(n+3)$.

Proof. The equality (7) with $T_0[\Delta]=1$ means that $h_q=1$ for all q with $0\le q\le n$. This implies that $\#\Sigma^{(n)}=n+1$ by (5). Putting the above values of h_q in Lemma 5.1 and using (6) we see that $\#\Sigma^{(1)}=e_1=n+1$.

Similarly the equality (8) with $T_0[\Delta]$ implies that $h_q=1$ for $q=0,n$ and $h_q=2$ for $1\le q\le n-1$. This implies that $\#\Sigma^{(n)}=2n$ by (5), and yields, together with Lemma 5.1 and (6), the equalities $\#\Sigma^{(1)}=e_1=n+2$ and $\#\Sigma^{(2)}=e_2=\frac{1}{2}n(n+3)$ in case $n\ge 3$. □

Corollary 5.4. *Let M be a complete toric variety of dimension n. If $c_1(M)$ is divisible by $n+1$, then M is isomorphic to the projective space $\mathbb{P}^n$ as a toric variety.*

Proof. By Proposition 5.2 and Lemma 5.3 the fan associated with M has $n+1$ 1-dimensional cones and $n+1$ n-dimensional cones. Such a fan is unique (up to automorphisms of the lattice L) and coincides with the fan associated with $\mathbb{P}^n$. Since a toric variety is determined by its fan, M must be $\mathbb{P}^n$. □

In order to handle the case $N=n$ we investigate the fan associated with a projective space bundle over a projective space. Let ξ denote the hyperplane bundle (dual of the tautological line bundle) over $\mathbb{P}^r$. Let $1\le r<n$ and set $\eta=(\sum_{i=r+1}^{n}\xi^{k_i})\oplus 1$ where k_i are integers and 1 denotes the trivial line bundle. The associated projective space bundle of η will be denoted by M. It is a complex manifold. A point of M is expressed in homogeneous coordinate

$$[z_0,z_1,\dots,z_r,w_{r+1},\dots,w_n,w_{n+1}] \tag{9}$$

where $z_i,w_j\in\mathbb{C}$, $(z_0,z_1,\dots,z_r)\ne(0,0,\dots,0)$, $(w_{r+1},\dots,w_n,w_{n+1})\ne(0,\dots,0,0)$, and if, $\alpha\in\mathbb{C}^*$, then

$$[\alpha z_0,\alpha z_1,\dots,\alpha z_r,\alpha^{k_{r+1}}w_{r+1},\dots,\alpha^{k_n}w_n,w_{n+1}]$$

and

$$[z_0,z_1,\dots,z_r,\alpha w_{r+1},\dots,\alpha w_n,\alpha w_{n+1}]$$

are identified with (9).

Let $(n+1)$-dimensional torus $T^{n+1}=S^1\times\cdots\times S^1$ act on M by

$$\begin{aligned}(t_0,t_1,\dots,t_n)&[z_0,z_1,\dots,z_r,w_{r+1},\dots,w_n,w_{n+1}]\\ &=[t_0z_0,t_1z_1,\dots,t_rz_r,t_{r+1}w_{r+1},\dots,t_nw_n,w_{n+1}]\end{aligned}$$

The action is a holomorphic action. The subgroup $D'=\{(t,\dots,t,t^{k_{r+1}},\dots,t^{k_n})\mid t\in S^1\}$ of T^{n+1} acts trivially on M. Hence the quotient $T=T^{n+1}/D'$ acts on M. Put

$$M_i=\begin{cases}\{[z_0,z_1,\dots,z_r,w_{r+1},\dots,w_n,w_{n+1}]\mid z_i=0\} & \text{for } 0\le i\le r\\ \{[z_0,z_1,\dots,z_r,w_{r+1},\dots,w_n,w_{n+1}]\mid w_i=0\} & \text{for } r+1\le i\le n+1.\end{cases}$$

We also put

$$S_i=\{(1,\dots,1,t_i,1,\dots,1)\in T^{n+1}\}\ \text{ for }\ 0\le i\le n$$

and

$$S_{n+1}=\{(1,1,\dots,1,t_{r+1},\dots,t_n)\in T^{n+1}\mid t_{r+1}=\cdots=t_n\}.$$

We shall denote by the same letter the image of S_i in T. It is easy to see that S_i pointwise fixes M_i, and there are no other circle subgroups of T which have (complex) codimension 1 fixed point set components.

Let $\tilde{v}_i \in \mathrm{Hom}(S^1, T^{n+1})$ denote the inclusion homomorphism of S^1 into the i-th factor of T^{n+1} for $0 \le i \le n$. Put $v_i = \pi_*(\tilde{v}_i) \in \mathrm{Hom}(S^1, T)$ where π_* is the homomophism induced by the projection $\pi : T^{n+1} \to T$. ¿From the definition it follows that there is a relation

$$v_0 + v_1 + \cdots + v_r + k_{r+1}v_{r+1} + \cdots + k_n v_n = 0. \tag{10}$$

We also put

$$v_{n+1} = -(v_{r+1} + \cdots + v_n) \in \mathrm{Hom}(S^1, T). \tag{11}$$

Then S_i is the image of $v_i : S^1 \to T$. Moreover S^1 acts via v_i on each fiber of the normal bundle of M_i in M by standard complex multiplication. Thus $\{v_i \mid i = 0, \ldots, n, n+1\}$ coincides with the set of primitive edge vectors of 1-dimensional cones of the multi-fan $\Delta(M) = (\Sigma(M), C(M), w(M)^{\pm})$ associated with the torus manifold M, and we have $\Sigma(M)^{(1)} = \{0, 1, \ldots, n, n+1\}$, cf. [HM].

To determine the whole augmented simplicial set $\Sigma(M)$, we need to look at the fixed point set M^T. For $i \in \{0, 1, \ldots r\}$, put $I_i = \{0, 1, \ldots r\} \setminus \{i\}$, and for $j \in \{r+1, \ldots n+1\}$, put $J_j = \{r+1, \ldots n+1\} \setminus \{j\}$. It is not difficult to see that M^T consists of points

$$M_{I_i} \cap M_{J_j}, \quad i \in \{0, 1, \ldots r\}, \quad j \in \{r+1, \ldots n+1\},$$

where $M_{I_i} = \cap_{k \in I_i} M_k$ and $M_{J_j} = \cap_{l \in J_j} M_l$. This implies that

$$\Sigma(M)^{(n)} = \{I_i \cup J_j \mid i \in \{0, 1, \ldots r\},\ j \in \{r+1, \ldots n+1\}\}.$$

In particular

$$\#\Sigma(M)^{(n)} = (r+1)(n-r+1).$$

It follows that

$$\#\Sigma(M)^{(n)} \ge 2n \quad \text{and} \quad \#\Sigma(M)^{(n)} = 2n \quad \text{if and only if} \quad r = 1 \quad \text{or} \quad n-1.$$

Let τ be the tautological line bundle over the projective space bundle M. Its dual τ^* restricts to the hyperplane bundle on each fiber of $\pi : M \to \mathbb{P}^r$. Let $\omega \in H^2(M)$ be the first Chern class of τ^*. Then, by the Leray-Hirsch theorem, $H^*(M)$ is a free $H^*(\mathbb{P}^r)$-module over generators $1, \omega, \omega^2, \ldots, \omega^{n-r}$. In particular, $H^2(M)$ is a free module over ω, ω', where ω' is the image of the canonical generator of $H^2(\mathbb{P}^r)$ by π^*. We have

Lemma 5.5.

$$c_1(M) = (n-r+1)\omega + (\sum_{i=r+1}^{n} k_i + r + 1)\omega'.$$

Proof. The tautological line bundle τ is a subbundle of $\pi^*\eta$, and the tangent bundle along the fibers $T_f M$ of $\pi : M \to \mathbb{P}^r$ is isomorphic to $\mathrm{Hom}(\tau, \pi^*\eta/\tau) = \tau^* \otimes (\pi^*\eta/\tau)$. Hence

$$\begin{aligned} c_1(T_f M) &= (n-r)c_1(\tau^*) + c_1(\pi^*\eta/\tau) \\ &= (n-r)\omega + c_1(\pi^*\eta) - c_1(\tau) \\ &= (n-r)\omega + (\sum_i k_i)\omega' + \omega \\ &= (n-r+1)\omega + (\sum_i k_i)\omega'. \end{aligned}$$

Since the tangent bundle TM is isomorphic to $\pi^*T\mathbb{P}^r \oplus T_f M$, and $c_1(\pi^*T\mathbb{P}^r) = (r+1)\omega'$, we have

$$c_1(M) = (n-r+1)\omega + (\sum_i k_i + r + 1)\omega'.$$

□

As an immediate consequence of Lemma 5.5 we obtain

Corollary 5.6. *Let $M = \mathbb{P}(\eta)$ be as above. Then $c_1(M)$ is divisible by n if and only if $r = 1$ and $\sum_{i=r+1}^n k_i + 2$ is divisible by n.*

We now consider complete non-singular multi-fans having first Chern class divisible by n.

Lemma 5.7. *Let $\Delta = (\Sigma, C, w^\pm)$ be a complete non-singular multi-fan of dimension n such that*

$$T_0(\Delta) = 1, \ w(I) = 1 \ \ \text{for all} \ \ I \in \Sigma^{(n)},$$

$$\#\Sigma^{(1)} = n+2, \quad \#\Sigma^{(n)} = 2n, \quad \text{and} \quad \#\Sigma^{(2)} = \frac{1}{2}n(n+3) \quad \text{in case} \quad n \geq 3.$$

Then it is equivalent to the multi-fan of a $\mathbb{P}^{n-1}$ bundle over $\mathbb{P}^1$ or a $\mathbb{P}^1$-bundle over $\mathbb{P}^{n-1}$.

Proof. Let $\{v_i\}_{i=0}^{n+1}$ be the primitive edge vectors of the 1-dimensional cones. In view of (10) and (11) it suffices to show that, under a suitable numbering, they satisfy the relations

$$v_1 + v_2 + \cdots + v_n = 0, \quad v_0 + v_{n+1} + \sum_{i=2}^n k_i v_i = 0, \tag{12}$$

or

$$v_1 + v_2 + \cdots + v_n + k v_{n+1} = 0, \quad v_0 + v_{n+1} = 0. \tag{13}$$

We first deal with the case $n \geq 3$. From the completeness we see that each 1-dimensional cone is a face of at least n 2-dimensional cones, and it is a face of at most $n+1$ 2-dimensional cones because the number of 1-dimensional cones are $n+2$. Since the number of 2-dimensional cones is $\frac{1}{2}n(n+3)$, we conclude that there are two edge vectors, say v_0 and v_{n+1} such that v_0 (v_{n+1} respectively) spans 2-dimensional cones with each of remaining vectors $v_1, v_2, \ldots, v_n$, and each v_i, $1 \leq i \leq n$, spans 2-dimensional cones with v_j, $j \neq i$. Thus the projected multi-fan $\Delta_{\{0\}}$ has exactly n 1-dimensional cones. It is complete and non-singular as a projected multi-fan of a complete non-singular multi-fan Δ. It follows that $\Delta_{\{0\}}$ is equivalent to the fan of $\mathbb{P}^{n-1}$, and the projected edge vectors $\bar{v}_i$, $1 \leq i \leq n$, satisfy the relation

$$\bar{v}_1 + \bar{v}_2 + \cdots + \bar{v}_n = 0.$$

This implies the relation

$$v_1 + v_2 + \cdots + v_n = k v_0.$$

Similarly we have

$$v_1 + v_2 + \cdots + v_n = k' v_{n+1}.$$

If $k=0$ then $k'=0$ since $v_{n+1}\neq 0$, and $v_1, v_2, \dots, v_n$ lie on a hyperplane $v_1+v_2+\cdots+v_n=0$. Since the multi-fan Δ is complete and non-singular, the primitive vectors v_0 and v_{n+1} lie on the different sides of that hyperplane and must satisfy a relation as (12).

If $k\neq 0$ then $k'\neq 0$ and v_0 and v_{n+1} are linearly dependent primitive vectors. Therefore we must have $v_0+v_{n+1}=0$. Thus (13) holds. This proves Lemma 5.7 in the case $n\geq 3$.

The case $n=2$ is similar and easier. We see that there are four primitive edge vectors v_0, v_1, v_2, v_3 in 2-dimensional vector space $V=L\otimes\mathbb{R}$ such that

$$v_1+v_2=kv_0=k'v_3.$$

By the same reasoning as above we derive

$$v_1+v_2=0,\ v_0+v_3+kv_2=0 \quad\text{or}\quad v_1+v_2+kv_3=0,\ v_0+v_3=0.$$

$\square$

Corollary 5.8. *Let M be a complete non-singular toric variety of dimension n. If $c_1(M)$ is divisible by n, then M is isomorphic to an $(n-1)$-dimensional projective space bundle over $\mathbb{P}^1$ of the form described in Corollary 5.6 as a toric variety.*

Proof. By Proposition 5.2 and Lemma 5.3 the fan $\Delta(M)$ associated with M has $n+2$ 1-dimensional cones and $2n$ n-dimensional cones. Moreover the number of 2-dimensional cones is $\frac{1}{2}n(n+3)$ in case $n\geq 3$. By Lemma 5.7 $\Delta(M)$ is equivalent to that of a $\mathbb{P}^{n-1}$-bundle over $\mathbb{P}^1$ or $\mathbb{P}^1$- bundle over $\mathbb{P}^{n-1}$. Among such manifolds those with c_1 divisible by n are of the form given in Corollary 5.6. $\square$

6. Remarks on singular case

So far we defined equivariant elliptic genus for complete non-singular multi-fans, and proved its rigidity and vanishing. We can define equivariant elliptic genus for general complete simplicial multi-fan eqipped with a set of edge vectors for 1-dimensional cones. However it seems rigiditiy property can not be expected in general case. In the sequel we shall briefly discuss these phenomena.

Using the θ function in Section 4 we set

$$\phi_{\alpha,y}(\lambda)=\frac{\theta_{\alpha y}(\lambda)}{\theta_{-\alpha}(\lambda)}.$$

Similarly to (4) we have

$$\phi_{\alpha,y}(q\lambda)=-y^{-1}\phi_{\alpha,y}(\lambda)\quad\text{or}\quad\phi_{\alpha,y}(q^{-1}\lambda)=-y\phi_{\alpha,y}(\lambda). \tag{14}$$

Let $\Delta=(\Sigma, C, w^{\pm})$ be a complete simplicial multi-fan, and $\mathcal{V}=\{v_i\mid i\in\Sigma^{(1)}\}$ a set of prescribed edge vectors. We do not assume that v_i's are primitive. For $I\in\Sigma^{(n)}$ we define L_I to be the submodule of L generated by $\{v_i\mid i\in I\}$. Let L_I^* be the dual lattice of L_I. We identify L_I^* with the lattice in $V^*=L^*\otimes\mathbb{Q}$ given by

$$L^*=\{u\in V^*\mid\langle v,u\rangle\in\mathbb{Z},\ \text{ for any }\ v\in L_I\}.$$

For $g\in L/L_I$ and $u\in L_I^*$ we define

$$\chi_I(u,g)=e^{2\pi\sqrt{-1}\langle v,u\rangle},$$

where $v \in L$ is a representative of g. If one fixes u, $g \mapsto \chi_I(u,g)$ gives a character of the group L/L_I. We set

$$\varphi_y(t) = \sum_{I\in\Sigma^{(n)}} \frac{w(I)}{|L/L_I|} \sum_{g\in L/L_I} \prod_{i\in I} \phi_{\chi_I(u_i^I,g)^{-1},y}(t^{-u_i^I}), \tag{15}$$

and call it the equivariant elliptic genus of the pair $(\Delta, \mathcal{V})$ of multi-fan and prescribed edge vectors. Let $v \in H_2(BT)$ be a generic vector such that $\langle u_i^I, v\rangle \in \mathbb{Z}$ for all $i \in I$ and $I \in \Sigma^{(n)}$. We set

$$\varphi_y^v(t) = \sum_{I\in\Sigma^{(n)}} \frac{w(I)}{|L/L_I|} \sum_{g\in L/L_I} \prod_{i\in I} \phi_{\chi_I(u_i^I,g)^{-1},y}(t^{-\langle u_i^I,v\rangle}).$$

If $(-y)^N = 1,\ -y \neq 1$, then $\varphi_y^v(t)$ is an elliptic function on $\mathbb{C}^*/q^N$, because $\phi_{\alpha,y}(t)$ is such a function by (14). In this case $\varphi_y^v(t)$ is called elliptic genus of level N.

The folloing proposition is a generalization of Lemma 4.1.

Proposition 6.1. *Let $\varphi_y(t) = \sum_{n=0}^\infty \varphi_{y,n}(t)q^n$ be the expansion into power series, then $\varphi_{y,n}(t)$ belongs to $R(T) \otimes \mathbb{C}$.*

The proof of Proposition 6.1 goes in a way similar to that of Lemma 4.1. We define the equivariant cohomology with real coefficients $H_T^*(\Delta;\mathbb{R})$ of a complete simplicial multi-fan Δ as the face ring of the simplicial complex Σ with real coefficients as in Section 2. We regard $H^2(BT;\mathbb{R})$ as a subspace of $H_T^2(\Delta,\mathbb{R})$ by the formula (1). Note that this definition depends not only on Δ but on $\mathcal{V}$. We also define the restriction homomorphism $i_I^* : H_T^*(\Delta;\mathbb{R}) \to H^*(BT;\mathbb{R})$ for each $I \in \Sigma^{(n)}$ by (2). Instead of Lemma 2.1 we use

Lemma 6.2. *For any $x = \sum_{i\in\Sigma^{(1)}} c_i x_i \in H_T^2(\Delta;\mathbb{R}),\ c_i \in \mathbb{Z}$, the element*

$$\sum_{I\in\Sigma^{(n)}} \frac{w(I)}{|L/L_I|} \sum_{g\in L/L_I} \frac{\chi_I(i_I^*(x),g)t^{i_I^*(x)}}{\prod_{i\in I}(1-\chi_I(u_i^I,g)^{-1}t^{-u_i^I})}$$

in a fractional ring containing $R(T)$ actually belongs to $R(T)$.

This was proved in Lemma 7.4 of [HM] with a further assumption that $i_I^*(x) \in H^2(BT;\mathbb{Z})$. The general case can be proved in a similar way. The formula was also given in Corollary 12.10 of [HM] when Δ is the multi-fan associated with a torus orbifold.

Let $\mathcal{O}$ be as in Section 2. Let α be in $\mathcal{O}$. If I is in $\Sigma^{(n)}$, then α is an orbit of the action of permutation group of I on linear forms over u_i^I. We set

$$\chi_I(\alpha,g)t_I^\alpha = \sum_{l\in\alpha} \chi_I(l,g)t^l.$$

We can deduce the following corollary from Lemma 6.2 just like we deduced Lemma 2.5 from Lemma 2.1.

Corollary 6.3. *For any $\alpha \in \mathcal{O}$ the expression*

$$\sum_{I\in\Sigma^{(n)}} \frac{w(I)}{|L/L_I|} \sum_{g\in N/N_I} \frac{\chi_I(\alpha,g)t_I^\alpha}{\prod_{i\in I}(1-\chi_I(u_i^I,g)^{-1}t^{-u_i^I})} \tag{16}$$

belongs to $R(T)$.

We are now ready to prove Proposition 6.1. We see that $\varphi_{y,n}(t)$ is a linear combination with complex coefficients of the expression of the form (16). Hence it belongs to $R(T)\otimes\mathbb{C}$ by Corollary 6.3. This proves Proposition 6.1.

It would seem reasonable to define $H_T^2(\Delta;\mathbb{Z})$ to be the submodule of $H_T^2(\Delta;\mathbb{Q})$ generated by $\mathcal{V}=\{v_i \mid i\in\Sigma^{(1)}\}$. Then $H^2(BT;\mathbb{Z})$ is contained in $H_T^2(\Delta;\mathbb{Z})$. It would be possible to define divisibility by N of the first Chern class as in Section 3. However Theorem 4.3 and Corollary 4.4 do not hold in this general setting. In fact Lemma 3.1 has no meaning as it is, because u_i^I's do not lie in $H^2(BT;\mathbb{Z})$ in general. Also the proof of Lemma 3.2 breaks down in general setting, and one can not define (v,m)-equivalence relation. However in the following favorable case we get rigidity and vanishing results.

We assume that the submodules $L_I\subset L=H_2(BT;\mathbb{Z})$ are independent of $I\in\Sigma^{(n)}$. We write $\tilde{L}=L_I$. It coincides with a submodule of L generated by $\mathcal{V}$. The dual lattice $\tilde{L}^*$ is contained in $H_T^2(\Delta;\mathbb{Z})$ as is seen from (1). The identification of L with $H_2(BT;\mathbb{Z})$ can be explained as follows. We interpret T as V/L where $V=L\otimes\mathbb{R}$. Then

$$L=\pi_1(T)=\pi_2(BT)=H_2(BT:\mathbb{Z}).$$

We put $\tilde{V}=\tilde{L}\otimes\mathbb{R}$ and $\tilde{T}=\tilde{V}/\tilde{L}$. This determines the identificaction $\tilde{L}=H_2(B\tilde{T};\mathbb{Z})$ and $\tilde{L}^*=H^2(B\tilde{T};\mathbb{Z})$. Let $p:\tilde{V}\to V$ and $p:\tilde{T}\to T$ denote the map induced by the inclusion $\tilde{L}\to L$. Then the kernel of $p:\tilde{T}\to T$ is identified with $H=L/\tilde{L}$. To distinguish we denote by $\tilde{v}_i$ the vector v_i when we consider it as lying in $\tilde{V}$ so that we have $p(\tilde{v}_i)=v_i$. We define a new multi-fan $\tilde{\Delta}=(\tilde{\Sigma},\tilde{C},\tilde{w}^{\pm})$ by setting $\tilde{\Sigma}=\Sigma$, $\tilde{w}^{\pm}=w^{\pm}$ and letting $\tilde{C}(i)$ to be the cone generated by $\tilde{v}_i$. Since $\{\tilde{v}_i \mid i\in I\}$ is a basis of $\tilde{L}$ for any $I\in\Sigma^{(n)}$, $\tilde{\Delta}$ is non-singular.

Let $\tilde{\varphi}_y(t)=\sum_{n=0}^{\infty}\tilde{\varphi}_{y,n}(t)q^n$ be the equivariant elliptic genus of $\tilde{\Delta}$. We have

$$\tilde{\varphi}_y(t)=\sum_{I\in\Sigma^{(n)}}w(I)\prod_{i\in I}\phi_y(t^{-u_i^I}) \tag{17}$$

By Lemma 4.1 $\tilde{\varphi}_{y,n}(t)$ is of the form

$$\tilde{\varphi}_{y,n}(t)=\sum_{u\in\tilde{L}^*}a_ut^u$$

where $a_u\in\mathbb{Z}[y,y^{-1}]$. Compairing (17) with (15) we obtain

$$\varphi_{y,n}(t)=\frac{1}{|L/\tilde{L}|}\sum_{u\in\tilde{L}^*}a_u\sum_{g\in L/\tilde{L}}\chi(u,g)t^u.$$

Since $\chi(u,\ )$ is a character of $L/\tilde{L}$,

$$\sum_{g\in L/\tilde{L}}\chi(u,g)t^u=\begin{cases}t^u & \text{if } \ u\in L^*\\ 0 & \text{if } \ u\notin L^*,\end{cases}$$

we see that

$$\varphi_{y,n}(t)=\sum_{u\in L^*}a_ut^u. \tag{18}$$

As a byproduct we obtain

Proposition 6.4. *Let Δ be a complete simplicial multi-fan and $\mathcal{V}$ a set of prescribed edge vectors such that L_I is independent of $I \in \Sigma^{(1)}$. Assume that $c_1(\Delta)$ is divisible by N in the sense that $c_1^T(\Delta) = \sum_{i\in\Sigma^{(1)}} x_i \in H_T^2(\Delta;\mathbb{Z})$ is written in the form*

$$c_1^T(\Delta) = Nx + u,\ x \in H_T^2(\Delta),\ u \in \tilde{L}^*,$$

then the equivariant elliptic genus φ_y of the pair $(\Delta,\mathcal{V})$ vanishes for $(-y)^N = 1,\ -y \neq 1$. In particular the T_y-genus $T_y(\Delta)$ vanishes for $(-y)^N = 1,\ \neq 1$.

Proof. The divisibility of $c_1(\Delta)$ by N eventually means that $c_1(\tilde{\Delta})$ is divisible by N. By Corollary 4.4 all the a_u vanish for $(-y)^N = 1, -y \neq 1$. Hence, by (18), φ_y of the pair $(\Delta,\mathcal{V})$ vanishes for $(-y)^N = 1,\ -y \neq 1$. □

Remark. In the situation of Proposition 6.4 we constructed a new non-singular multi-fan from the pair $(\Delta,\mathcal{V})$. We may call it the (ramified) covering of Δ with respect to $\mathcal{V}$. Conversely given a complete non-singular multi-fan $\tilde{\Delta} = (\tilde{\Sigma}, \tilde{C}, \tilde{w}^{\pm})$ with $\tilde{C} : \tilde{\Sigma}^{(1)} \to H_2(B\tilde{T};\mathbb{R})$ and a sublattice L of $H_2(B\tilde{T};\mathbb{R})$ such that $H_2(B\tilde{T};\mathbb{Z}) \subset L$ we can construct in an obvious way a new complete simplicial multi-fan Δ and a set of edge vectors $\mathcal{V}$ such that the covering of Δ with respect to $\mathcal{V}$ coincides with Δ. Geometric picture of this construction is making quotient torus orbifold M/H of a torus manifold M by a subgroup H of the torus T acting on M.

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