THE CLASSIFICATION PROBLEM OF CLOSED ORIENTABLE 3-MANIFOLDS

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Abstract

We show that every well-order in the set of integral vectors induces an embedding from the set of closed connected orientable 3-manifolds into the set of links which is a right inverse of the 0-surgery map and induces further two embeddings from the set of closed connected orientable 3-manifolds into the well-ordered set of integral vectors and into the set of link groups. In particular, the set of closed connected orientable 3-manifolds is a well-ordered set by a well-order inherited from the well-ordered set of integral vectors. To determine the embedding images of every 3-manifold, we take the canonical well-order in the set of integral vectors and propose a classification program on the well-ordered set of 3-manifolds which can be carried out inductively. As an exercise, we do the classification for the 3-manifolds with lengths up to 7. From this classification program, we find an answer to the classification problem on the 3-manifolds assuming inductive partial solutions of the homeomorphism problem on the 3-manifolds, the isomorphism problem on link groups and the decision problem on primeness of links.

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1. Introduction

The homeomorphism problem, namely the problem giving an effective procedure for determining whether two given 3-manifolds are homeomorphic and the classification problem, namely the problem generating effectively a list containing exactly one 3-manifold from every (unoriented) type of 3-manifolds are fundamental problems in the theory of 3-manifolds (see J. Hempel [6, p.169]). In this paper, we consider the classification problem on closed connected orientable 3-manifolds by establishing an embedding from the set of closed connected orientable 3-manifolds into the set of links in the 3-sphere S^3 which is a right inverse of the 0-surgery map. For this purpose, let $\mathbb{Z}$ be the set of integers, and $\mathbb{Z}^n$ the product of n copies of $\mathbb{Z}$ whose element $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{Z}^n$ is called an *integral vector* of *length* $\ell(\mathbf{x}) = n$. The *set* $\mathbb{X}$ *of integral vectors* is the disjoint union of $\mathbb{Z}^n$ for all $n = 1, 2, 3, \ldots$. Let Ω be a well-order in $\mathbb{X}$, although we define in §2 the *canonical order* Ω_c, a particular well-order with a property that every initial segment is finite. The class of oriented links L' in S^3 such that there is a homeomorphism $h : S^3 \to S^3$ sending L to L' is called the *unoriented link type* $[L]$ of an oriented link L in S^3, and the *oriented link type* $\langle L \rangle$ of L moreover if h is orientation-preserving both on the orientation of S^3 and on the orientations of L and L'. Let $\mathbb{L}$ and $\overrightarrow{\mathbb{L}}$ be the sets of unoriented link types and oriented link types in S^3, respectively. *A link type will be identified with a link belonging to the link type unless confusion might occur.* Thus, $\mathbb{L}$ and $\overrightarrow{\mathbb{L}}$ are understood as the sets of unoriented links and oriented links in S^3, respectively. We have a canonical map

$$\mathrm{cl}\beta : \mathbb{X} \longrightarrow \overrightarrow{\mathbb{L}} \xrightarrow{\iota} \mathbb{L}$$

sending an integral vector to the closured link of the associated braid (see §1 for the detail) which is independent of the well-order Ω, where $\iota : \overrightarrow{\mathbb{L}} \to \mathbb{L}$ denotes the forgetful surjection. On the other hand, the well-order Ω induces a map

$$\sigma : \mathbb{L} \longrightarrow \mathbb{X}$$

which is injective modulo split additions of trivial links, so that Ω defines a well-order in $\mathbb{L}$, also denoted by Ω. This construction of σ will be also done in §1. In §3, we define the concept of a minimal link by using this well-order Ω of $\mathbb{L}$. Let $\mathbb{L}^{\mathrm{m}}$ be the subset of $\mathbb{L}$ consisting of minimal links. Then we see that the restriction

$$\sigma|_{\mathbb{L}^{\mathrm{m}}} : \mathbb{L}^{\mathrm{m}} \longrightarrow \mathbb{X}$$

is an embedding (see Lemma 3.4), so that in the case of the canonical order $\Omega = \Omega_c$ every initial segment of $\mathbb{L}^{\mathrm{m}}$ as well as $\mathbb{X}$ is a finite set. The *link group* of a link L in S^3 is the fundamental group $\pi_1 E(L)$ of the exterior $E(L) = \mathrm{cl}(S^3 - N(L))$ of L with $N(L)$ a tubular neighborhood of L in S^3. Let $\mathbb{G}$ be the set of the isomorphism types of the link groups for the links in $\mathbb{L}$. *The isomorphism type of a group will be identified with a group belonging to the isomorphism type unless confusion might occur.* An *Artin presentation* is a finite group presentation

$$(x_1, x_2, \ldots, x_n \,|\, x_i = w_i x_{p(i)} w_i^{-1}, i = 1, 2, \ldots, n)$$

where $p(1), p(2), \ldots, p(n)$ are a permutation of $1, 2, \ldots, n$ and w_i $(i = 1, 2, \ldots, n)$ are words in $x_1, x_2, \ldots, x_n$ which satisfy the identity

$$\prod_{i=1}^{n} x_i = \prod_{i=1}^{n} w_i x_{p(i)} w_i^{-1}$$

in the free group F on the letters $x_1, x_2, \ldots, x_n$. Then we have a braid $b \in B_n$ corresponding to the automorphism φ of F defined by

$$\varphi(x_i) = w_i x_{p(i)} w_i^{-1} \quad (i = 1, 2, \ldots, n),$$

from which we see that the set $\mathbb{G}$ is characterized as the set of groups with Artin presentation (see for example [9; p.83] as well as J. S. Birman [2;p.46]). If the closured link $\mathrm{cl}(b)$ is prime or minimal, then we say that the Artin presentation is *prime* or *minimal*, respectively. For the map

$$\pi : \mathbb{L} \longrightarrow \mathbb{G}$$

sending every link L to the link group $\pi_1 E(L)$, we also see that the restriction

$$\pi|_{\mathbb{L}^{\mathrm{m}}} : \mathbb{L}^{\mathrm{m}} \longrightarrow \mathbb{G}$$

is an embedding (see Lemma 3.5). Let $\mathbb{M}$ and $\overrightarrow{\mathbb{M}}$ be the sets of unoriented types and oriented types of closed connected oriented 3-manifolds, respectively. *The type of a closed connected oriented 3-manifold will be identified with a 3-manifold belonging to the type unless confusion might occur.* Let

$$\chi_0 : \mathbb{L} \longrightarrow \mathbb{M}$$

be the map sending a link $L \in \mathbb{L}$ to the 0-surgery manifold $\chi(L, 0) \in \mathbb{M}$. The following theorem is our main theorem which is proved in §4:

Theorem 1.1. Any well-order Ω of $\mathbb{X}$ induces an embedding

$$\alpha : \mathbb{M} \longrightarrow \mathbb{L}^{\mathrm{m}} \subset \mathbb{L}$$

and hence two embeddings

$$\begin{aligned} \sigma_\alpha = \sigma\alpha : &\mathbb{M} \longrightarrow \mathbb{X}, \\ \pi_\alpha = \pi\alpha : &\mathbb{M} \longrightarrow \mathbb{G} \end{aligned}$$

which satisfy the following properties (1) and (2):

(1) $\chi_0 \alpha = 1$.
(2) If an integral vector $\sigma_\alpha(M) \in \mathbb{X}$ is given, then the minimal link $\alpha(M) \in \mathbb{L}$ with a braid presentation, the 3-manifold $M \in \mathbb{M}$ with a 0-surgery description along a minimal link and the link group $\pi_\alpha(M) \in \mathbb{G}$ with a minimal Artin presentation are determined.

Further taking a well-order Ω in $\mathbb{X}$ such that every initial segment is finite, we have the following properties (3) and (4):

(3) If a group $\pi_\alpha(M)$ with a prime Artin presentation is given, then the integral vector $\sigma_\alpha(M)$ is determined assuming a solution of the following problem:

Problem. Let $\mathbf{x}_1 < \mathbf{x}_2 < \cdots < \mathbf{x}_n = \mathbf{x}$ be the initial segment of the integral vector $\mathbf{x}$ induced from the prime Artin presentation of $\pi_\alpha(M)$. Find the smallest index i such that the link $\mathrm{cl}\beta(\mathbf{x}_i)$ is prime and there is an isomorphism $\pi_1 E(\mathrm{cl}\beta(\mathbf{x}_i)) \to \pi_\alpha(M)$.

(4) If a 3-manifold M with the 0-surgery description along a minimal link is given, then the integral vector $\sigma_\alpha(M)$ is determined assuming a solution of the following problem:

Problem. Let $\mathbf{x}$ be an integral vector induced from a minimal link L, and $\mathbf{x}_1 < \mathbf{x}_2 < \cdots < \mathbf{x}_n = \mathbf{x}$ be the initial segment of $\mathbf{x}$. Find the smallest index i such that the link $\mathrm{cl}\beta(\mathbf{x}_i)$ is minimal and the 0-surgery manifold $\chi(\mathrm{cl}\beta(\mathbf{x}_i), 0)$ is $\chi(L, 0)$.

The embedding σ_α makes the set $\mathbb{M}$ a well-ordered set by a well-order inherited from the well-order Ω of $\mathbb{L}$ and denoted also by Ω. The *length* of a 3-manifold $M \in \mathbb{M}$ is the length of the integral vector $\sigma_\alpha(M) \in \mathbb{X}$. In §5, to determine the images $\alpha(M)$, $\sigma_\alpha(M)$ and $\pi_\alpha(M)$ of every $M \in \mathbb{M}$, we take $\Omega = \Omega_c$ and propose a classification program on $\mathbb{M}$ based on Theorem 1.1 which we can carry out inductively. As an exercise, we carry out this classification for the 3-manifolds with lengths up to 7. *From this classification program, we find an answer to the classification problem on* $\mathbb{M}$ *assuming inductive partial solutions of the homeomorphism problem on* $\mathbb{M}$, *the decision problem on primeness of links and the isomorphism problem on* $\mathbb{G}$. A lifting of the embedding α to the oriented version is discussed in §6 together with an observation on a relationship between oriented 3-manifold invariants and oriented link invariants.

This paper is a growing up version of a part of the research announcement "*Link corresponding to closed 3-manifold*" a revised version of whose remaining part will appear in [10] (see http://www.sci.osaka-cu.ac.jp/˜kawauchi/index.htm).

2. Representing links in the set of integral vectors

For an integral vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ of length n, we denote the integral vector $(x_n, \ldots, x_2, x_1)$ of length n, the integers $\min_{1 \leqq i \leqq n} |x_i|$ and $\max_{1 \leqq i \leqq n} |x_i|$ by $\mathbf{x}^T$, $\min|\mathbf{x}|$ and $\max|\mathbf{x}|$, respectively. Further, for an integral vector $\mathbf{y} = (y_1, y_2, \ldots, y_m)$ of length m, we denote by $(\mathbf{x}, \mathbf{y})$ the integral vector

$$(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m)$$

of length $n+m$. Let $\overrightarrow{\mathbb{L}}$ be the set of oriented links. By the Alexander theorem (see J. S. Birman [2]), every oriented link L is represented by the closured link $\mathrm{cl}(b)$ of an s-string braid $b \in B_s$ for some $s \geqq 1$. Let σ_i $(i = 1, 2, \ldots, s-1)$ be the standard generators of the s-string braid group B_s. By convention, we regard the sign of the crossing point of the diagram σ_i as $+1$. When $b \neq 1$ in B_s, we write

$$b = \sigma_{i_1}^{\epsilon_1} \sigma_{i_2}^{\epsilon_2} \ldots \sigma_{i_r}^{\epsilon_r}, \quad \epsilon_i = \pm 1 \quad (i = 1, 2, \ldots, r).$$

Then we define the *integral vector* $\mathbf{x}(b)$ *of the braid* b by the identity

$$\mathbf{x}(b) = (\epsilon_1 i_1, \epsilon_2 i_2, \ldots, \epsilon_r i_r) \in \mathbb{Z}^r \subset \mathbb{X}.$$

When $b = 1$, we understand that $\mathbf{x}(b) = 0 \in \mathbb{Z} \subset \mathbb{X}$. For a non-zero integral vector $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{X}$, let x_{i_j} $(j = 1, 2, \ldots, m; i_1 < i_2 < \cdots < i_m)$ be the set of the non-zero integers in the corodinates x_i $(i = 1, 2, \ldots, n)$ of $\mathbf{x}$. Then the integral vector

$$\tilde{\mathbf{x}} = (x_{i_1}, x_{i_2}, \ldots, x_{i_m})$$

is called the *core* of $\mathbf{x}$. When $\mathbf{x}$ is a zero vector, we understand the core $\tilde{\mathbf{x}} = 0$. We note that for every non-zero integral vector $\mathbf{x}$, there is a unique braid $b \in B_s$ for every $s \geqq \max|\mathbf{x}| + 1$ such that $\mathbf{x}(b) = \tilde{\mathbf{x}}$. The braid b is called *the associated braid with index* s of $\mathbf{x}$ and denoted by $\beta^{(s)}(\mathbf{x})$, and in particular for $s = \max|\mathbf{x}| + 1$, called *the associated braid* of $\mathbf{x}$ and denoted by $\beta(\mathbf{x})$. The associated braid with index s of any zero vector of $\mathbb{X}$ is understood as $1 \in B_s$, and in particular the associated braid as $1 \in B_1$. Taking the closured link $\mathrm{cl}\beta(\mathbf{x})$ of the braid $\beta(\mathbf{x})$, we obtain a map

$$\mathrm{cl}\beta : \mathbf{X} \longrightarrow \vec{\mathbb{L}}.$$

By definition, the closured link $\mathrm{cl}\beta^{(s)}(\mathbf{x})$ with $s > \max|\mathbf{x}| + 1$ is obtained from the closured link $\mathrm{cl}\beta(\mathbf{x})$ by adding a trivial link of $(s - \max|\mathbf{x}| - 1)$ components. An equivalence relation $\sim$ in $\mathbb{X}$ is introduced as follows:

Definition 2.1. Two integral vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{X}$ are related as $\mathbf{x} \sim \mathbf{y}$ if we have $\mathrm{cl}\beta(\mathbf{x}) = \mathrm{cl}\beta(\mathbf{y})$ in $\vec{\mathbb{L}}$ modulo split additions of trivial links.

It is direct to see that the relation $\sim$ is an equivalence relation in $\mathbb{X}$. Let $\mathbb{X}/\sim$ be the quotient set of $\mathbb{X}$ by $\sim$, and $\langle\mathbf{x}\rangle$ the equivalence class of an integral vector $\mathbf{x} \in \mathbb{X}$ by $\sim$. We define a map

$$\tilde{\sigma} : \vec{\mathbb{L}} \longrightarrow \mathbb{X}/\sim$$

by

$$\tilde{\sigma}(\mathrm{cl}(b)) = \langle\mathbf{x}(b)\rangle,$$

which is a well-defined surjection and injective modulo split additions of trivial links.

We can describe the equivalence relation $\sim$ only in terms of $\mathbb{X}$ by using the braid group presentation and the Markov theorem (see J. S. Birman [2]), as stated in the following lemma:

Lemma 2.2. The relations (1)-(6) below are in the equivalence relation $\sim$ in $\mathbb{X}$. If $\mathbf{x} \sim \mathbf{y}$ in $\mathbb{X}$, then $\mathbf{y}$ is obtained from $\mathbf{x}$ by applying the relations (1)-(6) in a finite time.

(1) $(\mathbf{x}, 0) \sim_e \mathbf{x}$, $\mathbf{x} \sim_e (\mathbf{x}, 0)$ for all $\mathbf{x} \in \mathbb{X}$,

(2) $(\mathbf{x}, \mathbf{y}, -\mathbf{y}^T) \sim_e \mathbf{x},\ \mathbf{x} \sim_e (\mathbf{x}, \mathbf{y}, -\mathbf{y}^T)$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{X}$,
(3) $(\mathbf{x}, y) \sim_e \mathbf{x},\ \mathbf{x} \sim_e (\mathbf{x}, y)$ for all $\mathbf{x} \in \mathbb{X}$ and $y \in \mathbb{Z}$ such that $|y| > \max|\mathbf{x}|$,
(4) $(\mathbf{x}, \mathbf{y}, \mathbf{z}) \sim_e (\mathbf{x}, \mathbf{z}, \mathbf{y})$ for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{X}$ such that $\min|\mathbf{y}| > \max|\mathbf{z}| + 1$ or $\min|\mathbf{z}| > \max|\mathbf{y}| + 1$,
(5) $(\mathbf{x}, \varepsilon y, y+1, y) \sim_e (\mathbf{x}, y+1, y, \varepsilon(y+1))$ for all $\mathbf{x} \in \mathbb{X}$ and $y \in \mathbb{Z}$ such that $y(y+1) \neq 0$ and $\varepsilon = \pm 1$,
(6) $(\mathbf{x}, \mathbf{y}) \sim_e (\mathbf{y}, \mathbf{x})$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{X}$.

Proof. The relation (1) is in $\sim$ since $\beta(\mathbf{x}, 0) = \beta(\mathbf{x})$. For (2), we take $\beta^{(s)}(\mathbf{x})$ and $\beta^{(s)}(\mathbf{y})$ in B_s for some s. Then we have

$$\beta^{(s)}(\mathbf{x}, \mathbf{y}, -\mathbf{y}^T) = \beta^{(s)}(\mathbf{x})\beta^{(s)}(\mathbf{y})\beta^{(s)}(\mathbf{y})^{-1} = \beta^{(s)}(\mathbf{x})$$

in B_s and hence

$$\mathrm{cl}\beta(\mathbf{x}, \mathbf{y}, -\mathbf{y}^T) = \mathrm{cl}\beta(\mathbf{x})$$

in $\overrightarrow{\mathbb{L}}$ modulo split additions of trivial links, showing that (2) is in $\sim$. For (3), let $s = |y| + 1$. Then by the Markov theorem,

$$\mathrm{cl}\beta(\mathbf{x}, y) = \mathrm{cl}\beta^{(s)}(\mathbf{x})$$

in $\overrightarrow{\mathbb{L}}$ and the last link is equal to $\mathrm{cl}\beta(\mathbf{x})$ modulo split additions of trivial links, showing that (3) is in $\sim$. For (4), we take $\beta^{(s)}(\mathbf{x})$, $\beta^{(s)}(\mathbf{y})$ and $\beta^{(s)}(\mathbf{z})$ in B_s for some s. By the assumption on $\mathbf{y}$ and $\mathbf{z}$, we have

$$\beta^{(s)}(\mathbf{x}, \mathbf{z}, \mathbf{y}) = \beta^{(s)}(\mathbf{x})\beta^{(s)}(\mathbf{y})\beta^{(s)}(\mathbf{z}) = \beta^{(s)}(\mathbf{x})\beta^{(s)}(\mathbf{z})\beta^{(s)}(\mathbf{y}) = \beta^{(s)}(\mathbf{x}, \mathbf{z}, \mathbf{y})$$

in B_s which shows that

$$\mathrm{cl}\beta(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \mathrm{cl}\beta(\mathbf{x}, \mathbf{z}, \mathbf{y})$$

in $\overrightarrow{\mathbb{L}}$ modulo split additions of trivial links. Thus, (4) is in $\sim$. For (5), consider $\beta^{(s)}(\mathbf{x})$ and σ_j^{ε} ($j = |y|$, $\varepsilon' = \mathrm{sign}(y)$) in B_s for some s. Let $\varepsilon' = +1$. Then

$$\beta^{(s)}(\mathbf{x}, \varepsilon y, y+1, y) = \beta^{(s)}(\mathbf{x})\sigma_j^{\varepsilon}\sigma_{j+1}\sigma_j$$

and the last braid is equal to

$$\beta^{(s)}(\mathbf{x})\sigma_{j+1}\sigma_j\sigma_{j+1}^{\varepsilon} = \beta^{(s)}(\mathbf{x}, y+1, y, \varepsilon(y+1))$$

in B_s by a well-known braid relation. Hence we have

$$\mathrm{cl}\beta(\mathbf{x}, \varepsilon y, y+1, y) = \mathrm{cl}\beta(\mathbf{x}, y+1, y, \varepsilon(y+1))$$

in $\overrightarrow{\mathbb{L}}$ modulo split additions of trivial links, showing that (5) is in $\sim$. For $\varepsilon' = -1$, a similar argument gives the desired result since $\mathrm{sign}(y+1) = -1$ by assumption. For (6), let $\beta^{(s)}(\mathbf{x})$ and $\beta^{(s)}(\mathbf{y})$ in B_s for some s. Then we have

$$\mathrm{cl}\beta^{(s)}(\mathbf{x})\beta^{(s)}(\mathbf{y}) = \mathrm{cl}\beta^{(s)}(\mathbf{y})\beta^{(s)}(\mathbf{x})$$

by the Markov theorem and hence

$$\mathrm{cl}\beta(\mathbf{x},\mathbf{y}) = \mathrm{cl}\beta(\mathbf{y},\mathbf{x})$$

in $\overrightarrow{\mathbb{L}}$ modulo split additions of trivial links, showing that (6) is in $\sim$.

Next, we assume $\mathbf{x} \sim \mathbf{y}$. Let s and s' be the indices of the associated braids $b = \beta(\mathbf{x})$ and $b' = \beta(\mathbf{y})$, respectively. Applying the relation (3) to $\mathbf{x}$ or $\mathbf{y}$, we can assume that $s = s'$ and $\mathrm{cl}(b) = \mathrm{cl}(b')$ in $\overrightarrow{\mathbb{L}}$. Then the Markov theorem says that we have $b = b'$ in B_s with a suitable s after finitely many applications of the Markov equivalences:

$$b_1b_2 \leftrightarrow b_2b_1 \quad (b_1, b_2 \in B_m) \quad \text{and} \quad b\sigma_m^{\pm 1} \leftrightarrow b \quad (b \in B_m \subset B_{m+1})$$

for any m. This is equivalent to saying that $b = b' \in B_s$ after finitely many applications of the relations (3) and (6) to $\mathbf{x}$ and $\mathbf{y}$. Using a well-known group presentation of B_s with generators σ_i $(j = 1, 2, \ldots, s-1)$ and relators

$$\text{(i) } (b_ib_j)(b_jb_i)^{-1} \ (|i-j| \geqq 2) \quad \text{and} \quad \text{(ii) } (b_ib_{i+1}b_i)(b_{i+1}b_ib_{i+1})^{-1} \ (1 \leqq i \leqq s-1)$$

(see [2]), we can write $b(b')^{-1}$ in the free group F on the letters σ_i $(j = 1, 2, \ldots, s-1)$ as follows:

$$b(b')^{-1} = \prod_{k=1}^{n} R_k^{\varepsilon_k W_k},$$

where $R_k^{\varepsilon_k W_k} = W_k R^{\varepsilon_k} W_k^{-1}$ for $\varepsilon_k = \pm 1$ and R_k denotes a relator of (i) or (ii) and W_k is a word in the free group F. Since $b = b(b')^{-1}b'$ in F, the solution of the word problem on the free group guarantees us to change $\mathbf{x}$ into $(\mathbf{x}, -\mathbf{y}^T, \mathbf{y})$ by finitely many applications of (2) and (6). Noting that

$$\mathbf{x}(R_k^{\varepsilon_k W_k}) = (\mathbf{x}(W_k), \mathbf{x}(R_k^{\varepsilon_k}), -\mathbf{x}(W_k)^T),$$

and

$$\mathbf{x}(R_k) = (i, j, -i, -j), \quad \mathbf{x}(R_k^{-1}) = (j, i, -j, -i)$$

in (i) and

$$\mathbf{x}(R_k) = (i, i+1, i, -(i+1), -i, -(i+1)), \quad \mathbf{x}(R_k^{-1}) = (i+1, i, i+1, -i, -(i+1), -i)$$

in (ii), we can change $(\mathbf{x}, -\mathbf{y}^T, \mathbf{y})$ into $\mathbf{y}$ by finitely many applications of (2), (4), (5) and (6). $\square$

Using the forgetful surjection $\iota : \overrightarrow{\mathbb{L}} \to \mathbb{L}$ for the set $\mathbb{L}$ of unoriented links, we also introduce an equivalence relation $\approx$ in $\mathbb{X}$ more relaxed than $\sim$.

Definition 2.3. Two integral vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{X}$ are related as $\mathbf{x} \approx \mathbf{y}$ if we have $\mathrm{cl}\beta(\mathbf{x}) = \mathrm{cl}\beta(\mathbf{y})$ in $\mathbb{L}$ modulo split additions of trivial links.

It is direct to see that the relation $\approx$ is an equivalence relation in $\mathbb{X}$. By definition, we have the natural surjection

$$\tilde{\tilde{\sigma}} : \mathbb{L} \longrightarrow \mathbb{X}/\approx$$

which is injective modulo split additions of trivial links. Since $L = L'$ in $\overrightarrow{\mathbb{L}}$ means $L = L'$ in $\mathbb{L}$, we have also the natural surjection

$$\mathbb{X}/\sim \longrightarrow \mathbb{X}/\approx$$

denoted also by ι so that the following square is commutative:

$$\begin{array}{ccc} \overrightarrow{\mathbb{L}} & \xrightarrow{\tilde{\sigma}} & \mathbb{X}/\sim \\ {\scriptstyle\iota}\downarrow & & \downarrow{\scriptstyle\iota} \\ \mathbb{L} & \xrightarrow{\tilde{\tilde{\sigma}}} & \mathbb{X}/\approx . \end{array}$$

In this diagram, we denote $\tilde{\tilde{\sigma}}(\mathrm{cl}(b)) = [\mathbf{x}(b)]$ and $\iota\langle\mathbf{x}\rangle = [\mathbf{x}]$. To determine a class $[\mathbf{x}] \in \mathbb{X}/\approx$, it is desired to describe the equivalence relation $\approx$ only in terms of $\mathbb{X}$. At present, what we can say about $\approx$ is only the following lemma:

Lemma 2.4. We have the following (1) and (2):

(1) For any $\mathbf{x}, \mathbf{y} \in \mathbb{X}$ such that $\mathbf{x} \sim \mathbf{y}$, we have $\mathbf{x} \approx \mathbf{y}$.

(2) For all $\mathbf{x} \in \mathbb{X}$, we have

$$\mathbf{x} \approx \mathbf{x}^T \approx -\mathbf{x} \approx -\mathbf{x}^T.$$

Proof. (1) is direct from the surjection $\iota : \mathbb{X}/\sim \longrightarrow \mathbb{X}/\approx$. For (2), let $-L$ denote the inverse of an oriented link L, and $\pm\bar{L}$ the mirror image of $\pm L$. Then we have $L = -L = \bar{L} = -\bar{L}$ in $\mathbb{L}$. Taking $L = \mathrm{cl}(b)$ for a braid b, we have

$$\tilde{\sigma}(L) = \langle\mathbf{x}(b)\rangle,\ \tilde{\sigma}(-L) = \langle\mathbf{x}(b)^T\rangle,\ \tilde{\sigma}(\bar{L}) = \langle-\mathbf{x}(b)\rangle,\ \tilde{\sigma}(-\bar{L}) = \langle-\mathbf{x}(b)^T\rangle.$$

Then the commutative square preceding to Lemma 2.4 shows (2). $\square$

The following remark says that (1) and (2) of Lemma 2.4 are sufficient to characterize the equivalence relation $\approx$ in the set of knots:

Remark 2.5. Let $\mathbb{X}_1$ be the subset of $\mathbb{X}$ consisting of an integral vector $\mathbf{x}$ whose closured link $\mathrm{cl}\beta(\mathbf{x})$ is a knot. Then every relation $\mathbf{x} \approx \mathbf{y}$ for $\mathbf{x}, \mathbf{y} \in \mathbb{X}_1$ is generated by the equivalence relation $\sim$ and the relations in (2) of Lemma 2.4. In fact, let $K = \mathrm{cl}\beta(\mathbf{x})$ and $K' = \mathrm{cl}\beta(\mathbf{y})$. If $\mathbf{x} \approx \mathbf{y}$, then we have $[K] = [K']$ modulo split additions of trivial links. Then there is an oriented knot K'' which is one of the knots $\pm K$ and $\pm\bar{K}$ such that $K'' = K'$ in $\vec{\mathbb{L}}$ modulo split additions of trivial links. Thus, we have $\mathbf{x}'' \sim \mathbf{x}'$ for the integral vector $\mathbf{x}''$ which is one of $\pm\mathbf{x}$, $\pm\mathbf{x}^T$. More generally, for oriented links L, L' in S^3, we have $L = L'$ in $\mathbb{L}$ modulo split additions of trivial links if and only if we have $L = L'$ in $\vec{\mathbb{L}}$ modulo split additions of trivial links after a suitable choice of orientations of L and S^3. By counting Lemma 2.4, this implies that in order to know the class $\tilde{\tilde{\sigma}}(L) \in \mathbb{X}/\approx$ of an oriented link L in S^3 with $r(\geqq 2)$-components K_i $(i = 1, 2, \ldots, r)$, it suffices to know a braid presentation of the link $(-L') \cup (L - L')$ for every sublink L' of L with $1 \leqq \#L' \leqq \frac{r}{2}$ besides a braid presentation of L, where $\#L'$ denotes the number of components of L'.

The well-order Ω of $\mathbb{X}$ induces an embedding

$$\tilde{\tilde{\Omega}} : \mathbb{X}/\approx \longrightarrow \mathbb{X}$$

sending $[\mathbf{x}]$ to the initial element of the class $[\mathbf{x}]$ in the well-order Ω. By letting $\sigma = \tilde{\tilde{\Omega}}\tilde{\tilde{\sigma}}$, we obtain a map

$$\sigma : \mathbb{L} \longrightarrow \mathbb{X}$$

which is injective modulo split additions of trivial links. By the *length* of a link $L \in \mathbb{L}$, we mean the length of the integral vector $\sigma(L)$. By using σ, the well-order of $\mathbb{L}$ is defined as follows: Namely, two distinct links $L_1, L_2 \in \mathbb{L}$ have the order $L_1 < L_2$ if and only if either $\sigma(L_1) < \sigma(L_2)$ or $\sigma(L_1) = \sigma(L_2)$ and $\#(L_1) < \#(L_2)$ holds. Since the map σ is defined by Ω, we may say that this well-order in $\mathbb{L}$ is induced from Ω and denoted also by Ω. We now define the *canonical order* Ω_c in $\mathbb{X}$ as follows: Namely, the well-order in $\mathbb{Z}$ is defined by

$$0 < 1 < -1 < 2 < -2 < 3 < -3 < \ldots$$

which is understood as an order counted on the real line along a spiral curve in the complex plane starting from the origin and rounding counterclockwise. This order of $\mathbb{Z}$ is extended to a well-order in $\mathbb{Z}^n$ as the lexicographic order for every $n \geqq 2$. For any two elements $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{X}$ with $\ell(\mathbf{x}_1) < \ell(\mathbf{x}_2)$, we define $\mathbf{x}_1 < \mathbf{x}_2$. This order Ω_c makes $\mathbb{X}$ a well-ordered set. [In fact, let S be any non-empty subset of $\mathbb{X}$. Let S_ℓ be the subset of S consisting of integral vectors with the smallest length, say n. Since $\mathbb{Z}^n$ is a well-ordered set as defined above, we can find the initial integral vector of S_ℓ which is the initial integral vector of S by definition.] From the construction of σ, we see that *the length of a link L in $\Omega = \Omega_c$ is nothing but the minimal crossing number among the braids representing L.* As a consequence, we see that there are only finitely many non-splittable links with the same length and the prime links with up to n lengths for any given n are included in the prime links with up to n crossings. Thus, the classification table of prime links would be useful (see J. H. Conway [4], D. Rolfsen [12] for earlier link tables). Also, the braiding algorithm of S. Yamada [14] would be useful to deform a link into a closed braid form. In §5, we explain how to make the table of prime links with lengths up to any fixed number identified with the integral vectors in $\Omega = \Omega_c$, and as a demonstration, we make the table for the prime links with lengths up to 7.

3. Minimal links

Let $K_i (i = 1, 2, \dots, r)$ be the components of an oriented link L in S^3. A *coloring* f of L is a function

$$f : \{K_i |\, i = 1, 2, \dots, r\} \longrightarrow \mathbb{Q} \cup \{\infty\}.$$

By a *meridian-longitude system* of L on $N(L)$, we mean a pair of a meridian system $m(L) = \{m_i |\, i = 1, 2, \dots, r\}$ and a longitude system $\ell(L) = \{\ell_i |\, i = 1, 2, \dots, r\}$ on $N(L)$ such that (m_i, ℓ_i) is the meridian-longitude pair of K_i on $N(K_i)$ for every i. We can specify the orientations of $m(L)$ and $\ell(L)$ from those of L and S^3 uniquely. Let $f(K_i) = \frac{a_i}{b_i}$ for coprime integers a_i, b_i for every i where we take $a_i = \pm 1$ and $b_i = 0$ when $f(K_i) = \infty$. Then we have a (unique up to isotopies) simple loop s_i on $\partial N(K_i)$ with $[s_i] = a_i[m_i] + b_i[\ell_i]$ in the first integral homology $H_1(\partial N(K_i))$. We note that if the different choice $f(K_i) = \frac{-a_i}{-b_i}$ is made, then only the orientation of s_i is changed. The *Dehn surgery manifold* of a colored link (L, f) is the oriented 3-manifold

$$\chi(L, f) = E(L) \cup_{s_1 = 1\times\partial D_1^2} S^1 \times D_1^2 \cdots \cup_{s_r = 1\times\partial D_r^2} S^1 \times D_r^2$$

with the orientation induced from $E(L) \subset S^3$, where $\cup_{s_i=1\times\partial D_i^2}$ denotes a pasting of $S^1 \times \partial D_i^2$ to $\partial N(K_i)$ so that s_i is identified with $1 \times \partial D_i^2$. In this construction, the 3-manifold $\chi(L, f) \in \mathbb{M}$ is uniquely determined from the colored link (L, f). In this paper, we are particularly interested in the 0-surgery manifold, that is, $\chi(L, f)$ with $f = 0$. For every link $L \in \mathbb{L}$, we consider the subset

$$\{L\}_\pi = \{L' \in \mathbb{L} |\ \pi_1 E(L') = \pi_1 E(L)\}$$

of $\mathbb{L}$. Here are some examples on $\{L\}_\pi$.

Example 3.1. (1) For every prime knot $K \in \mathbb{L}$, we have $\{K\}_\pi = \{K\}$ by the Gordon-Luecke theorem [5] and W. Whitten [13]. However, for example if K is the trefoil knot, then $\{K\#K\}_\pi = \{K\#K, K\#\bar{K}\}$ where $\bar{K}$ denotes the mirror image of K.

(2) Let L be the Whitehead link obtained from the Hopf link $O \cup O'$ by replacing O' with the untwisted double D of O': $L = O \cup D$. Further, let L_m be the link obtained by replacing D with the m-full twist D_m of D along O for every $m \in \mathbb{Z}$ where we take $L_0 = L$. Then we have

$$\{L\}_\pi = \{L_m \mid m \in \mathbb{Z}\}.$$

To see (2), let $L' \in \{L\}_\pi$. Since $E(L)$ is a hyperbolic 3-manifold and hence $\pi_1 E(L) = \pi_1 E(L')$ means $E(L) = E(L')$ (see W. Jaco [7]), the meridian system on L' indicates a coloring f of L. Since the linking numer of O and D are 0, we have $f(O) = \frac{1}{m}$ and $f(D) = \frac{1}{n}$ for some integers $m, n \in \mathbb{Z}$. If m or n is not 0, then we can assume that $m \neq 0$ since O and D are interchangeable. If $m \neq 0$, then we obtain L_m by taking m full twists along O. Since any twisted doubled knot K' is non-trivial and $\chi(K', \frac{1}{n}) \neq S^3$ for $n \neq 0$, we must have $n = 0$, giving the desired result. On this example, one may note that since the linking numer of L_m is 0, the longitude system of L_m coincides with the longitude system of L in $\partial E(L)$, so that $\chi(L_m, 0) = \chi(L, 0)$ for every m.

We consider $\mathbb{L}$ as a well-ordered set by the well-order Ω (defined from the well-order Ω of $\mathbb{X}$ in §2). The following definition is needed to choose exactly one link in the set $\{L\}_\pi$ for a link $L \in \mathbb{L}$:

Definition 3.2. A link $L \in \mathbb{L}$ is *minimal* if L is the initial element of the prime link subset of $\{L\}_\pi$ in the well-order Ω.

The following remark gives a reason why we restrict ourselves to a link in S^3:

Remark 3.3. For a certain torus knot $L \in \mathbb{L}$, there are homotopy torus knot spaces E', not the exterior of any knot in S^3, such that $\pi_1(E') = \pi_1 E(L)$ (see J. Hempel [6,p.152]).

Let $\mathbb{L}^{\mathrm{m}}$ be the subset of $\mathbb{L}$ consisting of minimal links. Since a minimal link is a prime link by definition, we have the following lemma:

Lemma 3.4. The restriction

$$\sigma|\mathbb{L}^{\mathrm{m}} : \mathbb{L}^{\mathrm{m}} \longrightarrow \mathbb{X}$$

is injective.

For the map $\pi : \mathbb{L} \to \mathbb{G}$ sending a link to the link group, we have the following lemma:

Lemma 3.5. The restriction

$$\pi|\mathbb{L}^{\mathrm{m}} : \mathbb{L}^{\mathrm{m}} \longrightarrow \mathbb{G}$$

is injective.

Proof. For $L, L' \in \mathbb{L}^{\mathrm{m}}$, assume that $\pi_1 E(L) = \pi_1 E(L')$. Since both L and L' are minimal in $\{L\}_\pi = \{L'\}_\pi$, we have $L \leqq L'$ and $L \geqq L'$ by definition. Hence $L = L'$. $\square$

The following question is related to determining when a given prime link is minimal:

Question 3.6 When does $\pi_1 E(L) = \pi_1 E(L')$ mean $E(L) = E(L')$ for prime links $L, L' \in \mathbb{L}$?

This question is known to be yes for a large class of prime links, including all prime knots by W. Whitten [13], and prime links L such that $E(L)$ does not contain any essential embedded annulus, in particular, hyperbolic links, by the Johannson Theorem (see W. Jaco [7]). Here is another class of links.

Proposition 3.7 For links $L, L' \in \mathbb{L}$, assume that $E(L)$ is a special Seifert manifold (that is, a Seifert manifold without essential embedded torus) and there is an isomorphism $\pi_1 E(L) \to \pi_1 E(L')$. Then there is a homeomorphism $E(L) \to E(L')$.

Proof. By a classification result of G. Burde-K. Murasugi [3], the Seifert structure of $E(L)$ comes from a Seifert structure on S^3. By [7], the orbit surface of the Seifert manifold $E(L)$ is

(i) the disk with at most two exceptional fibers,
(ii) the annulus with at most one exceptional fiber, or
(iii) the disk with two holes and no exceptional fibers.

In particular, $\pi_1 E(L)$ and hence $\pi_1 E(L')$ are groups with non-trivial centers, so that $E(L')$ is also a special Seifert fibered manifold with the same orbit data as $E(L)$. In the case (i), both L and L' are torus knots and $\pi_1 E(L) \cong \pi_1 E(L')$ implies $L = L'$ (confirmed for example by the Alexander polynomials) and hence $E(L) = E(L')$. In the cases of (ii) without exceptional fiber and (iii), we have $E(L) = E(L') = S^1 \times C$ for the annulus or the disk with two holes C. Assume that $E(L)$ and $E(L')$ are in the case of (ii) with one exceptional fiber. Let (p, q) and (r, s) be the types of the exceptional fibers of $E(L)$ and $E(L')$, respectively, where $p, r \geqq 2$, $(p, q) = 1$, $(r, s) = 1$. Let

$$\begin{aligned} \pi_1 E(L) &= (t, a, b | ta = bt, tb = bt, t^q = a^p) \quad \text{and} \\ \pi_1 E(L') &= (t, a, b | ta = bt, tb = bt, t^s = a^r) \end{aligned}$$

be the fundamental group presentations of $E(L)$ and $E(L')$, respectively, obtained from $S^1 \times C$ with C the disk with two holes by adjoining a fibered solid torus around the exceptional fiber. Let $\psi : \pi_1 E(L) \to \pi_1 E(L')$ be an isomorphism. Considering the central group which is the infinite cyclic group generated by t, we see that $\psi(t) = t^{\pm 1}$. Replacing $-s$ with s if necessarily, we may have $\psi(t) = t$. In the quotient groups, ψ induces an isomorphism

$$\psi_* : (a | a^p = 1) * (b | -) \cong (a | a^r = 1) * (b | -).$$

Hence $p = r$ and $\psi(a) = t^m a^\varepsilon$ for some integer m and $\varepsilon = \pm 1$. Then

$$t^q = \psi(a^p) = t^{mp} a^{\varepsilon p} = t^{mp} a^{\varepsilon r} = t^{mp+\varepsilon s}$$

and hence $q \equiv \pm s \pmod p$, which shows the types (p, q) and (r, s) are equivalent. Thus, there is a fiber-preserving homeomorphism $E(L) \to E(L')$. $\square$

Here is a remark on minimal links.

Remark 3.8 The connected sum L of two copies of the Hopf link, and L' the $(3, 3)$-torus link. Then $\sigma(L) = (1, 1, 2, 2)$ and $\sigma(L') = (1, 1, 2, 1, 1, 2)$ in the order $\Omega = \Omega_c$ (cf. §5). Although $E(L) = E(L')$ and $L < L'$, the link L' is a minimal link. We note that $\chi(L, 0) = S^3$ and $\chi(L', 0) = P^3$ (the projective 3-space) (cf. §5).

4. Proof of Theorem 1.1

The following lemma is a folklore result obtained by the Kirby calculus (see R. Kirby [11]):

Lemma 4.1. The map $\chi_0 : \mathbb{L} \to \mathbb{M}$ is a surjection.

Proof. For every $M \in \mathbb{M}$, we have a colored link (L, f) with the components K_i $(i = 1, 2, \ldots, r)$ such that $\chi(L, f) = M$ and $f(K_i) = m_i$ is an even integer for all i (see S. J. Kaplan [8]). Let $L_1 = L \cup L_0$ be the split union of the oriented link L and a negative Hopf link L_0. Let f_1 be the coloring of L_1 obtained from f and the 0-coloring of L_0. If $\mathrm{sign}(m_i) = +1$, then we take a fusion K'_i of K_i and $\frac{m_i}{2}$ parallel copies of L_0. If $\mathrm{sign}(m_i) = -1$, then we take a fusion K'_i of K_i and $\frac{|m_i|}{2}$ parallel copies of L_0 with the orientations of all the parallel copies of one component of L_0 reversed. Replacing K_i with K'_i for all i with $m_i \neq 0$, we obtain an oriented link L'_1 from L_1 such that $\chi(L'_1, 0) = \chi(L_1, f_1) = M$. $\square$

Let $\mathbb{L}^{\mathrm{m}}(M)$ be the subset of $\mathbb{L}^{\mathrm{m}}$ consisting of a minimal link L such that $\chi(L, 0) = M$. It is not so easy to find a minimal link in $\mathbb{L}^{\mathrm{m}}(M)$ for a given $M \in \mathbb{M}$ in general. If one consider a prime link $L \in \mathbb{L}$ with $\chi(L, 0) = M$ and then take the initial element L_0 of the set $\{L\}_\pi$, then the link L_0 need not be prime, as it is noted in Remark 3.8. Thus, L_0 should be taken as the initial element of the prime link subset of $\{L\}_\pi$. In this case, L_0 is a minimal link in $\mathbb{L}^{\mathrm{m}}(\chi(L_0))$, but in general we cannot guarantee that $\chi(L_0, 0) = M$, as we note in the following example:

Example 4.2. There are hyperbolic links $L, L' \in \mathbb{L}$ such that $E(L) = E(L')$, $\chi(L, 0) \neq \chi(L', 0)$ and $\{L\}_\pi = \{L'\}_\pi = \{L, L'\}$. Thus, if $L < L'$ in the well-order Ω, then the link L is minimal, but $L \notin \mathbb{L}(\chi(L', 0))$. To see this assertion, let $L_H = O_1 \cup O_2$ be the Hopf link with coloring f such that $f(O_1) = 0, f(O_2) = 1$. Then $\chi(L_H, f) = S^3$ and the dual colored link (L'_H, f') is given by $L'_H = L_H$ and

$f'(O_1) = -1$ and $f'(O_2) = 0$. By Remark 3.7 of [10], we have a normal imitation $q : (S^3, L_H^*) \to (S^3, L_H)$ and dual normal imitation $q' : (S^3, L_H'^*) \to (S^3, L_H')$ such that $E(L_H'^*) = E(L_H^*)$, the links L_H^* and $L_H'^*$ are totally hyperbolic componentwise distinct, and every homeomorphism $h : E(L'') \to E(L_H^*)$ extends to a homeomorphism $h^+ : (S^3, L'') \to (S^3, L_H^*)$ or $h^{+\prime} : (S^3, L'') \to (S^3, L_H'^*)$. Then $\chi(L_H^*, 0)$ and $\chi(L_H'^*, 0)$ are homology 3-spheres, for they are normal imitations of $\chi(L_H, 0) = \chi(L_H', 0) = S^3$. Since $\chi(L_H', 0) = \chi(L_H, f'')$ with $f''(O_1) = -1, f''(O_2) = \infty$, we can assume from Theorem 3.1(2) of [10] that $\chi(L_H^*, 0)$ and $\chi(L_H^*, f''q) = \chi(L_H'^*, 0)$ are distinct and hyperbolic because $f \neq 0, f''$. Thus, we can take L_H^* and $L_H'^*$ as L and L', respectively.

In spite of Example 4.2, we can show the following lemma:

Lemma 4.3. For every $M \in \mathbb{M}$, the set $\mathbb{L}^{\mathrm{m}}(M)$ is an infinite set.

Proof. By Lemma 4.1, we take a disconnected link L in S^3 such that $\chi(L, 0) = M$. Let $M \neq S^3$. By a result of [10], there are infinitely many normal imitations

$$q_i : (S^3, L_i^*) \longrightarrow (S^3, L) \quad (i = 1, 2, 3, \dots)$$

such that

(1) $\chi(L_i^*, 0) = \chi(L, 0) = M$,
(2) L_i^* is (totally) hyperbolic, and
(3) every homeomorphism $h : E(L_i^*) \to E(L')$ for a link L' in S^3 extends to a homeomorphism $h^+ : (S^3, L_i^*) \to (S^3, L')$.

Then L_i^* is minimal by (2) and (3), so that $L_i^* \in \mathbb{L}^{\mathrm{m}}(M)$, $i = 1, 2, 3, \dots$. For $M = S^3$, let L be a Hopf link. Then $\chi(L, 0) = S^3$ and the dual link L' of the Dehn surgery is also the Hopf link. By Remark 3.7 of [10], there are infinitely many pairs of normal imitations

$$\begin{aligned} q_i :& (S^3, L_i^*) \longrightarrow (S^3, L), \\ q_i' :& (S^3, L_i'^*) \longrightarrow (S^3, L') \quad (i = 1, 2, 3, \dots) \end{aligned}$$

such that

(1) $\chi(L_i^*, 0) = \chi(L, 0) = S^3 = \chi(L', 0) = \chi(L_i'^*, 0)$,
(2) $E(L^*) = E(L_i'^*)$,
(3) L_i^* and $L_i'^*$ are (totally) hyperbolic,
(4) every homeomorphism $h : E(L_i^*) \to E(L'')$ for a link L'' in S^3 extends to a homeomorphism $h^+ : (S^3, L_i^*) \to (S^3, L'')$ or $h'^+ : (S^3, L_i'^*) \to (S^3, L'')$.

Thus, $\{L_i^*\}_\pi = \{L_i^*, L_i'^*\}$ for every i, and we can take a minimal link, say L_i^* in $\{L_i^*\}_\pi$ for every i, so that $L_i^* \in \mathbb{L}^{\mathrm{m}}(S^3)$, $i = 1, 2, 3, \dots$. $\square$

We are in a position to prove Theorem 1.1.

Proof of Theorem 1.1. Since $\mathbb{L}^{\mathrm{m}}(M) \neq \emptyset$ by Lemma 4.3, we can take the initial element $L^{\mathrm{m}}(M)$ of $\mathbb{L}^{\mathrm{m}}(M)$ for every $M \in \mathbb{M}$. Using that the set $\mathbb{L}^{\mathrm{m}}(M)$ is uniquely determined by M and Ω, we see that the well-order Ω of $\mathbb{X}$ induces a map

$$\alpha : \mathbb{M} \longrightarrow \mathbb{L}$$

sending a 3-manifold M to the link $L^{\mathrm{m}}(M)$. This map α must be injective because of the 0-surgery manifold $\chi(\alpha(M), 0) = M$. Combining this result with Lemmas 3.4 and 3.5, we obtain the embeddings σ_α and π_α. If an integral vector $\mathbf{x} = \sigma_\alpha(M)$ is given, then we obtain the link $\alpha(M) = \mathrm{cl}\beta(\mathbf{x})$ with braid presentation, the 3-manifold $M = \chi(\mathrm{cl}\beta(\mathbf{x}), 0)$ with 0-surgery description and the link group $\pi_1 E(\mathrm{cl}\beta(\mathbf{x}))$ with Artin presentation associated with the braid $\beta(\sigma_\alpha(M))$, completing the proof of the first half. If a link group $G = \pi_\alpha(M)$ with a prime Artin presentation is given, then we have a braid b such that G is the link group of the prime closured link $\mathrm{cl}(b)$. Let $\mathbf{x}_i$ $(i = 1, 2, \dots, n)$ be the integral vectors smaller than or equal to the integral vector $\mathbf{x}(b)$. By using a solution of the problem in (3), let $\mathbf{x}_{i_0}$ be the smallest integral vector such that $\mathrm{cl}\beta(\mathbf{x}_{i_0})$ is a prime link and there is an isomorphism $\pi_1 E(\mathrm{cl}\beta(\mathbf{x}_{i_0})) \to G$ among $\mathbf{x}_i$ $(i = 1, 2, \dots, n)$. Then the link $\mathrm{cl}\beta(\mathbf{x}_{i_0})$ is minimal by this construction. Thus, the desired integral vector $\sigma_\alpha(M) = \mathbf{x}_{i_0}$ is obtained, proving (3). If a minimal link L with $\chi(L, 0) = M$ is given, let $\mathbf{x}$ be an integral vector induced from a braid b representing L, and x_i $(i = 1, 2, \dots, n)$ be the integral vectors smaller than or equal to $\mathbf{x}$. By using a solution of the problem in (4), we take the smallest integral vector $\mathbf{x}_{i_0}$ in the integral vectors x_i $(i = 1, 2, \dots, n)$ such that the link $\mathrm{cl}\beta(\mathbf{x}_{i_0})$ is a minimal link and $\chi(\mathrm{cl}\beta(\mathbf{x}_{i_0}), 0) = M$. Thus, the desired integral vector $\sigma_\alpha(M) = \mathbf{x}_{i_0}$ is obtained, proving (4). $\square$

As a matter of fact, we can construct many variants of the embedding $\alpha : \mathbb{M} \to \mathbb{L}$. Here are remarks on constructing some embeddings α.

Remark 4.4. If we take the subset $\mathbb{L}^{\mathrm{h}} \subset \mathbb{L}$ consisting of hyperbolic links (possibly with infinite volume) which are determined by the exteriors and $\mathbb{L}^{\mathrm{h}}(M) = \{L \in \mathbb{L}^{\mathrm{h}} \mid \chi(L, 0) = M\}$, then we still have an embedding

$$\alpha : \mathbb{M} \to \mathbb{L}^{\mathrm{h}} \subset \mathbb{L}$$

with $\chi_0 \alpha = 1$ such that σ_α and π_α are embeddings by the proof of Theorem 1.1 using $\mathbb{L}^{\mathrm{h}}(M)$ instead of $\mathbb{L}^{\mathrm{m}}(M)$. (For this proof, we use that $\mathbb{L}^{\mathrm{h}}(S^3)$ contains the Hopf link and the set $\mathbb{L}^{\mathrm{h}}(M)$ for $M \neq S^3$ is infinite by Lemma 4.3.) In this case, the links $\alpha(S^1 \times S^2)$, $\alpha(S^3)$ and $\alpha(M)$ for every $M \neq S^1 \times S^2, S^3$ are the trivial knot, the Hopf link and a hyperbolic link of finite volume, respectively. If we take the subset $\mathbb{L}(M) \subset \mathbb{L}$ consisting of links L with $\chi(L, 0) = M$, then the proof of Theorem 1.1 using $\mathbb{L}(M)$ instead of $\mathbb{L}^{\mathrm{m}}(M)$ shows the existence of an embedding $\alpha : \mathbb{M} \to \mathbb{L}$

with $\chi_0\alpha = 1$. However, in this case, the maps σ_α and π_α are no longer injective. In fact, in the canonical order $\Omega = \Omega_c$, if M_n is the connected sum of n copies of $S^1 \times S^2$, then $\sigma_\alpha(M_n) = 0$ for every n. If $K\#K$ is the granny knot and $K\#\bar{K}$ is the square knot where K is a trefoil knot, then we see that $\alpha(\chi(K\#K, 0)) = K\#K$ and $\alpha(\chi(K\#\bar{K}, 0)) = K\#\bar{K}$. Thus, we have $\pi_\alpha(\chi(K\#K, 0)) = \pi_\alpha(\chi(K\#\bar{K}, 0))$ although $\chi(K\#K, 0)) \neq \chi(K\#\bar{K}, 0)$.

5. A classification program.

We consider the following mutually related three embeddings already established:

$$\begin{aligned} \alpha :&\mathbb{M} \longrightarrow \mathbb{L}, \\ \sigma_\alpha :&\mathbb{M} \longrightarrow \mathbb{X}, \\ \pi_\alpha :&\mathbb{M} \longrightarrow \mathbb{G}. \end{aligned}$$

Throughout this section, we take the canonical order $\Omega = \Omega_c$. By the embedding σ_α and a property of Ω_c, we can attach without overlapping to every 3-manifold M in $\mathbb{M}$ a lable (n, i) where n denotes the length of M and i denotes that M appears as the ith 3-manifold of length n, so that we have

$$M_{n,1} < M_{n,2} < \cdots < M_{n,m_n}$$

for a positive integer $m_n < \infty$. Let

$$\alpha(M_{n,i}) = L_{n,i} \in \mathbb{L}, \quad \pi_\alpha(M_{n,i}) = G_{n,i} \in \mathbb{G} \quad \text{and} \quad \sigma_\alpha(M_{n,i}) = \mathbf{x}_{n,i} \in \mathbb{X}.$$

Our classification program is to enumerate the 3-manifolds $M_{n,i}$ for all $n = 1, 2, \dots$ and $i = 1, 2, \dots, m_n$ together with the data $L_{n,i}, G_{n,i}$ and $\mathbf{x}_{n,i}$, but by Theorem 1.1 (2) it is sufficient to give the integral vector $\mathbf{x}_{n,i}$, because we can easily construct $L_{n,i}$, $M_{n,i}$ and $G_{n,i}$ by $L_{n,i} = \mathrm{cl}\beta(\mathbf{x}_{n,i})$, $M_{n,i} = \chi(L_{n,i})$ and $G_{n,i} = \pi_1 E(L_{n,i})$. We proceed the argument by induction on the length n. For any integer x, we have $x \sim 0$. In fact, if $x \neq 0$, then

$$x \sim (x, 0) \sim (0, x) \sim 0$$

by (1),(3) and (6) of Lemma 2.2. Since the trivial knot O is the closured link of the associated braid of 0 and $\chi(O, 0) = S^1 \times S^2$, we have the classification of $\mathbb{M}$ with length 1:

$$m_1 = 1, \quad M_{1,1} = S^1 \times S^2, \quad L_{1,1} = O \quad \text{and} \quad G_{1,1} = \mathbb{Z}.$$

To explain our classification of $\mathbb{M}$ with any length n, we assume that the classification of $\mathbb{M}$ with lengths $\leqq n-1$ are done. To enumerate integral vectors needed for our purpose, we need some notinos.

Definition 5.1. An integral vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is *reducible* if

(1) $\min |\mathbf{x}| = 0$ and $\ell(\mathbf{x}) > 1$,
(2) there is an integer m such that $\min |\mathbf{x}| < m < \max |\mathbf{x}|$ and $m \neq |x_i|$ for all i, or
(3) $\min \mathbf{x} < \max \mathbf{x}$ and there is a pair of integers k, s with $1 \leqq k \leqq s \leqq n$ such that $|x_k| = |x_i| = |x_s|$ for all i with $k \leqq i \leqq s$ and $|x_j| \neq |x_s|$ for all j with $j < k$ or $j > s$.

In Definition 5.1, we note that the core $\tilde{\mathbf{x}}$ of $\mathbf{x}$ has a shorter length in (1), the closured link $L = \mathrm{cl}\beta(\mathbf{x})$ is a split link in (2), and L is not a prime link or we have $\mathbf{x} \sim (x_k, x_{k+1}, \ldots, x_s)$ whose length is shorter than n in (3). Here is another relation.

Lemma 5.2 (Duality relation).

$$\mathbf{x} = (x_1, x_2, \ldots, x_n) \sim \mathbf{x}' = (x'_1, x'_2, \ldots, x'_n),$$

where

$$x'_i = \begin{cases} \mathrm{sign}(x_i)(\max |\mathbf{x}| + 1 - |x_i|) & x_i \neq 0 \\ 0 & x_i = 0. \end{cases}$$

Proof. The integral vector $\mathbf{x}'$ is obtained by changing the numbering $1, 2, \ldots, m$ of the strings of the associated braid b of $\mathbf{x}$ (with $m = \max |\mathbf{x}| + 1$) into $m, m-1, \ldots, 1$ and then overturning the braid diagram. Since this deformation does not change the link type of $\mathrm{cl}(b)$ in $\overrightarrow{\mathbb{L}}$, we have $\mathbf{x} \sim \mathbf{x}'$ by Definition 2.1 □

We also use the following notion:

Definition 5.3. An integral vector $\mathbf{x} \in \mathbb{X}$ is *quasi-minimal* if it is minimal in the class $[\mathbf{x}]$ with respect to the deformations given in Lemmas 2.2, 2.4 and the duality relation except for the three deformations increasing the length stated in (1),(2),(3) of Lemma 2.2.

We note that a quasi-minimal integral vector $\mathbf{x}$ is not necessarily the initial element of the class $[\mathbf{x}]$. The first step of our classification program is as follows:

Step 1. Enumerate a set of integral vectors of length n containing all the irreducible quasi-minimal integral vectors of length n.

It would make our work simple to take a set of integral vectors in Step 1 as small as possible. It is recommended to enumerate first the integral vectors $\mathbf{x}$ with entries x_i $(i = 1, 2, \ldots, n)$ in the following conditions (because every irreducible quasi-minimal integral vector except 0 has these conditions):

(1) $x_1 = 1$ and the number of i with $|x_i| = 1$ is greater than or equal to the number of j with $|x_j| = |\mathbf{x}|$.

(2) Except the casse that $|x_i| = 1$ for all i, there is no pair k, s with $k \leqq s$ such that

$$|x_k| = |x_i| = |x_s| \quad (k \leqq i \leqq s) \quad \text{and} \quad |x_j| \neq |x_k| = |x_s| \quad (j < k, s < j).$$

(3) $\max(|x_{i-1}| - 1, 1) \leqq |x_i| \leqq \frac{n}{2} \quad (i \geqq 2)$.

Then we select integral vectors as small as possible by using Lemmas 2.2, 2.4 and the duality relation. The reason why we can impose these conditions on the integral vectors comes from the duality relation and Lemmas 2.2, 2.4 for (1), the reducibility condition for (2) and Lemma 2.4, the reducibility condition and the duality relation for (3).

The following list of integral vectors of lengths $\leqq 7$ is thus made.

Example 5.4. The following list contains all the irreducible quasi-minimal integral vectors of lengths $\leqq 7$:

length 1: 0,
length 2: $(1, 1)$,
length 3: $(1, 1, 1)$,
length 4: $(1, 1, 1, 1), (1, -2, 1, -2)$,
length 5: $(1, 1, 1, 1, 1), (1, 1, 2, -1, 2), (1, 1, -2, 1, -2)$,
length 6: $(1, 1, 1, 1, 1, 1), (1, 1, 1, 2, -1, 2), (1, 1, 1, -2, 1, -2), (1, 1, 2, 1, 1, 2),$
$(1, 1, 2, -1, -1, 2), (1, 1, -2, 1, 1, -2), (1, 1, -2, 1, -2, -2),$
$(1, -2, 1, -2, 1, -2), (1, -2, 1, 3, -2, 3)$,
length 7: $(1, 1, 1, 1, 1, 1, 1), (1, 1, 1, 1, 2, -1, 2), (1, 1, 1, 1, -2, 1, -2),$
$(1, 1, 1, 2, 1, 1, 2), (1, 1, 1, 2, -1, -1, 2), (1, 1, 1, -2, 1, 1, -2),$
$(1, 1, 1, -2, 1, -2, -2), (1, 1, 1, -2, -1, -1, -2), (1, 1, 2, -1, -3, 2, -3),$
$(1, 1, -2, 1, 1, -2, -2), (1, 1, -2, 1, -2, 1, -2), (1, 1, -2, 1, 3, -2, 3),$
$(1, -2, 1, -2, 3, -2, 3)$.

Let $\bar{\mathbb{D}}_n$ be the set consisting of the closured link diagram $\mathrm{cl}\beta(\mathbf{x})$ of an integral vector $\mathbf{x}$ of length n in Step 1. Let $\mathbb{L}_j$ be the subset of $\mathbb{L}$ consisting of prime links of length j. The set $\mathbb{L}_n$ consists of an element represented by a member of $\bar{\mathbb{D}}_n$ but not belonging to the sets $\mathbb{L}_j$ $(j = 1, 2, \ldots, n-1)$ (already constructed by our inductive hypothesis). Step 2 is the following procedure:

Step 2. Construct $\mathbb{L}_n$ from $\bar{\mathbb{D}}_n$.

The closuredlink $\mathrm{cl}\beta(\mathbf{x})$ of an integral vector $\mathbf{x}$ of length n admits a braided link diagram with crossing number $\leqq n$. Thus, if a list of prime links with crossing number up to n is available, then this enumeration procedure would not be so difficult. In the following example, the main work is only to identify the integral vectors of length $n \leqq 7$ in Example 5.4 with prime links in Rolfsen's table [12].

Example 5.5. The following list gives the elements of the sets $\mathbb{L}_n$ of lengths $n \leqq 7$ together with the corresponding integral vectors.

$\mathbb{L}_1: \ O \qquad \sigma(O) = 0.$

$\mathbb{L}_2: \ 2_1^2 \qquad \sigma(2_1^2) = (1,1).$

$\mathbb{L}_3: \ 3_1 \qquad \sigma(3_1) = (1,1,1).$

$\mathbb{L}_4: \ 4_1^2 < 4_1$
$\sigma(4_1^2) = (1,1,1,1),$
$\sigma(4_1) = (1,-2,1,-2).$

$\mathbb{L}_5: \ 5_1 < 5_1^2$
$\sigma(5_1) = (1,1,1,1,1),$
$\sigma(5_1^2) = (1,1,-2,1,-2).$

$\mathbb{L}_6: \ 6_1^2 < 5_2 < 6_2 < 6_3^3 < 6_1^3 < 6_3 < 6_2^3 < 6_3^2$
$\sigma(6_1^2) = (1,1,1,1,1,1),$
$\sigma(5_2) = (1,1,1,2,-1,2),$
$\sigma(6_2) = (1,1,1,-2,1,-2),$
$\sigma(6_3^3) = (1,1,2,1,1,2),$
$\sigma(6_1^3) = (1,1,-2,1,1,-2),$
$\sigma(6_3) = (1,1,-2,1,-2,-2),$
$\sigma(6_2^3) = (1,-2,1,-2,1,-2),$
$\sigma(6_3^2) = (1,-2,1,3,-2,3).$

$\mathbb{L}_7: \ 7_1 < 6_2^2 < 7_1^2 < 7_7^2 < 7_8^2 < 7_4^2 < 7_2^2 < 6_1 < 7_5^2 < 7_6^2 < 7_6 < 7_1^3$
$\sigma(7_1) = (1,1,1,1,1,1,1),$
$\sigma(6_2^2) = (1,1,1,1,2,-1,2),$
$\sigma(7_1^2) = (1,1,1,1,-2,1,-2),$
$\sigma(7_7^2) = (1,1,1,2,1,1,2),$
$\sigma(7_8^2) = (1,1,1,2,-1,-1,2),$
$\sigma(7_4^2) = (1,1,1,-2,1,1,-2),$
$\sigma(7_2^2) = (1,1,1,-2,1,-2,-2),$
$\sigma(6_1) = (1,1,2,-1,-3,2,-3),$
$\sigma(7_5^2) = (1,1,-2,1,1,-2,-2),$

$\sigma(7_6^2) = (1, 1, -2, 1, -2, 1, -2),$
$\sigma(7_6) = (1, 1, -2, 1, 3, -2, 3),$
$\sigma(7_1^3) = (1, -2, 1, -2, 3, -2, 3).$

The integral vectors $(1, 1, 2, -1, 2)$, $(1, 1, 2, -1, -1, 2)$ and $(1, 1, 1, -2, -1, -1, -2)$ of Example 5.4 are removed from the list, since the closured links are seen to be 4_1^2, 6_3^3, 7_7^2, respectively. The links $7_2, 7_3, 7_4, 7_5, 7_7, 7_3^2$ in Rolfsen's table of [12] are also excluded from the list since these links turn out to have the lengths greater than 7. In Steps 3 and 4, powers of low dimensional topology techniques will be seriously tested.

Step 3. Construct the subset $\mathbb{L}_n^{\mathrm{m}} \subset \mathbb{L}_n$ by removing every link $L \in \mathbb{L}_n$ such that $\pi_1 E(L) = \pi_1 E(L')$ for a link $L' \in \mathbb{L}_j$ with $j < n$ or $L' \in \mathbb{L}_n$ with $L' < L$.

From construction, we see that the set $\mathbb{L}_n^{\mathrm{m}}$ consists of minimal links of length n. Among the links in Example 5.5, we see that $E(4_1^2) = E(7_7^2)$ and $E(5_1^2) = E(7_8^2)$ by taking one full twist along a component and that except these identities, all the links have mutually distinct link groups by using the following lemma on the Alexander polynomials:

Lemma 5.6. Let $A(t_1, t_2, \ldots, t_r)$ and $A'(t_1, t_2, \ldots, t_r)$ be the Alexander polynomials of oriented links L and L' with r components. If there is a homeomorphism $E(L) \to E(L')$, then there is an automorphism ψ of the multiplicative free abelian group with generators t_i $(i = 1, 2, \ldots, r)$ such that

$$A'(t_1, t_2, \ldots, t_r) = \pm t_1^{s_1} t_2^{s_2} \ldots t_r^{s_r} A(\psi(t_1), \psi(t_2), \ldots, \psi(t_r)),\ s_i \in \mathbb{Z}(i = 1, 2, \ldots, r).$$

The proof of this lemma is direct from the definition of Alexander polynomial(see [9]). Thus, we obtain the following example:

Example 5.7. We have $\mathbb{L}_n^{\mathrm{m}} = \mathbb{L}_n$ for $n \leqq 6$ and

$$\mathbb{L}_7^{\mathrm{m}} = \{7_1, 6_2^2, 7_1^2, 7_4^2, 7_2^2, 6_1, 7_5^2, 7_6^2, 7_6, 7_1^3\}.$$

Let $\mathbb{M}_n$ be the subset of $\mathbb{M}$ consisting of a 3-manifold of length n, and $\mathbb{L}_n^{\mathbb{M}}$ the set obtained from $\mathbb{L}_n^{\mathrm{m}}$ by removing a minimal link L such that $\chi(L, 0) \in \mathbb{M}_j$ for an index $j < n$ or $\chi(L, 0) = \chi(L', 0)$ for a minimal link $L' \in \mathbb{L}_n^{\mathrm{m}}$ with $L' < L$. The following step is the final step of our classification program:

Step 4. Construct the set $\mathbb{L}_n^{\mathbb{M}}$.

Let $L_i\,(i = 1, 2, \ldots, r)$ be the minimal links of the set $\mathbb{L}_n^{\mathbb{M}}$, ordered by Ω_c. Then we have $M_{n,i} = \chi(L_i, 0)$, $\alpha(M_{n,i}) = L_i$ $(i = 1, 2, \ldots, r)$. An important notice is that every 3-manifold in $\mathbb{M}$ appears once as $M_{n,i}$ without overlaps. As we shall show later, the 0-surgery manifolds of the minimal links of Example 5.7 are mutually non-homeomorphic, so that we have the complete list of 3-manifolds in $\mathbb{M}$ with lengths $\leqq 7$ as it is stated in Example 5.8.

Example 5.8.

$M_{1,1} = \chi(O, 0)$, $\quad \mathbf{x}_{1,1} = 0$,

$M_{2,1} = \chi(2_1^2, 0)$, $\quad \mathbf{x}_{2,1} = (1, 1)$,

$M_{3,1} = \chi(3_1, 0)$, $\quad \mathbf{x}_{3,1} = (1, 1, 1)$,

$M_{4,1} = \chi(4_1^2, 0)$, $\quad \mathbf{x}_{4,1} = (1, 1, 1, 1)$,
$M_{4,2} = \chi(4_1, 0)$, $\quad \mathbf{x}_{4,2} = (1, -2, 1, -2)$,

$M_{5,1} = \chi(5_1, 0)$, $\quad \mathbf{x}_{5,1} = (1, 1, 1, 1, 1)$,
$M_{5,2} = \chi(5_1^2, 0)$, $\quad \mathbf{x}_{5,2} = (1, 1, -2, 1, -2)$,

$M_{6,1} = \chi(6_1^2, 0)$, $\quad \mathbf{x}_{6,1} = (1, 1, 1, 1, 1, 1)$,
$M_{6,2} = \chi(5_2, 0)$, $\quad \mathbf{x}_{6,2} = (1, 1, 1, 2, -1, 2)$,
$M_{6,3} = \chi(6_2, 0)$, $\quad \mathbf{x}_{6,3} = (1, 1, 1, -2, 1, -2)$,
$M_{6,4} = \chi(6_3^3, 0)$, $\quad \mathbf{x}_{6,4} = (1, 1, 2, 1, 1, 2)$,
$M_{6,5} = \chi(6_1^3, 0)$, $\quad \mathbf{x}_{6,5} = (1, 1, -2, 1, 1, -2)$,
$M_{6,6} = \chi(6_3, 0)$, $\quad \mathbf{x}_{6,6} = (1, 1, -2, 1, -2, -2)$,
$M_{6,7} = \chi(6_2^3, 0)$, $\quad \mathbf{x}_{6,7} = (1, -2, 1, -2, 1, -2)$,
$M_{6,8} = \chi(6_3^2, 0)$, $\quad \mathbf{x}_{6,8} = (1, -2, 1, 3, -2, 3)$.

$M_{7,1} = \chi(7_1, 0)$, $\quad \mathbf{x}_{7,1} = (1, 1, 1, 1, 1, 1, 1)$,
$M_{7,2} = \chi(6_2^2, 0)$, $\quad \mathbf{x}_{7,2} = (1, 1, 1, 1, 2, -1, 2)$,
$M_{7,3} = \chi(7_1^2, 0)$, $\quad \mathbf{x}_{7,3} = (1, 1, 1, 1, -2, 1, -2)$,
$M_{7,4} = \chi(7_4^2, 0)$, $\quad \mathbf{x}_{7,4} = (1, 1, 1, -2, 1, 1, -2)$,
$M_{7,5} = \chi(7_2^2, 0)$, $\quad \mathbf{x}_{7,5} = (1, 1, 1, -2, 1, -2, -2)$,
$M_{7,6} = \chi(6_1, 0)$, $\quad \mathbf{x}_{7,6} = (1, 1, 2, -1, -3, 2, -3)$,
$M_{7,7} = \chi(7_5^2, 0)$, $\quad \mathbf{x}_{7,7} = (1, 1, -2, 1, 1, -2, -2)$,
$M_{7,8} = \chi(7_6^2, 0)$, $\quad \mathbf{x}_{7,8} = (1, 1, -2, 1, -2, 1, -2)$,
$M_{7,9} = \chi(7_6, 0)$, $\quad \mathbf{x}_{7,9} = (1, 1, -2, 1, 3, -2, 3)$,
$M_{7,10} = \chi(7_1^3, 0)$, $\quad \mathbf{x}_{7,10} = (1, -2, 1, -2, 3, -2, 3)$.

To see that the 3-manifolds in Example 5.8 are mutually non-homeomorphic, we first check the first integral homology. It is computed as follows:

(1) $H_1 = \mathbb{Z}$: $M_{1,1}$, $M_{3,1}$, $M_{4,2}$, $M_{5,1}$, $M_{6,2}$, $M_{6,3}$, $M_{6,6}$, $M_{7,1}$, $M_{7,6}$, $M_{7,9}$.
(2) $H_1 = \mathbb{Z} \oplus \mathbb{Z}$: $M_{5,2}$, $M_{7,4}$, $M_{7,8}$.

(3) $H_1 = \mathbb{Z}_2$: $M_{6,4}$, $M_{6,5}$, $M_{7,10}$.
(4) $H_1 = \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}$: $M_{6,7}$.
(5) $H_1 = \mathbb{Z}_2\oplus\mathbb{Z}_2$: $M_{4,1}$, $M_{6,8}$, $M_{7,7}$.
(6) $H_1 = \mathbb{Z}_3\oplus\mathbb{Z}_3$: $M_{6,1}$, $M_{7,2}$.
(7) $H_1 = 0$: $M_{2,1}$, $M_{7,3}$, $M_{7,5}$.

For (1), since the Alexander polynomial of a knot K is an invariant of the homology handle $\chi(K,0)$, we see that the homology handles of (1) are mutually distinct. For (2), since the Alexander polynomial of an oriented link L with all the linking numbers 0 is also an invariant of $\chi(L,0)$ in the sense of Lemma 5.6, these 3-manifolds are mutually distinct. For (3), we note that $M_{6,4} = P^3$ the projective 3-space, $M_{6,5} = \chi(3_1,-2)$ (where we take 3_1 the positive trefoil knot) and $M_{7,10} = \chi(4_1,-2)$. We take the connected double covering spaces $\tilde{M}$ of $M = M_{6,4}$, $M_{6,5}$ and $M_{7,10}$. The homology $H_1(\tilde{M})$ for $M = M_{6,4}$, $M_{6,5}$ or $M_{7,10}$ is respectively computed as 0, $\mathbb{Z}_3$, $\mathbb{Z}_5$, showing that these 3-manifolds are mutually distinct. For (4), we have nothing to prove. Note that $M_{6,7} = T^3$. For (5), we compare the first integral homologies of the three kinds of connected double coverings of every $M = M_{4,1}$, $M_{6,8}$, $M_{7,7}$. For $M = M_{4,1}$, it is the quaternion space Q and we have $H_1(\tilde{M}) = \mathbb{Z}_4$ for every connected double covering space $\tilde{M}$ of M. For $M = M_{6,8}$, we have $H_1(\tilde{M};\mathbb{Z}_3) = \mathbb{Z}_3$ for every connected double covering space $\tilde{M}$ of M. On the other hand, for $M = M_{7,7}$, we have $H_1(\tilde{M}) = \mathbb{Z}_{16}$ and $H_1(\tilde{M};\mathbb{Z}_3) = 0$ for some connected double covering space $\tilde{M}$ of M. Thus, these 3-manifolds are mutually distinct. For(6), we use the following lemma:

Lemma 5.9. Let $T = \mathbb{Z}_p\oplus\mathbb{Z}_p$ for an odd prime $p > 1$. If the linking form $\ell : T\times T \longrightarrow \mathbb{Q}/\mathbb{Z}$ is hyperbolic, then the hyperbolic $\mathbb{Z}_p$-basis e_1, e_2 of T is unique up to unit multiplications of $\mathbb{Z}_p$.

Proof. Let e_1', e_2' be another hyperbolic $\mathbb{Z}_n$-basis of T. Let $e_i' = a_{i1}e_1 + a_{i2}e_2$. Then

$$\begin{aligned} 0 = \ell(e_i', e_i') &= \frac{2a_{i1}a_{i2}}{p} \quad \pmod 1, \\ \frac{1}{p} = \ell(e_1', e_2') &= \frac{a_{11}a_{22}+a_{12}a_{21}}{p} \quad \pmod 1. \end{aligned}$$

By these identities, we have either $e_1' = a_{11}e_1$ and $e_2' = a_{22}e_2$ with $a_{11}a_{22} = 1$ in $\mathbb{Z}_p$ or $e_1' = a_{12}e_2$ and $e_2' = a_{21}e_1$ with $a_{12}a_{21} = 1$ in $\mathbb{Z}_p$. □

By this lemma, there are just two connected $\mathbb{Z}_3$-coverings $\tilde{M}$ of every $M = M_{6,1}$, $M_{7,2}$ associated with a hyperbolic direct summand $\mathbb{Z}_3$ of $H_1(M) = \mathbb{Z}_3\oplus\mathbb{Z}_3$. In other words, the covering $\tilde{M}$ is associated with a $\mathbb{Z}_3$-covering covering of the exterior $E(L)$ lifting one torus boundary component trivially, where $L = 6_1^2, 6_2^2$. Since the link L is interchangeable, it is sufficient to check one covering for each M. For

$M = M_{6,1}$ we have $H_1(\tilde{M}) = \mathbb{Z}_{12} \oplus \mathbb{Z}_4$ and for $M = M_{7,2}$ we have $H_1(\tilde{M}) = \mathbb{Z} \oplus \mathbb{Z}$. Thus, these 3-manifolds are distinct. For (7), the Dehn surgery manifolds $chi(7^2_1, 0)$ and $\chi(7^2_2)$ are the boundaries of Mazur manifolds (which are normal imitations of S^3) and identified with the Brieskorn homology 3-spheres $\Sigma(2,3,13)$, $\Sigma(2,5,7)$ by S. Akbult and R. Kirby [1]. Hence, we have $M_{2,1} = S^3$, $M_{7,3} = \Sigma(2,3,13)$ and $M_{7,5} = \Sigma(2,5,7)$ and these 3-manifolds are mutually distinct. Thus, we see that the 3-manifolds of Example 5.8 are mutually distinct.

For the Poincaré homology 3-sphere Σ which is not a normal imitation of S^3, the prime link $\alpha(\Sigma)$ must have at least 10 components. [To see this, assume that $\alpha(\Sigma)$ has r components. Using that Σ is a homology 3-sphere and $\Sigma = \chi(\alpha(\Sigma), 0)$, we see that Σ bounds a simply connected 4-manifold W with an $r \times r$ non-singular intersection matrix whose diagonal entries are all 0. Since the Rochlin invariant of Σ is non-trivial, it follows that the signature of W is an odd multiple of 8 and r is even. Hence $r \geqq 8$. If $r = 8$, then the intersection matrix is a positive or negative definite matrix which is not in our case. Thus, we have $r \geqq 10$.] Since $\chi(3_1, 1) = \Sigma$ for the positive trefoil knot 3_1, an answer to the following question on Kirby calculus (see [8, 11,12]) will help in understanding the link $\alpha(\Sigma)$:

Question. How is the canonical order Ω_c understood among the colored links ?

Examining Steps 1-4, we have the following corollary to the classification program:

Corollary 5.10. The classification problem on $\mathbb{M}$ is solved assuming inductive partial solutions of the homeomorphism problem on $\mathbb{M}$, the decision problem on primeness of links and the isomorphism problem on $\mathbb{G}$.

6. Notes on the oriented version and oriented 3-manifold invariants.

Let $\vec{\mathbb{M}}$ be the set of closed connected oriented 3-manifolds. Combining the map $\tilde{\sigma} : \vec{\mathbb{L}} \to \mathbb{X}/\sim$ with the embedding $\vec{\Omega} : \mathbb{X}/\sim \to \mathbb{X}$ sending the class $\langle \mathbf{x} \rangle$ to the initial element of $\langle \mathbf{x} \rangle$, we have a map

$$\vec{\sigma} : \vec{\mathbb{L}} \longrightarrow \mathbb{X}$$

which is injective modulo split additions of trivial links. Using this map $\vec{\sigma}$, we have a well-order in $\vec{\mathbb{L}}$ induced from a well-order Ω in $\mathbb{X}$ by a method similar to the unoriented version. This well-order in $\vec{\mathbb{L}}$ is also denoted by Ω. Writing

$$\vec{\mathbb{L}}^{\mathrm{m}} = \iota^{-1}\mathbb{L}^{\mathrm{m}} \subset \vec{\mathbb{L}},$$

we see that the embedding $\alpha : \mathbb{M} \to \mathbb{L}$ in Theorem 1.1 lifts to an embedding

$$\vec{\alpha} : \vec{\mathbb{M}} \longrightarrow \vec{\mathbb{L}}$$

such that $\chi_0\vec{\alpha} = 1$ and $\vec{\alpha}(-M) = -\overline{\vec{\alpha}(M)}$ for every $M \in \vec{\mathbb{M}}$, where the map $\chi_0 : \vec{\mathbb{L}} \to \vec{\mathbb{M}}$ denotes a natural lift of the map $\chi_0 : \mathbb{L} \to \mathbb{M}$. In fact, for any $M \in \vec{\mathbb{M}}$, the link $L_0 = \mathrm{cl}\beta\sigma_\alpha(M)$ is canonically oriented and we have $\chi(L_0, 0) = \pm M$, where $-M$ denotes the same M but with the orientation reversed. If $M = -M$, then we define $\vec{\alpha}(M) = L_0$. If $M \neq -M$, then we define $\vec{\alpha}(M)$ so as to satisfy

$$\{\vec{\alpha}(M), \vec{\alpha}(-M)\} = \{L_0, -\bar{L}_0\}$$

and $\chi(\vec{\alpha}(M), 0) = M$ as desired. As a related question, it would be interesting to know *whether or not there is an oriented link $L \in \vec{\mathbb{L}}$ with* $L = -\bar{L}$ and $\chi(L, 0) = M$ *for every $M \in \vec{\mathbb{M}}$ with $M = -M$.*

For an algebraic system Λ, an *oriented 3-manifold invariant* in Λ is a map $\vec{\mathbb{M}} \to \Lambda$ and *an oriented link invariant* in Λ is a map $\vec{\mathbb{L}} \to \Lambda$. Let $\mathrm{Inv}(\vec{\mathbb{M}}, \Lambda)$ and $\mathrm{Inv}(\vec{\mathbb{L}}, \Lambda)$ be the sets of oriented 3-manifold invariants and oriented link invariants in Λ, respectively. Then we have $\chi_0\vec{\alpha} = 1$. We consider the following sequence

$$\mathrm{Inv}(\vec{\mathbb{M}}, \Lambda) \xrightarrow{\chi_0^{\#}} \mathrm{Inv}(\vec{\mathbb{L}}, \Lambda) \xrightarrow{\vec{\alpha}^{\#}} \mathrm{Inv}(\vec{\mathbb{M}}, \Lambda)$$

of the dual maps $\vec{\alpha}^{\#}$ and $\chi_0^{\#}$ of $\vec{\alpha}$ and χ_0. Since the composite $\vec{\alpha}^{\#}\chi_0^{\#} = 1$, we see that $\chi_0^{\#}$ is injective and $\vec{\alpha}^{\#}$ is surjective, both of which imply that *every oriented 3-manifold invariant can be obtained from an oriented link invariant.* More precisely, if I is an oriented 3-manifold invariant, then $\chi_0^{\#}(I)$ is an oriented link invariant. Conversely, if J is an oriented link invariant, then $\vec{\alpha}^{\#}(J)$ is an oriented 3-manifold invariant and every oriented 3-manifold invarint is obtained in this way. Here are two examples creating an oriented 3-manifold invariant from an oriented link invariant when we use the right inverse $\vec{\alpha}$ of χ_0, defined by the canonical order Ω_c.

Example 6.1. Let $\lambda \in \mathrm{Inv}(\vec{\mathbb{L}}, \mathbb{Z})$ be the signature invariant $\mathrm{sign}(V + V')$, and $P \in \mathrm{Inv}(\vec{\mathbb{L}}, \mathbb{Z}[t, t^{-1}])$ the one variable Alexander polynomial $\det(tV - V')$ (an invariant up to units $\pm t^m$, $m \in \mathbb{Z}$) of an oriented link, where V denotes a Seifert matrix associated with a connected Seifert surface of the link (see [9]). For the right inverse $\vec{\alpha}$ of $\vec{\chi}$, defined by the canonical order Ω_c, we have the oriented 3-manifold invariants

$$\lambda_{\vec{\alpha}} = \vec{\alpha}^{\#}(\lambda) \in \mathrm{Inv}(\vec{\mathbb{M}}, \mathbb{Z}) \quad \text{and} \quad P_{\vec{\alpha}} = \vec{\alpha}^{\#}(P) \in \mathrm{Inv}(\vec{\mathbb{M}}, \mathbb{Z}).$$

For some 3-manifolds, these invariants are calculated as follows:

(6.1.1) $\lambda_{\vec{\alpha}}(S^1 \times S^2) = 0$, $P_{\vec{\alpha}}(S^1 \times S^2) = 1$.
(6.1.2) $\lambda_{\vec{\alpha}}(S^3) = -1$, $P_{\vec{\alpha}}(S^3) = t - 1$.
(6.1.3) $\lambda_{\vec{\alpha}}(\pm Q) = \mp 3$, $P_{\vec{\alpha}}(\pm Q) = (t - 1)(t^2 + 1)$ (we note that $Q \neq -Q$).
(6.1.4) $\lambda_{\vec{\alpha}}(P^3) = -4$, $P_{\vec{\alpha}}(P^3) = (t - 1)^2$.
(6.1.5) $\lambda_{\vec{\alpha}}(T^3) = 0$, $P_{\vec{\alpha}}(T^3) = (t - 1)^4$.

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