

# HAMILTONIAN STABILITY OF PARALLEL LAGRANGIAN SUBMANIFOLDS IN COMPLEX SPACE FORMS (A SURVEY)

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## INTRODUCTION

Let  $(M, \omega)$  be a  $2n$ -dimensional symplectic manifold with a symplectic form  $\omega$ . An immersion  $\varphi : L \rightarrow M$  of an  $n$ -dimensional smooth manifold is called an *Lagrangian immersion* if the 2-form  $\varphi^*\omega$  on  $L$  pulled back by the immersion  $\varphi : L \rightarrow M$  vanishes. Then  $L$  is called a *Lagrangian submanifold* immersed in a symplectic manifold  $M$ .

We say that a compact Lagrangian submanifold immersed in a Kähler manifold  $M$  is a *H-minimal* Lagrangian submanifold if it has extremal volume under all Hamiltonian variations of the Lagrangian immersion. Note that if a Lagrangian submanifold is a minimal submanifold in the usual sense, then it is H-minimal. A compact H-minimal Lagrangian submanifold in a Kähler manifold  $M$  is called *Hamiltonian stable* if the second variation for the volume is nonnegative for all Hamiltonian deformations of the Lagrangian immersion.

Y. G. Oh (1990-94) provided the fundamental theory for Hamiltonian stability of H-minimal Lagrangian submanifolds of Kähler manifolds. We know that every compact minimal Lagrangian submanifold  $L$  in an Einstein-Kähler manifold  $M$  with nonpositive Ricci curvature is stable (B. Y. Chen(1981), Oh(1990)) and hence Hamiltonian stable. It is an interesting problem to study Hamiltonian stability of compact minimal Lagrangian submanifolds  $L$  in an Einstein-Kähler manifold  $M$  with positive Ricci curvature.

### 1. MINIMAL LAGRANGIAN SUBMANIFOLDS IN $CP^n$

It is known that the real projective subspaces  $RP^n$  and the Clifford tori  $T^n$  are Hamiltonian stable compact minimal Lagrangian submanifolds embedded in  $CP^n$  ([9]).

Let  $B$  denote the second fundamental form of a Lagrangian submanifold  $L$  in a Kähler manifold  $M$  with Kähler metric  $g$  and complex structure  $J$ . Then

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<sup>1</sup>In Report of the Fukuoka University Geometry meeting celebrating the sixtieth birthday of Professor Yoshihiko Suyama, "Geometry and Something" 2005.10.7-10, Fukuoka

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we define a symmetric tensor field  $S$  on  $L$  of degree 3 by

$$S(X, Y, Z) := g(JB(X, Y), Z)$$

for each tangent vector  $X, Y, Z$  on  $L$ . If  $S$  is parallel with respect to the Levi-Civita connection  $\nabla$  of  $L$ , namely,  $S$  satisfies the equation  $\nabla S = 0$ , then we say that  $L$  has *parallel second fundamental form*, or a Lagrangian submanifold  $L$  in a Kähler manifold  $M$  is called a *parallel Lagrangian submanifold* of  $M$ . The parallel Lagrangian submanifolds in complex space forms were classified by H. Naitoh and M. Takeuchi.

Urbano (1993) and S. Chang (2000) showed that an Hamiltonian stable compact minimal Lagrangian torus in  $\mathbf{C}P^2$  is only the Clifford tori  $T^2$ . The Clifford tori  $T^2$  is a parallel Lagrangian submanifold in  $\mathbf{C}P^2$ .

We gave a wider class of Hamiltonian stable compact minimal Lagrangian submanifolds embedded in  $\mathbf{C}P^n$  as follows:

**Theorem** ([3]). *Let  $L$  be a compact  $n$ -dimensional totally real minimal submanifold embedded in  $\mathbf{C}P^n$  with the parallel second fundamental form in the following list :*

- (1)  $SU(p)/\mathbf{Z}_p$ ,  $n = p^2 - 1$ .
- (2)  $SU(p)/SO(p)\mathbf{Z}_p$ ,  $n = \frac{(p-1)(p+2)}{2}$ .
- (3)  $SU(2p)/Sp(p)\mathbf{Z}_{2p}$ ,  $n = (p-1)(2p+1)$ .
- (4)  $E_6/F_4\mathbf{Z}_3$ ,  $n = 26$ .

*Then  $L$  is a compact Hamiltonian stable minimal Lagrangian submanifold embedded in  $\mathbf{C}P^n$ .*

## 2. REAL FORMS OF HERMITIAN SYMMETRIC SPACES

Here we mention about Hamiltonian stability of all real forms embedded in compact irreducible Hermitian symmetric spaces, using M. Takeuchi's results ([13],[9],[3]). Real forms are totally geodesic Lagrangian submanifolds.

Here we give a complete list of Hamiltonian stability of all irreducible symmetric  $R$ -spaces of non-Hermitian type which are canonically embedded in Hermitian symmetric spaces. (There were a few of careless mistakes in the list of the paper [3]. The following is a correct table.)

$M$	$L$	Einstein	$\mu_1$	H-stable	stable
$G_{p,q}(\mathbf{C}), q \geq p \geq 1$	$G_{p,q}(\mathbf{R})$	Yes	$\frac{1}{2}$	Yes	No
$G_{2p,2q}(\mathbf{C}), q \geq p \geq 1$	$G_{p,q}(\mathbf{H})$	Yes	$\frac{1}{2}$	Yes	Yes
$G_{m,m}(\mathbf{C}), m \geq 2$	$U(m)$	No	$\frac{1}{2}$	Yes	No
$\frac{SO(2m)}{U(m)}$	$SO(m), m \geq 5$	Yes	$\frac{1}{2}$	Yes	No
$\frac{SO(4m)}{U(2m)}, m \geq 3$	$\frac{U(2m)}{Sp(m)}$	No	$\frac{m}{4m-2}$	No	No
$\frac{Sp(2m)}{U(2m)}$	$Sp(m), m \geq 2$	Yes	$\frac{1}{2}$	Yes	Yes
$\frac{Sp(m)}{U(m)}, m \geq 3$	$\frac{U(m)}{O(m)}$	No	$\frac{1}{2}$	Yes	No
$Q_{p+q-2}(\mathbf{C}), q-p \geq 3$	$Q_{p,q}(\mathbf{R}), p \geq 2$	No	$\frac{p}{p+q-2}$	No	No
$Q_{2p-2}(\mathbf{C})$	$Q_{p,p}(\mathbf{R}), p \geq 2$	Yes	$\frac{1}{2}$	Yes	No
$Q_{2p+k-2}(\mathbf{C}), k=1,2$	$Q_{p,p+k}(\mathbf{R}), p \geq 2$	No	$\frac{1}{2}$	Yes	No
$Q_{q-1}(\mathbf{C}), q \geq 4$	$Q_{1,q}(\mathbf{R})$	Yes	$\frac{1}{2}$	Yes	Yes(!)
$E_6/T \cdot Spin(10)$	$P_2(\mathbf{K})$	Yes	$\frac{1}{2}$	Yes	Yes
$E_6/T \cdot Spin(10)$	$G_{2,2}(\mathbf{H})/\mathbf{Z}_2$	Yes	$\frac{1}{2}$	Yes	No
$E_7/T \cdot E_6$	$SU(8)/Sp(4)\mathbf{Z}_2$	Yes	$\frac{1}{2}$	Yes	No
$E_7/T \cdot E_6$	$T \cdot E_6/F_4$	No	$\frac{1}{6}$	No	No

Here  $G_{p,q}(\mathbf{F})$  : Grassmanian manifold of all  $p$ -dimensional subspaces of  $\mathbf{F}^{p+q}$ , for each  $\mathbf{F} = \mathbf{R}, \mathbf{C}, \mathbf{H}$ ,

$\mathbf{P}_2(K)$  : Cayley projective plane,

$Q_n(\mathbf{C})$  : complex quadric of dimension  $n$ .

Note that the heading of the third column indicates whether  $L$  is Einstein or not.

### 3. PARALLEL LAGRANGIAN SUBMANIFOLDS IN COMPLEX SPACE FORMS

In [4] we have discussed about Hamiltonian stability of symmetric  $R$ -spaces which can be embedded in complex Euclidean spaces as  $H$ -minimal Lagrangian submanifolds (NEVER minimal !). It is known that each symmetric  $R$ -space

of  $U(r)$  type can be embedded in the complex Euclidean space as a Lagrangian submanifold by the standard embedding of an  $R$ -space. Since such standard embedding has parallel second fundamental form, it gives a model of a compact  $H$ -minimal Lagrangian submanifold embedded in the complex Euclidean space. The following is a list of all irreducible symmetric  $R$ -spaces of  $U(r)$  type

$$Q_{2,q}(\mathbf{R}), U(p), U(p)/O(p), U(2p)/Sp(p), T \cdot E_6/F_4.$$

**Theorem** ([4]). *Let  $L$  be an irreducible symmetric  $R$ -spaces of  $U(r)$  type standardly embedded in  $\mathbf{C}^m$ . Then  $L$  is a Hamiltonian stable compact  $H$ -minimal Lagrangian submanifold embedded in  $\mathbf{C}^m$ .*

Using this result and the classification theory of totally real parallel submanifolds in complex space forms, we obtain

**Theorem** ([5]). *Any compact Lagrangian submanifold embedded in complex space forms with parallel second fundamental form is Hamiltonian stable.*

#### 4. A HAMILTONIAN STABLE NON-PARALLEL MINIMAL LAGRANGIAN SUBMANIFOLD IN $\mathbf{C}P^3$

On the other hand, there exists a 3-dimensional Hamiltonian stable compact homogeneous minimal Lagrangian submanifold in  $\mathbf{C}P^3$ , which does never have parallel second fundamental form ([7]).

Let  $V^3$  be the complex vector space of all complex homogeneous polynomials with two variables  $z_1, z_2$  of degree 3. We equip the standard Hermitian inner product such that

$$\{v_k = \frac{1}{\sqrt{k!(3-k)!}} z_1^{3-k} z_2^k \mid k = 0, 1, 2, 3\}$$

is a unitary basis of  $V^3$ , and  $V^3 \cong \mathbf{C}^4 \cong \mathbf{R}^8$ . We know that  $V^3$  is an irreducible unitary representation of  $SU(2)$ . Then the orbit  $\rho_3(SU(2)) \frac{1}{\sqrt{2}}(v_0 + v_3) \subset S^7(1)$  is a 3-dimensional minimal Legendrian submanifold embedded in  $S^7(1)$ . Let  $\pi : S^7(1) \rightarrow \mathbf{C}P^3$  be the Hopf fibration.

**Theorem** ([7]). *The orbit  $\pi(\rho_3(SU(2))(\frac{1}{\sqrt{2}}(v_0 + v_3))) \subset \mathbf{C}P^3$  is a 3-dimensional Hamiltonian stable compact homogeneous minimal Lagrangian submanifold embedded in  $\mathbf{C}P^3$ , which does neither have parallel second fundamental form nor have constant sectional curvature.*

Palmer ([11], [12]) showed that the Gauss maps of compact oriented minimal surfaces  $N$  in the sphere  $S^3(1)$  and isoparametric hypersurfaces  $N$  in the sphere  $S^{n+1}(1)$  provide compact minimal Lagrangian submanifolds immersed in the complex hyperquadrics  $Q_n(\mathbf{C}) \cong \tilde{G}_2(\mathbf{R}^{n+2}) = SO(n+2)/SO(2) \times SO(n)$  and

they are not Hamiltonian stable if  $N$  is not a great or small sphere. Suppose that  $N$  is an  $n$ -dimensional compact isoparametric hypersurface embedded in  $S^{n+1}(1)$  with  $g$  distinct principal curvatures. Since  $N$  is orientable, we can take a unit normal vector field  $\mathbf{n}$  on  $N$ . The Gauss map  $\mathcal{G} : N \rightarrow Q_n(\mathbf{C}) = \tilde{G}_2(\mathbf{R}^{n+2})$  is defined by

$$\mathcal{G}(\mathbf{x}) := [\mathbf{x} + \sqrt{-1}\mathbf{n}]$$

=subspace spanned by  $\mathbf{x}$  and  $\mathbf{n}$  with the orientation

for each position vector  $\mathbf{x} \in N$ . Then  $\mathcal{G}(N) = N/\mathbf{Z}_g$  is a compact minimal Lagrangian submanifold embedded in  $Q_n(\mathbf{C})$ .

In the case  $g = 1$ ,  $\mathcal{G}(N) = N/\mathbf{Z}_g$  coincides with a real form  $Q_{1,n+1}(\mathbf{R})$  of  $Q_n(\mathbf{C})$  in the above table.

In the case  $g = 2$ ,  $\mathcal{G}(N) = N/\mathbf{Z}_g$  coincides with a real form  $Q_{p,q}(\mathbf{R})$  of  $Q_n(\mathbf{C})$  in the above table. Here  $N$  is the Clifford hypersurface  $S^{p-1} \times S^{q-1}$  for some  $p, q \geq 2$ .

Their Hamiltonian stabilities are already determined as in the above table.

We consider the case  $g = 3$ . The first simplest case is the case when  $N$  is the Cartan hypersurface  $N = SO(3)/\mathbf{Z}_2 \oplus \mathbf{Z}_2$  with  $n = 3$ .

In a joint work with Ma Hui (in progress), we obtain

**Theorem** ([6]). *If  $N$  is the Cartan hypersurface  $N = SO(3)/\mathbf{Z}_2 \oplus \mathbf{Z}_2$  with  $n = 3$ , then  $\mathcal{G}(N) = N/\mathbf{Z}_3$  is a 3-dimensional Hamiltonian stable compact minimal Lagrangian submanifold embedded in  $Q_3(\mathbf{C})$ , which has constant positive sectional curvature  $\frac{1}{24}$ .*

Here we equip  $Q_3(\mathbf{C})$  with the Hermitian symmetric metric of Einstein constant  $\frac{1}{2}$ . We remark that  $\mathcal{G}(N) = N/\mathbf{Z}_3$  is never totally geodesic and does never have parallel second fundamental form.

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