STABILITY AND RIGIDITY OF SPECIAL LAGRANGIAN CONES OVER CERTAIN MINIMAL LEGENDRIAN ORBITS

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Dedicated to Professor Katsuei Kenmotsu on his retirement from Tohoku University

Abstract. Special Lagrangian cones in complex Euclidean spaces are obtained as cones over compact minimal Legendrian submanifolds in the odd dimensional standard hypersphere. The notion of the stability, the Legendrian stability and the rigidity of special Lagrangian cones were recently introduced and investigated by D. Joyce, M. Haskins etc. In this paper we determine explicitly the stability-index, the Legendrian-index, and the rigidity of special Lagrangian cones over compact irreducible symmetric spaces of type $A$ obtained as minimal Legendrian orbits and over a minimal Legendrian $SU(2)$-orbit. We obtain the examples of stable and rigid special Lagrangian cones in higher dimensions. Moreover we discuss a relationship of these properties with the Hamiltonian stability of minimal Lagrangian submanifolds in complex projective spaces.

Introduction

A special Lagrangian submanifold in a Ricci-flat Kähler manifold, a so-called Calabi-Yau manifold, has two aspects of a Lagrangian submanifold in symplectic geometry and a calibrated submanifold in Riemannian geometry. A calibrated submanifold is a minimal submanifold in the sense that the mean curvature vector field vanishes, and more strongly it is a real homologically volume minimizing submanifold.

Recently D. Joyce provided the profound theory on special Lagrangian submanifolds with isolated conical singularities in (almost) Calabi-Yau manifolds and their deformations, moduli spaces in a series of his papers. His work emphasizes so much the importance of investigation of special Lagrangian cones in complex Euclidean spaces.

The notion of the stability-index, the stability and the rigidity of special Lagrangian cones were introduced by D. Joyce. They are closely related to the deformation of special Lagrangian submanifolds with isolated conical singularities and the regularity of special Lagrangian integral currents. A special Lagrangian cone is obtained as a cone over a compact minimal Legendrian submanifold in the odd dimensional standard sphere. By the Hopf fibration

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a minimal Legendrian submanifold can be locally projected to a minimal Lagrangian submanifold in the complex projective space.

The most fundamental and typical examples are special Lagrangian cones $C^m_{HL}$ over minimal Legendrian orbits of the maximal torus $T^{m-1}$ of the special unitary group $SU(m)$ given by Harvey and Lawson ([8]). M. Haskins showed that a stable special Lagrangian cone in $\mathbb{C}^3$ over a compact minimal Legendrian surface of genus 1 in $S^5$ is only $C^3_{HL}$ ([7]). The further research on stable special Lagrangian cones in higher dimensions and the stability-index of higher dimensional homogeneous examples are suggested in the paper [7, p62].

Now we assume that $\Sigma$ is one of compact irreducible symmetric spaces standardly embedded in the odd dimensional standard sphere $S^{2m-1}(1)$ as minimal Legendrian submanifolds in the standard way (see Section 2):

$$\Sigma = SU(p), SU(p)/SO(p), SU(2p)/Sp(p) \ (p \geq 3), \text{ or } E_6/F_4.$$ 

Note that the rank of these symmetric spaces is equal to $p-1$ and the rank of $E_6/F_4$ is equal to 2. Let $C\Sigma$ be the special Lagrangian cone in $\mathbb{C}^m$ over $\Sigma$. Then we shall show the following.

**Theorem.** (1) $C\Sigma$ are all RIGID.

(2) If $\Sigma = SU(3), SU(3)/SO(3), SU(6)/Sp(3) \ (p = 3), E_6/F_4$, then $C\Sigma$ is STABLE, and hence Legendrian stable.

(3) If $\Sigma = SU(p), SU(p)/SO(p), SU(2p)/Sp(p), p \geq 4$, $C\Sigma$ is NOT stable, in fact NOT Legendrian stable.

The properties of these minimal Legendrian submanifolds will be discussed in detail and their stability-indices will be determined explicitly. In the last section of this paper we shall discuss such properties of a special Lagrangian cone over a minimal Legendrian $SU(2)$-orbit in $\mathbb{C}^4$.

The results in this paper were partially announced in [17]. In November 2004, Mark Haskins has visited Kyushu University and Tokyo Metropolitan University. The author could have nice discussion with him about this subject there. The author would like to thank Mark Haskins for his valuable suggestion of a problem on the existence of stable special Lagrangian cones in higher dimensions.

### 1. Special Lagrangian cones and their stability-indices

In this section we shall describe some fundamental definitions and properties which are necessary in the later sections (cf. [6],[7], [10],[11],[12]).

#### 1.1. Special Lagrangian submanifolds of Calabi-Yau manifolds.

In complex Euclidean space $\mathbb{C}^m \cong \mathbb{R}^{2m}$, we recall the notion of special Lagrangian submanifolds. The natural group action of $SU(m) \subset U(m)$ preserves the standard Kähler form (symplectic form) defined as

$$\omega := \sqrt{-1} \sum_{i=1}^{m} dz^i \wedge d\bar{z}^i$$
and the standard complex volume form defined by
\[ \Omega := dz^1 \wedge \cdots \wedge dz^m. \]

We decompose \( \Omega \) into real and imaginary parts by
\[ \Omega = \Re(\Omega) + \sqrt{-1} \Im(\Omega). \]
Then \( \Re(\Omega) \) and \( \Im(\Omega) \) are parallel real \( n \)-forms on \( \mathbb{C}^m \).

The calibrated submanifolds by \( \Re(\Omega) \) are characterized by the condition that the restrictions of \( \omega \) and \( \Im(\Omega) \) to the submanifold vanish. The special Lagrangian submanifold in \( \mathbb{C}^m \) is defined as such a submanifold Harvey and Lawson showed that a minimal Lagrangian submanifold in \( \mathbb{C}^m \) is a special Lagrangian submanifold.

In general, suppose that \((M, g)\) is a Riemannian manifold with holonomy group contained in \( SU(m) \), and such a Riemannian manifold becomes a Calabi-Yau Kähler manifold of complex dimension \( m \). Then the parallel Kähler form \( \omega \) and the parallel complex volume form \( \Omega \) are defined on the whole \( M \), and \( \Re(\Omega) \) defines a calibration on \( M \). The calibrated submanifolds with respect to \( \Re(\Omega) \) are characterized by the condition that the pull-backs of \( \omega \) and \( \Im(\Omega) \) to the submanifold vanish. An \( m \)-dimensional submanifold \( X \) in a Calabi-Yau manifold is called a special Lagrangian submanifold if the pull-backs of both \( \omega \) and \( \Im(\Omega) \) to \( X \) vanish.

For each constant \( \theta \in \mathbb{R} \), we also can consider a calibration defined by \( \Re(e^{\sqrt{-1}\theta}\Omega) \) and its corresponding calibrated submanifolds. We also call such a calibrated submanifold a special Lagrangian submanifolds (with respect to \( \Re(e^{\sqrt{-1}\theta}\Omega) \)) if the pull-backs of both \( \omega \) and \( \Im(e^{\sqrt{-1}\theta}\Omega) \) to \( X \) vanish. Let \( X \) be a Lagrangian submanifold immersed in a Calabi-Yau manifold \( M \). Then we know that \( X \) is a minimal submanifold in \( M \) if and only if \( X \) is a special Lagrangian submanifold with respect to the calibration \( \Re(e^{\sqrt{-1}\theta}\Omega) \) for some \( \theta \in \mathbb{R} \).

1.2. Special Lagrangian cones. Let \( S^{2m-1}(1) \) denote the unit standard hypersphere of \( \mathbb{C}^m \). Let \( \Sigma \) be an \((m - 1)\)-dimensional smooth submanifold immersed in \( S^{2m-1}(1) \) defined by an immersion \( \varphi : \Sigma \rightarrow S^{2m-1}(1) \). The cone \( C = C\Sigma \) over \( \Sigma \) in \( \mathbb{C}^m \) is defined by an immersion
\[ \Phi : \Sigma \times [0, \infty) \ni (\sigma, t) \mapsto t\varphi(\sigma) \in \mathbb{C}^m. \]
Then \( C \) has an isolated singularity at the origin 0 and \( C' := C \setminus \{0\} \) is an \( m \)-dimensional smooth submanifold immersed in \( \mathbb{C}^m \) defined by the immersion
\[ \Phi' : \Sigma \times (0, \infty) \ni (\sigma, t) \mapsto t\varphi(\sigma) \in \mathbb{C}^m. \]
Let \( \pi : S^{2m-1}(1) \rightarrow \mathbb{C}P^{m-1} \) be the Hopf fibration, which is a Riemannian submersion onto the \((m - 1)\)-dimensional complex projective space \( \mathbb{C}P^{m-1} \) of constant holomorphic sectional curvature 4. Then \( C\Sigma \) is a Lagrangian cone with an isolated singularity at 0 if and only if \( \Sigma \) is a Legendrian submanifold in \( S^{2m-1}(1) \) with the standard contact structure, and then the immersion \( \pi \circ \varphi : \Sigma \rightarrow S^{2m-1}(1) \) defines a Lagrangian submanifold immersed in \( \mathbb{C}P^{m-1} \).
Moreover since the mean curvature vectors of these submanifolds correspond each other, we know the following fundamental fact (cf. [7]).

**Proposition 1.1.** The following three conditions on local properties of these submanifolds are equivalent each other:

(a) $C^\Sigma$ is a special Lagrangian cone in $\mathbb{C}^m$.

(b) $\Sigma$ is a minimal Legendrian submanifold in $S^{2m-1}(1)$ with respect to its standard contact structure.

(c) $\pi(\Sigma)$ is a minimal Lagrangian submanifold in $\mathbb{C}P^m$.

**Example 1.1.** In Harvey-Lawson [8] the following example of a special Lagrangian cone in $\mathbb{C}^m$ was given as

$$C^m_{HL} := \{(z_1, \cdots, z_m) \in \mathbb{C}^m \mid (\sqrt{-1})^{m+1}z_1 \cdots z_m \in \mathbb{R}, |z_1| = \cdots = |z_m|\}.$$  

Then $\Sigma^m_{HL} := C^m_{HL} \cap S^{2m-1}(1) \subset S^{2m-1}(1)$ is a minimal Legendrian orbit of the maximal torus of $SU(m)$, which is isometric to an $(m-1)$-dimensional flat torus $T^{m-1}$.

Let $\Delta$ and $\Delta^\Sigma$ be the Laplacians of $(C', g)$ and $(\Sigma, g^\Sigma)$ on functions, respectively. A function $u$ on $C'$ is called a *homogeneous function of order* $\alpha$ on $C'$ if $u$ satisfies $u \circ t = t^\alpha u$ for each $t > 0$. Then such a function can be expressed as $u(r\sigma) = r^\alpha v(\sigma)$ for some function $v$ on $\Sigma$. The relationship between $\Delta$ and $\Delta^\Sigma$ is given by the formula

$$\Delta u(r\sigma) = r^{\alpha-2}(\Delta^\Sigma v(\sigma) - \alpha(\alpha + m - 2)v(\sigma)).$$

(1.1) Hence we see that $u$ is harmonic if and only if $v$ is an eigenfunction on $\Sigma$ with eigenvalue $\alpha(\alpha + m - 2)$.

Assume that $m > 2$. Set

$$D_\Sigma := \{\alpha \in \mathbb{R} \mid \alpha(\alpha + m - 2) \text{ is an eigenvalue of } \Delta^\Sigma\},$$

which is a countable and discrete subset of $\mathbb{R}$. For each $\alpha \in D_\Sigma$, we denote by $m_\Sigma(\alpha)$ the multiplicity for eigenvalue $\alpha(\alpha + m - 2)$ of $\Delta^\Sigma$, which is equal to the dimension of vector space of all homogeneous harmonic functions of order $\alpha$ on $C'$. Then we define a monotone increasing, upper semi-continuous function $N_\Sigma: \mathbb{R} \rightarrow \mathbb{Z}$ as

$$N_\Sigma(\delta) := - \sum_{\alpha \in D_\Sigma \cap (\delta, 0)} m_\Sigma(\alpha)$$

if $\delta < 0$ and

$$N_\Sigma(\delta) := \sum_{\alpha \in D_\Sigma \cap [0, \delta]} m_\Sigma(\alpha)$$

if $\delta \geq 0$.

**Definition 1.1.** The *stability-index* of a special Lagrangian cone $C$ is defined by

$$s\text{-ind}(C) := N_\Sigma(2) - b^0(\Sigma) - m^2 - 2m + 1 + \text{dim } G^\Sigma.$$  

(1.2)
where $b^0(\Sigma)$ denotes the 0-th Betti number of $\Sigma$, i.e. the number of connected components of $\Sigma$ and $G_\Sigma$ denotes a maximal compact subgroup of $SU(m)$ preserving the special Lagrangian cone $C$, or equivalently the minimal Legendrian submanifold $\Sigma$.

Note that $m_\Sigma(0) = b^0(\Sigma)$, $m_\Sigma(1) \geq 2m$ if $\Sigma$ is not totally geodesic, $m_\Sigma(2) \geq m^2 - 1 - \dim G_\Sigma$. Since $N_\Sigma(2) \geq m_\Sigma(0) + m_\Sigma(1) + m_\Sigma(2)$, we have $s\text{-ind}(C) \geq 0$ if $\Sigma$ is not totally geodesic. If $\Sigma$ is totally geodesic, then $s\text{-ind}(C) = -m$.

A special Lagrangian cone $C$ is called **stable** if $s\text{-ind}(C) = 0$. A special Lagrangian cone $C$ is called **rigid** if $m_\Sigma(2) = m^2 - 1 - \dim G_\Sigma$. We see that a special Lagrangian cone $C$ is stable if and only if the following three conditions are satisfied

1. $N_\Sigma(2) = m_\Sigma(0) + m_\Sigma(1) + m_\Sigma(2)$,
2. $m_\Sigma(1) = 2m$,
3. $m_\Sigma(2) = m^2 - 1 - \dim G_\Sigma$.

The **Legendrian-index** of a special Lagrangian cone $C$ ([7]) is defined by

$$l\text{-ind}(C) := \sum_{\alpha \in D_\Sigma \cap (0,2)} m_\Sigma(\alpha).$$

A special Lagrangian cone $C$ is **Legendrian-stable** ([7]) if $l\text{-ind}(C) = 2m$. A special Lagrangian cone $C$ is Legendrian-stable if and only if $N_\Sigma(2) = m_\Sigma(0) + m_\Sigma(1) + m_\Sigma(2)$ and $m_\Sigma(1) = 2m$. By the definitions $C$ is stable if and only if $C$ is rigid and Legendrian-stable.

Here we shall mention a relationship of the stability of special Lagrangian cones with the Hamiltonian stability of minimal Lagrangian submanifolds in complex projective spaces (cf. [1]).

Assume that $\psi : L \longrightarrow \mathbb{C}P^{m-1}$ is a minimal Lagrangian immersion of an $(m - 1)$-dimensional connected compact smooth manifold $L$ into a complex projective space. Since the pull-back $S^1$-bundle $\psi^{-1}\pi : \psi^{-1}S^{2m-1}(1) \rightarrow L$ is flat, there is a connected integral manifold $\Sigma$ of the horizontal distribution on $\psi^{-1}S^{2m-1}(1)$, and hence it gives a minimal Legendrian immersion $\varphi : \Sigma \rightarrow S^{2m-1}(1)$ and a covering map $\psi^{-1}\pi : \Sigma \rightarrow L$. We denote by $\rho : \pi_1(L) \rightarrow S^1$ the holonomy homomorphism of the flat $S^1$-bundle $\psi^{-1}S^{2m-1}(1)$ over $L$. Then the following holds.

**Proposition 1.2.** Suppose that $\rho$ is nontrivial. If the special Lagrangian cone $C\Sigma$ over $\Sigma$ in $\mathbb{C}^m$ is stable, then a minimal Lagrangian submanifold $L$ in $\mathbb{C}P^{m-1}$ is Hamiltonian stable.

**Proof.** We may assume that $\varphi$ is not totally geodesic. For each $v \in \mathbb{C}^m$, we define a smooth function $f_v$ on $\Sigma$ by

$$(f_v)(x) := \langle \varphi(x), v \rangle \quad (x \in \Sigma).$$

Let $\rho : \pi_1(\Sigma) \rightarrow S^1$ be the holonomy homomorphism of the pull-back $S^1$-bundle from the Hopf $S^1$-bundle $\pi : S^{2m-1}(1) \rightarrow \mathbb{C}P^{m-1}$ by the Lagrangian immersion $\psi$. Here $S^1$ is considered as the center of the unitary group $U(m)$. 5
Set $\Gamma := \rho(\pi_1(\Sigma))$, which is a finite subgroup of $S^1$. Let $\Gamma$ be the deck transformation group of the covering map $\psi^{-1}: \Sigma \to L$. Suppose that there is a vector $v \in \mathbb{C}^m$ such that $f_v(xc) = f_v(x)$ for each $c \in \Gamma$ and each $x \in \Sigma$. Since $\varphi(xc) = \varphi(x)\rho(c)$, we have $\langle \varphi(x)a, v \rangle = \langle \varphi(x), v \rangle$ for each $a \in \rho(\Gamma)$ and each $x \in \Sigma$. By the non-triviality of $\Gamma$, there is $a \in \Gamma$ with $a \neq 1$. Since $\langle \varphi(x), va^{-1} - v \rangle = 0$ for all $x \in \Sigma$, by the fullness of $\varphi$ we have $va^{-1} = v$. As $a \neq 1$, $v$ must be zero and thus $f_v = 0$. Hence by the assumption on the stability we conclude that $\Sigma$ has no nonzero eigenvalue smaller than $2m$. Therefore $\Sigma$ is Hamiltonian stable.

It can happen that $L$ becomes Hamiltonian stable even if $C\Sigma$ is not stable. Such examples will be shown in the later sections.

1.3. Special Lagrangian submanifolds with isolated conical singularities. Here we mention the results of Joyce on the deformation of a compact special Lagrangian submanifold $X$ with isolated conical singularities or the local structure of moduli spaces around $X$, and the regularity of special Lagrangian varieties, which are described in terms of the stability-index and the rigidity of special Lagrangian cones.

Let $M$ be the moduli space of compact special Lagrangian submanifolds with isolated conical singularities embedded in $M$. McLean [14] showed that if $X \in M$ is smooth (i.e. without singularities), then the moduli space $M$ is a smooth manifold of dimension $b^1(X)$ around $X$.

Joyce [11] showed that if $X$ is a special Lagrangian submanifold with isolated conical singularities $C_1, \cdots, C_k$, then the dimension of the obstruction space $O_X$ of $X$ is equal to the sum of stability-indices of special Lagrangian cones $C_1, \cdots, C_k$:

$$\dim O_X = \sum_{i=1}^k s\text{-ind}(C_i).$$

This means that $s\text{-ind}(C)$ of a special Lagrangian cone $C$ is the dimension of the obstruction space to deforming a special Lagrangian submanifold $X$ in a Calabi-Yau manifold with a conical singularity with cone $C$, and that if $C$ is stable then the deformation theory of $X$ simplifies.

That a special Lagrangian cone $C$ is rigid means that if all infinitesimal deformations of $C$ as a special Lagrangian cone comes from rotations of $C$ by $SU(m)$. Next we mention the Joyce’s regularity results of special Lagrangian integral currents, or special Lagrangian varieties. Geometric measure theory implies the compactness of the space of such objects. Suppose that $X$ is a special Lagrangian integral current and has the multiplicity 1 tangent cone at $x \in \text{supp}X$. Joyce showed that if the tangent cone of $X$ at $x$ is a rigid special Lagrangian cone, then $X$ has an isolated conical singularity at $x$.

So it is actually interesting and important to investigate explicitly the stability and rigidity of special Lagrangian cones.
Joyce and Marshall proved that $C^{3}_{HL}$ is stable and $C^{m}_{HL}$ is unstable if $m \geq 4$, and $C^{m}_{HL}$ is rigid if and only if $m \neq 8, 9$, and they determined their stability-indices and Legendrian-indices explicitly (cf. [11]). By the spectral analysis on surfaces Haskins showed that a stable special Lagrangian cone in $C^{3}$ over a minimal Legendrian torus in $S^{5}$ is only $C^{3}_{HL}$ ([7]).

**Problem.** Construct and classify stable special Lagrangian cones in complex Euclidean spaces.

2. **Stability-index of special Lagrangian cones over certain compact irreducible symmetric spaces**

In this section we shall discuss a class of special Lagrangian cones constructed by the Lie theoretic method including the Harvey-Lawson cones $C^{m}_{HL}$. Let $(U, G)$ be an Hermitian symmetric pair of compact type with the canonical decomposition $u = g + p$. Set dim($U/G$) = $2m$. Let $\langle , \rangle_{u}$ denote the Ad($U$)-invariant inner product of $u$ defined by $(-1)$-times Killing-Cartan form of $u$. We decompose $g$ into the direct sum of the semisimple part $g_{ss}$ and the center $c(g)$ as follows: $g = g_{ss} \oplus c(g)$. There is an element $Z \in c(g)$ such that ad$Z$ defines the invariant complex structure of $(U, G)$. Relative to the complex structure the subspace $p$ can be identified with a complex Euclidean space $C^{m}$. We take the decomposition of $(U, G)$ into irreducible Hermitian symmetric pairs of compact type:

$$
(U, G) = (U_{1}, G_{1}) \oplus \cdots \oplus (U_{s}, G_{s}).
$$

(2.1)

Set dim$(U_{i}/G_{i}) = 2m_{i}$ for $i = 1, \cdots, s$. Let $u_{i} = g_{i} + p_{i}$ be the canonical decomposition of $(U_{i}, G_{i})$ for each $i = 1, 2, \cdots, s$. Assume that there is an element $\eta_{i} \in p_{i}$ satisfying the condition $(\text{ad} \eta_{i})^{3} + 4(\text{ad} \eta_{i}) = 0$. Choose positive numbers $c_{1} > 0, \cdots, c_{s} > 0$ with $\sum_{i=1}^{s} 1/c_{i} = 1/c$. Put $a_{i} = 1/\sqrt{2c_{i}m_{i}}$ for each $i = 1, \cdots, s$. Set $L_{i} = \text{Ad}(G_{i})(a_{i}\eta_{i}) \subset S^{2m_{i} - 1}(c_{i}/4) \subset p_{i}$, which is an irreducible symmetric $R$-space standard embedded in a complex Euclidean space $C^{m}$. We take the decomposition of $(U, G)$ into irreducible Hermitian symmetric pairs of compact type:

(2.2) $L = L_{1} \times \cdots \times L_{s} \subset S^{2m_{1} - 1}(c_{1}/4) \times \cdots \times S^{2m_{s} - 1}(c_{s}/4) \subset S^{2m - 1}(c/4)$.

Note that we have the inclusions

Note that $L$ is a compact $H$-minimal Lagrangian submanifold embedded in $C^{m}$ (see [3]).

We take an orthogonal decomposition $c(g) = c^{0} \oplus \{Z\} \cup R$ of $c(g)$. Let $g^{0} := g_{ss} \oplus c^{0}$ and $C^{g^{0}}$ denote the analytic subgroup of $G$ generated by $g^{0}$. Set $\Sigma = \text{Ad}(G^{0})(\eta) \cong G^{0}/K^{0} \subset S^{2m - 1}(c/4) \subset p$, where $K^{0} = \{a \in G^{0} | \text{Ad}(a)(\eta) = \eta\}$. Then $\Sigma$ is a Legendrian submanifold in $S^{2m - 1}(c/4)$. Moreover $\Sigma$ is a minimal submanifold in $S^{2m - 1}(c/4)$ if and only if $c_{i}m_{i} = cm$ for each $i = 1, 2, \cdots, s$. Thus we obtain
Proposition 2.1. \( C_\Sigma \) is a special Lagrangian cone in \( \mathbb{C}^m \) if and only if the condition \( c_i m_i = cm \) is satisfied for each \( i = 1, 2, \ldots, s \).

In the case when \( (U_i, G_i) = (SU(2), S(U(1) \times U(1))) \) for all \( i \), the above special Lagrangian cone \( C_\Sigma \) coincides with the Harvey-Lawson’s special Lagrangian cone \( C_{HL}^m \).

In the case when \( (U_i, G_i) \) is irreducible, i.e. \( s = 1 \), from the classification theory of symmetric \( R \)-spaces, \( \Sigma \) is one of symmetric spaces of compact type in the following list:

(a) \( S^{m-1} \).
(b) \( SU(p) \), \( m = p^2 \).
(c) \( SU(p)/SO(p) \), \( m = \frac{(p-1)(p+2)}{2} + 1 \).
(d) \( SU(2p)/Sp(p) \), \( m = (p-1)(2p+1) + 1 \).
(e) \( E_6/F_4 \), \( m = 27 \).

Here \( p \geq 3 \). Note that they are connected, simply connected and compact irreducible symmetric spaces whose restricted root systems are of type \( A \), and the rank of the symmetric spaces is equal to \( p - 1 \) and the rank of \( E_6/F_4 \) is 2. They are the standard embeddings by the first eigenfunctions of the Laplacian (cf. [16]).

Suppose that \( \Sigma \) is a compact embedded minimal Legendrian submanifold of \( S^{2m-1}(1) \) given by the standard embedding of the above symmetric spaces of compact type. Let \( C_\Sigma \) be a special Lagrangian cone over \( \Sigma \) in \( \mathbb{C}^m \). Then we shall show

Theorem 2.1. (1) They all \( C_\Sigma \) are rigid.
(2) If \( \Sigma = SU(3), SU(3)/SO(3), SU(6)/Sp(3)(p = 3), E_6/F_4 \), then \( C_\Sigma \) is stable and thus Legendrian stable.
(3) If \( \Sigma = SU(p), SU(p)/SO(p), SU(2p)/Sp(p), p \geq 4 \) then \( C_\Sigma \) is not Legendrian stable and thus not stable.

Remark. In case (a), \( \Sigma = S^{m-1} \) is a totally geodesic Legendrian submanifold embedded in \( S^{2m-1}(1) \) and thus \( C_\Sigma \) is a Lagrangian vector subspace of \( \mathbb{C}^m \).

In order to determine the stability-indices of special Lagrangian cones over these minimal Legendrian submanifolds \( \Sigma = G^0/K^0 \), we shall examine explicitly the eigenvalues and their multiplicities of the Laplacian of compact irreducible symmetric spaces \( G^0/K^0 \) by the theory of spherical functions on compact symmetric spaces (cf. [19]). In the calculation we use the results described in [1].

First we prepare a useful algebraic lemma for our calculation. Let \( (m_1, \ldots, m_p) \) be a \( p \)-tuple of real numbers satisfying the conditions

\[
(2.3) \quad \sum_{i=1}^{p} m_i = 0 \quad \text{and} \quad 0 \leq m_i - m_{i+1} \in \mathbb{Z} \quad \text{for each} \quad i = 1, 2, \ldots, p-1.
\]
Then note that \( m_i \in (1/p)\mathbb{Z} \) for each \( i = 1, 2, \ldots, p - 1 \). In fact, if we set \( k_i := m_i - m_{i+1} \geq 0 \), then we have

\[
m_p = -\frac{1}{p} \sum_{j=1}^{p-1} jk_j,
\]

\[
m_i = k_i + \cdots + k_{p-1} - \frac{1}{p} \sum_{j=1}^{p-1} jk_j \quad (i = 1, 2, \ldots, p - 1).
\]

**Lemma 2.1.** Fix a positive real number \( t > 0 \). Define a function \( Q \) with respect to \( m_1, \ldots, m_p \) or \( k_1, \ldots, k_{p-1} \) by

\[
Q := \sum_{i=1}^{p} (m_i)^2 - t \sum_{i=1}^{p} im_i.
\]

1. If \( (m_1, \ldots, m_p) = (1, 0, \ldots, 0, -1) \) i.e. \( (k_1, \ldots, k_{p-1}) = (1, 0, \ldots, 0, 1) \), then \( Q \) attains \( Q = 2 + t(p-1) \).
2. If \( (m_1, \ldots, m_p) = \left( \frac{p-1}{p}, \frac{1}{p}, \ldots, \frac{1}{p} \right) \) i.e. \( (k_1, \ldots, k_{p-1}) = (0, 1, 0, \ldots, 0) \), or \( (m_1, \ldots, m_p) = \left( \frac{1}{p}, \ldots, \frac{1}{p}, -\frac{p-1}{p} \right) \) \( (k_1, \ldots, k_{p-1}) = (0, \ldots, 0, 1, 0) \), then \( Q \) attains

\[
Q = \frac{p-1}{p} + t\frac{p-1}{2} < 2 + t(p-1).
\]

3. Assume that \( p \geq 4 \). If \( (m_1, \ldots, m_p) = \left( \frac{p-2}{p}, \frac{p-2}{p}, -\frac{2}{p}, \ldots, -\frac{2}{p} \right) \) i.e. \( (k_1, \ldots, k_{p-1}) = (0, 1, 0, \ldots, 0) \) or \( (m_1, \ldots, m_p) = \left( \frac{2}{p}, \ldots, \frac{2}{p}, -\frac{p-2}{p}, -\frac{p-2}{p} \right) \) i.e. \( (k_1, \ldots, k_{p-1}) = (0, \ldots, 0, 1, 0) \), then \( Q \) attains

\[
Q = \frac{p-1}{p} + t\frac{p-1}{2} < Q = \frac{2(p-2)}{p} + t(p-2) < 2 + t(p-1).
\]

4. \( Q = 2 + t(p-1) \) if and only if \( (m_1, \ldots, m_p) = (1, 0, \ldots, 0, -1) \) i.e. \( (k_1, \ldots, k_{p-1}) = (1, 0, \ldots, 0, 1) \).
5. \( Q < 2 + t(p-1) \) if and only if \( (m_1, \ldots, m_p) \) or \( (k_1, \ldots, k_{p-1}) \) is one of the following table:

<table>
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<th>( p )</th>
<th>( (k_1, \ldots, k_{p-1}) )</th>
<th>( (m_1, \ldots, m_p) )</th>
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<td>( (1, 0, \ldots, 0, 0) )</td>
<td>( \left( \frac{p-1}{p}, \frac{1}{p}, \ldots, -\frac{1}{p} \right) )</td>
<td>( \frac{p-1}{p} + t\frac{p-1}{2} )</td>
</tr>
<tr>
<td>( \geq 3 )</td>
<td>( (0, 0, \ldots, 0, 1) )</td>
<td>( \left( \frac{1}{p}, \frac{1}{p}, \ldots, \frac{1}{p}, -\frac{p-1}{p} \right) )</td>
<td>( \frac{p-1}{p} + t\frac{p-1}{2} )</td>
</tr>
<tr>
<td>( 4 )</td>
<td>( (0, 1, 0) )</td>
<td>( \left( \frac{1}{p}, \frac{1}{p}, \frac{1}{p}, -\frac{1}{p}, -\frac{1}{p} \right) )</td>
<td>( 1 + 2t )</td>
</tr>
<tr>
<td>( \geq 5 )</td>
<td>( (0, 1, 0, \ldots, 0) )</td>
<td>( \left( \frac{p-2}{p}, \frac{p-2}{p}, -\frac{2}{p}, \ldots, -\frac{2}{p} \right) )</td>
<td>( \frac{2(p-2)}{p} + t(p-2) )</td>
</tr>
<tr>
<td>( \geq 5 )</td>
<td>( (0, \ldots, 0, 1, 0) )</td>
<td>( \left( \frac{2}{p}, \ldots, \frac{2}{p}, -\frac{p-2}{p}, -\frac{p-2}{p} \right) )</td>
<td>( \frac{2(p-2)}{p} + t(p-2) )</td>
</tr>
<tr>
<td>( 6 )</td>
<td>( (0, 0, 1, 0, 0) )</td>
<td>( \left( \frac{1}{p}, \frac{1}{p}, \frac{1}{p}, -\frac{p-2}{p}, -\frac{p-2}{p} \right) )</td>
<td>( \frac{p-2}{p} + 6t )</td>
</tr>
<tr>
<td>( 7 )</td>
<td>( (0, 0, 1, 0, 0, 0) )</td>
<td>( \left( \frac{1}{p}, \frac{1}{p}, \frac{1}{p}, \frac{1}{p}, \frac{1}{p}, -\frac{5}{p}, -\frac{5}{p} \right) )</td>
<td>( \frac{p-2}{p} + 6t )</td>
</tr>
</tbody>
</table>
Proof. The statements (1), (2) and (3) are obtained by direct computations. The function \( Q \) can be described in terms of \( k_1, \ldots, k_{p-1} \) as the formula:

\[
(2.5) \quad Q = \sum_{i=1}^{p-1} \left\{ \sum_{j=1}^{i} \left( 1 - \frac{i}{p} \right) jk_j + \sum_{j=i+1}^{p-1} i(1 - \frac{j}{p})k_j \right\} k_i + t \sum_{i=1}^{p-1} \frac{i(p-i)}{2} k_i.
\]

The statements (4), (5) follow from this formula. \( \Box \)

The case \( \Sigma = (SU(p) \times SU(p))/SU(p) \): In this case note that \( m - 1 = \dim \Sigma = p^2 - 1, \ m = p^2, \ 2m = 2p^2 \) and \( m^2 - 1 - \dim G_\Sigma = m^2 - 1 - \dim (SU(p) \times SU(p)) = (p^2 - 1)^2 \).

Let \( \{ \varepsilon_1, \ldots, \varepsilon_p \} \) be the standard orthonormal basis of a \( p \)-dimensional Euclidean vector space \( \mathbb{R}^p \). Set

\[
D(SU(p)) = \left\{ \sum_{i=1}^{p} m_i \varepsilon_i \mid \sum_{i=1}^{p} m_i = 0, 0 \leq m_i - m_{i+1} \in \mathbb{Z} \ (i = 1, 2, \ldots, p-1) \right\}
\]

\[
(2.6) \quad = \left\{ \sum_{i=1}^{p-1} k_i \Lambda_i \mid 0 \leq k_i \in \mathbb{Z} \ (i = 1, 2, \ldots, p-1) \right\}.
\]

Here \( k_i = m_i - m_{i+1} \ (i = 1, \ldots, p-1) \) and \( \{ \Lambda_1, \ldots, \Lambda_{p-1} \} \) is the fundamental weight system of \( SU(p) \) defined by

\[
\Lambda_i = \varepsilon_1 + \cdots + \varepsilon_i - \frac{i}{p} \sum_{j=1}^{p} \varepsilon_j \quad (i = 1, 2, \ldots, p-1).
\]

We know that there is a bijective correspondence between \( D(SU(p)) \) and the complete set of all inequivalent complex irreducible representations of \( SU(p) \). Then for each \( \Lambda = \sum_{i=1}^{p} m_i \varepsilon_i \in D(SU(p)) \) the eigenvalue \( a_\Lambda \) of the Casimir operator on a complex irreducible representation with highest weight \( \Lambda \) is equal to

\[
(2.7) \quad -a_\Lambda = \sum_{i=1}^{p} (m_i)^2 - 2 \sum_{i=1}^{p} im_i
\]

and the corresponding eigenvalue of \( \Delta_\Sigma \) is given by

\[
(2.8) \quad \lambda = (-a_\Lambda)^{\frac{1}{2p}} \cdot 2C^{-1} = (-a_\Lambda)^{\frac{1}{2p}} \cdot 2 \cdot p^2 = (-a_\Lambda)p = pQ
\]

because of \( C = \frac{4}{p^2c} = \frac{1}{p^2} \) by [1, p594]. Here \( Q \) is a function defined in Lemma 2.1. For each \( \Lambda \in D(SU(p)) \), we denote by \( d_\Lambda \) the dimension of a complex irreducible representation with highest weight \( \Lambda \). The dimension \( d_\Lambda \) is given by the Weyl’s dimension formula. The multiplicity \( m(\lambda) \), i.e. the dimension of the eigenspace, for the eigenvalue \( \lambda \) of the Laplacian \( \Delta_\Sigma \) is equal to

\[
m(\lambda) = \sum_{\Lambda \in D(SU(p)), \lambda = (-a_\Lambda)p} (d_\Lambda)^2.
\]
First we consider the case \( p = 3 \). Then (2.7) becomes

\[
-a_{\Lambda} = \frac{2}{3}(k_{1}^{2} + k_{1}k_{2} + k_{2}^{2}) + 2(k_{1} + k_{2}).
\]

(2.9)

(1) If \((k, k_2) = (1, 0)\) or \((0, 1)\), then \((-a_{\Lambda}) \cdot 3 = \frac{8}{3}3 = 8\) and \(d_{\Lambda} = 3\).

(2) If \((k, k_2) = (1, 1)\), then \((-a_{\Lambda}) \cdot 3 = 6 \cdot 3 = 18\) and \(d_{\Lambda} = 8\).

(3) If \((k, k_2) = \) otherwise, then \((-a_{\Lambda}) \cdot 3 \geq 20 > 18\).

Thus all eigenvalues \( \lambda \) and their multiplicity \( m(\lambda) \) of \( \Delta_{\Sigma} \) between 0 and \( 2m = 18 \) are determined as follows:

<table>
<thead>
<tr>
<th>( \lambda )</th>
<th>( \alpha \in D_{\Sigma} \cap [0, 2] )</th>
<th>( m_{\Sigma}(\alpha) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>8</td>
<td>1</td>
<td>18</td>
</tr>
<tr>
<td>18</td>
<td>2</td>
<td>64</td>
</tr>
</tbody>
</table>

Hence we have \( N_{\Sigma}(2) = m_{\Sigma}(0) + m_{\Sigma}(1) + m_{\Sigma}(2), m_{\Sigma}(0) = 1 = b^{0}(\Sigma), m_{\Sigma}(1) = 18 = 2m, \) and \( m^{2} - 1 - \dim G_{\Sigma} = 9^{2} - 1 - (8 + 8) = 64 = m_{\Sigma}(2). \) Therefore we conclude that \( s\text{-ind}(C) = 0. \)

Next we treat the case \( p \geq 4. \) By Lemma 2.1 we obtain the following table of all \( \Lambda \in D(SU(p)) \) corresponding to eigenvalues \( \lambda \leq 2m = 2p^{2}: \)

<table>
<thead>
<tr>
<th>( p )</th>
<th>( \Lambda )</th>
<th>( (k_1, \cdots, k_{p-1}) )</th>
<th>( \lambda = pQ )</th>
<th>( d_{\Lambda} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \geq 3 )</td>
<td>( \Lambda_{1} + \Lambda_{p-1} )</td>
<td>( (1, 0, \cdots, 0, 1) )</td>
<td>( 2p^{2} )</td>
<td>( p^{2} - 1 )</td>
</tr>
<tr>
<td>( \geq 3 )</td>
<td>( \Lambda_{1} )</td>
<td>( (1, 0, \cdots, 0, 0) )</td>
<td>( p^{2} - 1 )</td>
<td>( p )</td>
</tr>
<tr>
<td>( \geq 3 )</td>
<td>( \Lambda_{p-1} )</td>
<td>( (0, 0, \cdots, 0, 1) )</td>
<td>( p^{2} - 1 )</td>
<td>( p )</td>
</tr>
<tr>
<td>4</td>
<td>( \Lambda_{2} )</td>
<td>( (0, 1, 0, \cdots, 0) )</td>
<td>20</td>
<td>6</td>
</tr>
<tr>
<td>( \geq 5 )</td>
<td>( \Lambda_{2} )</td>
<td>( (0, 1, 0, \cdots, 0) )</td>
<td>( 2(p + 1)(p - 2) )</td>
<td>( \frac{p(p - 1)}{2} )</td>
</tr>
<tr>
<td>( \geq 5 )</td>
<td>( \Lambda_{p-2} )</td>
<td>( (0, \cdots, 0, 1, 0) )</td>
<td>( 2(p + 1)(p - 2) )</td>
<td>( \frac{p(p - 1)}{2} )</td>
</tr>
<tr>
<td>6</td>
<td>( \Lambda_{3} )</td>
<td>( (0, 0, 1, 0, 0, 0) )</td>
<td>63</td>
<td>20</td>
</tr>
<tr>
<td>7</td>
<td>( \Lambda_{3} )</td>
<td>( (0, 0, 1, 0, 0, 0) )</td>
<td>96</td>
<td>35</td>
</tr>
<tr>
<td>7</td>
<td>( \Lambda_{4} )</td>
<td>( (0, 0, 0, 1, 0, 0) )</td>
<td>96</td>
<td>35</td>
</tr>
</tbody>
</table>

Note that the (nonzero) first eigenvalue of \( \Delta_{\Sigma} \) is \( p^{2} - 1 = \dim \Sigma. \)

By using these results, we determine all \( \alpha \in D_{\Sigma} \cap [0, 2] \) by \( \lambda = \alpha(\alpha + m - 2), \)

that is, \( \alpha = (\sqrt{(m - 2)^{2} + 4\lambda} - (m - 2))/2 \) as follows:

If \( p \geq 8, \) then we have

<table>
<thead>
<tr>
<th>( \lambda )</th>
<th>( \alpha \in D_{\Sigma} \cap [0, 2] )</th>
<th>( m_{\Sigma}(\alpha) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>( p^{2} - 1 )</td>
<td>1</td>
<td>( 2p^{2} )</td>
</tr>
<tr>
<td>( 2(p + 1)(p - 2) )</td>
<td>( \sqrt{(p^{2} - 2)^{2} + 8(p + 1)(p - 2) - (p^{2} - 2)} )</td>
<td>( \frac{p^{2}(p - 1)^{2}}{2} )</td>
</tr>
<tr>
<td>( 2p^{2} )</td>
<td>( 2 )</td>
<td>( (p^{2} - 1)^{2} )</td>
</tr>
</tbody>
</table>

If \( p = 7, \) then we have
If \( p = 6 \), then we have

<table>
<thead>
<tr>
<th>( \lambda )</th>
<th>( \alpha \in D_\Sigma \cap [0, 2] )</th>
<th>( m_\Sigma(\alpha) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>48</td>
<td>2</td>
<td>882</td>
</tr>
<tr>
<td>96</td>
<td>(( \sqrt{2593} - 47 ))/2</td>
<td>2450</td>
</tr>
<tr>
<td>98</td>
<td>2</td>
<td>2304</td>
</tr>
</tbody>
</table>

If \( p = 5 \), then we have

<table>
<thead>
<tr>
<th>( \lambda )</th>
<th>( \alpha \in D_\Sigma \cap [0, 2] )</th>
<th>( m_\Sigma(\alpha) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>24</td>
<td>1</td>
<td>50</td>
</tr>
<tr>
<td>36</td>
<td>(( \sqrt{673} - 23 ))/2</td>
<td>200</td>
</tr>
<tr>
<td>50</td>
<td>2</td>
<td>576</td>
</tr>
</tbody>
</table>

If \( p = 4 \), then we have

<table>
<thead>
<tr>
<th>( \lambda )</th>
<th>( \alpha \in D_\Sigma \cap [0, 2] )</th>
<th>( m_\Sigma(\alpha) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>15</td>
<td>1</td>
<td>32</td>
</tr>
<tr>
<td>20</td>
<td>( \sqrt{69} - 7 )</td>
<td>36</td>
</tr>
<tr>
<td>32</td>
<td>2</td>
<td>225</td>
</tr>
</tbody>
</table>

We obtain \( s\text{-ind}(C) > 0 \) and thus \( C \) is not stable.

The case \( \Sigma = SU(p)/SO(p) \) : In this case note that \( m - 1 = \text{dim} \Sigma = (p - 1)(p + 2)/2 \), \( m = p(p + 1)/2 \), \( 2m = p(p + 1) \) and \( m^2 - 1 - \text{dim} G_\Sigma = m^2 - 1 - \text{dim} SU(p) = p^2(p + 3)(p - 1)/4 \).

The subset \( D(SU(p), SO(p)) \subset D(SU(p)) \) is defined by

\[
D(SU(p), SO(p)) = \left\{ \sum_{i=1}^{p} m_i \varepsilon_i \mid \sum_{i=1}^{p} m_i = 0, 0 \leq m_i - m_{i+1} \in \mathbb{Z} \; (i = 1, 2, \ldots, p - 1) \right\}
\]

\[
(2.10)
\]

\[
= \left\{ \sum_{i=1}^{p-1} k_i M_i \mid 0 \leq k_i \in \mathbb{Z} \; (i = 1, 2, \ldots, p - 1) \right\}
\]
Here $k_i = m_i - m_{i+1}$ ($i = 1, \cdots , p - 1$) and $\{M_i \mid i = 1, \cdots , p - 1\}$ is the fundamental weight system of $(SU(p), SO(p))$ defined by

$$M_i = 2\Lambda_i = 2(\varepsilon_1 + \cdots + \varepsilon_i - \frac{i}{p} \sum_{j=1}^{p} \varepsilon_j) \quad (i = 1, 2, \cdots , p - 1).$$

We know that there is a bijective correspondence between $D(SU(p), SO(p))$ and the complete set of all inequivalent spherical representations of the compact symmetric pair $(SU(p), SO(p))$. Then for each $\Lambda = 2\sum_{i=1}^{p} m_i \varepsilon_i \in D(SU(p), SO(p))$ we have

\begin{equation}
-a_\Lambda = 4 \sum_{i=1}^{p} (m_i)^2 - 4 \sum_{i=1}^{p} im_i
\end{equation}

and the corresponding eigenvalue of $\Delta_\Sigma$ is given by

\begin{equation}
\lambda = (-a_\Lambda) \frac{1}{2p} C^{-1} \cdot \frac{p^2}{2} = (-a_\Lambda) \frac{p}{4} = pQ
\end{equation}

because of $C = \frac{8}{p^2} \cdot \frac{p}{2}$ by [1, p594]. Here $Q$ is a function defined in Lemma 2.1. The multiplicity $m(\lambda)$, i.e. the dimension of the eigenspace, with eigenvalue $\lambda$ of the Laplacian $\Delta_\Sigma$ is equal to

$$m(\lambda) = \sum_{\Lambda \in D(SU(p), SO(p)) \setminus \{\lambda = (-a_\Lambda)p/4\}} d_\Lambda.$$  

First we consider the case $p = 3$. Then (2.11) becomes

\begin{equation}
-a_\Lambda = \frac{8}{3}(k_1^2 + k_1 k_2 + k_2^2) + 4(k_1 + k_2).
\end{equation}

(1) If $(k_1, k_2) = (1, 0)$ or $(0, 1)$, then $(-a_\Lambda) = \frac{3}{4} = \frac{20}{8} = 5$ and $d_\Lambda = 6$.

(2) If $(k_1, k_2) = (1, 1)$, then $(-a_\Lambda) = \frac{3}{4} = 16 \cdot \frac{3}{4} = 12$ and $d_\Lambda = 27$.

(3) If $(k_1, k_2) = \text{otherwise}$, then $(-a_\Lambda) > \frac{3}{4} > 13 > 12$.

Thus all eigenvalues $\lambda$ and their multiplicities of $\Delta_\Sigma$ between 0 and $2m = 12$ are determined as follows:

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$\alpha \in D_\Sigma \cap [0, 2]$</th>
<th>$m_\Sigma(\alpha)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>12</td>
</tr>
<tr>
<td>12</td>
<td>2</td>
<td>27</td>
</tr>
</tbody>
</table>

Hence we have $N_\Sigma(2) = m_\Sigma(0) + m_\Sigma(1) + m_\Sigma(2)$, $m_\Sigma(0) = 1 = b^0(\Sigma)$, $m_\Sigma(1) = 12 = 2m$, and $m^2 - 1 - \dim G_\Sigma = 6^2 - 1 - (9 - 1) = 27 = m_\Sigma(2)$. Therefore we conclude that $s$-ind$(C) = 0$.

Next we treat the case $p \geq 4$. By Lemma 2.1 we obtain the following table of all $\Lambda \in D(SU(p), SO(p))$ corresponding to eigenvalues $\lambda \leq 2m = p(p + 1)$:
Note that the (nonzero) first eigenvalue of $\Delta_{\Sigma}$ is $(p - 1)(p + 2)/2 = \dim \Sigma$.

By using these results, we determine all $\alpha \in D_{\Sigma} \cap [0, 2]$ as follows:

If $p \geq 8$, then we have

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$\alpha \in D_{\Sigma} \cap [0, 2]$</th>
<th>$m_{\Sigma}(\alpha)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0$</td>
<td>$0$</td>
<td>$1$</td>
</tr>
<tr>
<td>$(p - 1)(p + 2)/2$</td>
<td>$1$</td>
<td>$p(p + 1)$</td>
</tr>
<tr>
<td>$(p - 2)(p + 2)$</td>
<td>$\sqrt{p(p+1)/2-2}^2+4(p+2)(p-2)-(p(p+1)/2-2)$</td>
<td>$2 \frac{p(p-1)}{3} \frac{(p+1)p}{2^2}$</td>
</tr>
<tr>
<td>$p(p + 2)$</td>
<td>$2$</td>
<td>$\frac{(p+1)p}{2^2}$</td>
</tr>
</tbody>
</table>

If $p = 7$, then we have

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$\alpha \in D_{\Sigma} \cap [0, 2]$</th>
<th>$m_{\Sigma}(\alpha)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0$</td>
<td>$0$</td>
<td>$1$</td>
</tr>
<tr>
<td>$27$</td>
<td>$1$</td>
<td>$56$</td>
</tr>
<tr>
<td>$45$</td>
<td>$\sqrt{214 - 13}$</td>
<td>$392$</td>
</tr>
<tr>
<td>$54$</td>
<td>$\sqrt{223 - 13}$</td>
<td>$980$</td>
</tr>
<tr>
<td>$56$</td>
<td>$2$</td>
<td>$735$</td>
</tr>
</tbody>
</table>

If $p = 6$, then we have

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$\alpha \in D_{\Sigma} \cap [0, 2]$</th>
<th>$m_{\Sigma}(\alpha)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0$</td>
<td>$0$</td>
<td>$1$</td>
</tr>
<tr>
<td>$20$</td>
<td>$1$</td>
<td>$42$</td>
</tr>
<tr>
<td>$32$</td>
<td>$(\sqrt{489 - 19})/2$</td>
<td>$210$</td>
</tr>
<tr>
<td>$36$</td>
<td>$(\sqrt{505 - 19})/2$</td>
<td>$175$</td>
</tr>
<tr>
<td>$42$</td>
<td>$2$</td>
<td>$405$</td>
</tr>
</tbody>
</table>

If $p = 5$, then we have

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$\alpha \in D_{\Sigma} \cap [0, 2]$</th>
<th>$m_{\Sigma}(\alpha)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0$</td>
<td>$0$</td>
<td>$1$</td>
</tr>
<tr>
<td>$14$</td>
<td>$1$</td>
<td>$30$</td>
</tr>
<tr>
<td>$21$</td>
<td>$(\sqrt{253 - 13})/2$</td>
<td>$100$</td>
</tr>
<tr>
<td>$30$</td>
<td>$2$</td>
<td>$200$</td>
</tr>
</tbody>
</table>

If $p = 4$, then we have
The case $\Sigma = SU(2p)/Sp(p)$: In this case note that $m - 1 = \dim \Sigma = (p-1)(2p+1)$, $m = 2p^2 - p = p(2p-1)$, $2m = 2p(2p-1)$ and $m^2 - 1 - \dim SU(2p) = p^2(2p-3)(2p+1)$. Set

$$D(SU(2p))$$

\[= \left\{ \sum_{i=1}^{2p} m_i \epsilon_i \mid \sum_{i=1}^{2p} m_i = 0, 0 \leq m_i - m_{i+1} \in \mathbb{Z} \ (i = 1, 2, \ldots, 2p-1) \right\} \]

\[= \left\{ \sum_{i=1}^{2p-1} k_i \Lambda_i \mid 0 \leq k_i \in \mathbb{Z} \ (i = 1, 2, \ldots, 2p-1) \right\}. \]

Here $k_i = m_i - m_{i+1}$ $(i = 1, \ldots, 2p-1)$ and $\{ \Lambda_i \mid i = 1, \ldots, 2p-1 \}$ is the fundamental weight system of $SU(2p)$ defined by

$$\Lambda_i = \epsilon_1 + \cdots + \epsilon_i - \frac{i}{p} \sum_{j=1}^{2p} \epsilon_j \ (i = 1, 2, \ldots, 2p-1).$$

Now we define $f_i \in \mathbb{R}^{2p}$ by

$$f_i := \frac{1}{\sqrt{2}}(\epsilon_{2i-1} + \epsilon_{2i}) \quad (i = 1, 2, \ldots, p-1, p).$$

The subset $D(SU(2p), Sp(p)) \subset D(SU(2p))$ is defined by

$$D(SU(2p), Sp(p))$$

\[= \left\{ \sqrt{2} \sum_{i=1}^{p} m_i f_i \mid \sum_{i=1}^{p} m_i = 0, 0 \leq m_i - m_{i+1} \in \mathbb{Z} (i = 1, 2, \ldots, p-1) \right\} \]

\[= \left\{ \sum_{i=1}^{p-1} k_i M_i \mid 0 \leq k_i \in \mathbb{Z} (i = 1, 2, \ldots, p-1) \right\}. \]

Here $k_i = m_i - m_{i+1}$ $(i = 1, \ldots, p-1)$ and $\{ M_i \mid i = 1, \ldots, p-1 \}$ is the fundamental weight system of $(SU(2p), Sp(p))$ defined by $M_i = \Lambda_{2i}$. We know that there is a bijective correspondence between $D(SU(2p), Sp(p))$ and the complete set of all inequivalent spherical representations of the compact symmetric pair $(SU(2p), Sp(p))$. Then for each $\Lambda = \sqrt{2} \sum_{i=1}^{p} m_i f_i \in D(SU(2p), Sp(p))$ we have

\[a_\Lambda = 2 \sum_{i=1}^{p} (m_i)^2 - 8 \sum_{i=1}^{p} im_i \]

The table below shows the relationship between $\lambda$, $\alpha \in D_{\Sigma} \cap [0, 2]$, and $m_{\Sigma}(\alpha)$.

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$\alpha \in D_{\Sigma} \cap [0, 2]$</th>
<th>$m_{\Sigma}(\alpha)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>9</td>
<td>1</td>
<td>20</td>
</tr>
<tr>
<td>12</td>
<td>$2(\sqrt{7} - 2)$</td>
<td>20</td>
</tr>
<tr>
<td>20</td>
<td>$2$</td>
<td>84</td>
</tr>
</tbody>
</table>
and the corresponding eigenvalue of $\Delta_\Sigma$ is given by

\[(2.17) \quad \lambda = (-a_\Lambda) \frac{1}{4p} C^{-1} = (-a_\Lambda) \frac{1}{4p} \cdot 2p^2 = (-a_\Lambda) \frac{p}{2} = pQ\]

because of $C = \frac{2}{p^2} = \frac{1}{2p^2}$ by [1, p594]. Here $Q$ is a function defined in Lemma 2.1. The multiplicity $m(\lambda)$, i.e. the dimension of the eigenspace, for the eigenvalue $\lambda$ of the Laplacian $\Delta_\Sigma$ is equal to

\[m(\lambda) = \sum_{\Lambda \in D(SU(2p), Sp(p)), \lambda = -a_\Lambda} d_\Lambda.\]

First we consider the case $p = 3$. Then (2.16) becomes

\[(2.18) \quad -a_\Lambda = \frac{4}{3} (k_1^2 + k_1k_2 + k_2^2) + 8(k_1 + k_2).\]

(1) If $(k_1, k_2) = (1, 0)$ or $(0, 1)$, the $(-a_\Lambda) \cdot \frac{3}{2} = 28\frac{3}{2} = 14$ and $d_\Lambda = 15$.

(2) If $(k_1, k_2) = (1, 1)$, the $(-a_\Lambda) \cdot \frac{3}{2} = 20 \cdot \frac{3}{2} = 30$ and $d_\Lambda = 189$.

(3) If $(k_1, k_2)$ = otherwise, then $(-a_\Lambda) \cdot \frac{3}{2} \geq 32 > 30$.

Thus all eigenvalues $\lambda$ and their multiplicity $m(\lambda)$ of $\Delta_\Sigma$ between 0 and $2m = 30$ are determined as follows:

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$\alpha \in D_\Sigma \cap [0, 2]$</th>
<th>$m_\Sigma(\alpha)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td>0</td>
</tr>
<tr>
<td>14</td>
<td></td>
<td>1</td>
</tr>
<tr>
<td>30</td>
<td></td>
<td>2</td>
</tr>
</tbody>
</table>

Hence we have $N_\Sigma(2) = m_\Sigma(0) + m_\Sigma(1) + m_\Sigma(2)$, $m_\Sigma(0) = 1 = t^0(\Sigma)$, $m_\Sigma(1) = 30 = 2m$, $m_\Sigma(2) = 189$. On the other hand, $m^2 - 1 - \dim G_\Sigma = 15^2 - 1 - (36 - 1) = 189$. Therefore we conclude that $\text{s-ind}(C) = 0$.

Next we treat the case $p \geq 4$ by Lemma 2.1 we obtain the following table of all $\Lambda \in D(SU(2p), Sp(p))$ corresponding to eigenvalues $\lambda \leq 2m = 2p(2p - 1)$.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$\Lambda$</th>
<th>$\lambda = pQ$</th>
<th>$d_\Lambda$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\geq 3$</td>
<td>$\Lambda_2 + \Lambda_{2p-2}$</td>
<td>$2p(2p - 1)$</td>
<td>$p^2(2p - 3)(2p + 1)$</td>
</tr>
<tr>
<td>$\geq 3$</td>
<td>$\Lambda_2$</td>
<td>$(1, 0, \ldots, 0, 1)$</td>
<td>$2p+1)(p - 1)$</td>
</tr>
<tr>
<td>$\geq 3$</td>
<td>$\Lambda_{2p-2}$</td>
<td>$(0, 0, \ldots, 0, 1)$</td>
<td>$(2p+1)(p - 1)$</td>
</tr>
<tr>
<td>4</td>
<td>$\Lambda_4$</td>
<td>$(0, 1, 0)$</td>
<td>36</td>
</tr>
<tr>
<td>$\geq 5$</td>
<td>$\Lambda_4$</td>
<td>$(0, 1, 0, \ldots, 0)$</td>
<td>$(2p+1)(p - 2)$</td>
</tr>
<tr>
<td>$\geq 5$</td>
<td>$\Lambda_2(p-2)$</td>
<td>$(0, \ldots, 0, 1, 0)$</td>
<td>$(2p+1)(p - 2)$</td>
</tr>
<tr>
<td>6</td>
<td>$\Lambda_6$</td>
<td>$(0, 0, 1, 0, 0)$</td>
<td>117</td>
</tr>
<tr>
<td>7</td>
<td>$\Lambda_6$</td>
<td>$(0, 0, 1, 0, 0, 0)$</td>
<td>180</td>
</tr>
<tr>
<td>7</td>
<td>$\Lambda_6$</td>
<td>$(0, 0, 0, 1, 0, 0)$</td>
<td>180</td>
</tr>
</tbody>
</table>

Note that the (nonzero) first eigenvalue of $\Delta_\Sigma$ is $(2p + 1)(p - 1) = \dim \Sigma$.

By using these results, we determine all $\alpha \in D_\Sigma \cap [0, 2]$ as follows:

If $p \geq 8$, then we have
If \( p = 7 \), then we have

<table>
<thead>
<tr>
<th>( \lambda )</th>
<th>( \alpha \in D_\Sigma \cap [0,2] )</th>
<th>( m_\Sigma(\alpha) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>90</td>
<td>1</td>
<td>182</td>
</tr>
<tr>
<td>150</td>
<td>( \sqrt{8521 - 89}/2 )</td>
<td>2002</td>
</tr>
<tr>
<td>180</td>
<td>( \sqrt{8641 - 89}/2 )</td>
<td>6006</td>
</tr>
<tr>
<td>182</td>
<td>2</td>
<td>8085</td>
</tr>
</tbody>
</table>

If \( p = 6 \), then we have

<table>
<thead>
<tr>
<th>( \lambda )</th>
<th>( \alpha \in D_\Sigma \cap [0,2] )</th>
<th>( m_\Sigma(\alpha) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>65</td>
<td>1</td>
<td>132</td>
</tr>
<tr>
<td>104</td>
<td>( \sqrt{1128 - 32} )</td>
<td>990</td>
</tr>
<tr>
<td>117</td>
<td>( \sqrt{1141 - 32} )</td>
<td>924</td>
</tr>
<tr>
<td>132</td>
<td>2</td>
<td>4212</td>
</tr>
</tbody>
</table>

If \( p = 5 \), then we have

<table>
<thead>
<tr>
<th>( \lambda )</th>
<th>( \alpha \in D_\Sigma \cap [0,2] )</th>
<th>( m_\Sigma(\alpha) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>44</td>
<td>1</td>
<td>90</td>
</tr>
<tr>
<td>66</td>
<td>( \sqrt{253 - 13}/2 )</td>
<td>420</td>
</tr>
<tr>
<td>90</td>
<td>2</td>
<td>1925</td>
</tr>
</tbody>
</table>

If \( p = 4 \), then we have

<table>
<thead>
<tr>
<th>( \lambda )</th>
<th>( \alpha \in D_\Sigma \cap [0,2] )</th>
<th>( m_\Sigma(\alpha) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>27</td>
<td>1</td>
<td>56</td>
</tr>
<tr>
<td>36</td>
<td>( \sqrt{205 - 13} )</td>
<td>70</td>
</tr>
<tr>
<td>56</td>
<td>2</td>
<td>720</td>
</tr>
</tbody>
</table>

The case \( \Sigma = E_6/F_4 \) : In this case \( m - 1 = \dim \Sigma = 26 \), \( m = 27 \), \( 2m = 54 \) and \( m^2 - 1 - \dim G_\Sigma = m^2 - 1 - \dim E_6 = 27^2 - 1 - 78 = 650 \). Let \( \{M_1, M_2\} \) be the fundamental weight system of \( (E_6, F_4) \) defined by

\[
M_1 = \Lambda_1 = \frac{2}{3} (\varepsilon_8 - \varepsilon_7 - \varepsilon_6),
\]

\[
M_2 = \Lambda_6 = \frac{1}{3} (\varepsilon_8 - \varepsilon_7 - \varepsilon_6) + \varepsilon_5,
\]

where \( \{\varepsilon_i \mid i = 1, \ldots, 8\} \) denotes the standard orthonormal basis of \( \mathbb{R}^8 \) and \( \{\Lambda_i \mid i = 1, \ldots, 6\} \) denotes the fundamental weight system of \( E_6 \) (cf.[4],[1,
Then for each $\Lambda = k_1 M_1 + k_2 M_2 \in D(E_6, F_4)$ we have
\[ -a_\Lambda = 4k_1 \left( \frac{1}{3}k_1 + 4 \right) + 4 \frac{4}{3}k_1 k_2 + 4k_2 \left( \frac{1}{3}k_2 + 4 \right). \]
and the corresponding eigenvalue of $\Delta_\Sigma$ is given by
\[ \lambda = \left( -a_\Lambda \right) \frac{1}{24} C^{-1} = \left( -a_\Lambda \right) \frac{1}{24} \cdot 36 = \left( -a_\Lambda \right) \frac{3}{2}, \]
because of $C = \frac{1}{24} = \frac{1}{36}$ by [1, p594]. Thus we determine all $\Lambda \in D(E_6, F_4)$ corresponding to eigenvalues $\lambda \leq 2m = 54$ and their multiplicities $d_\Lambda$ (cf.[13]) as follows:

(1) If $(k_1, k_2) = (1, 0)$ or $(0, 1)$, then we have $\lambda = \left( -a_\Lambda \right) \cdot \frac{3}{2} = \frac{52}{3} \frac{3}{2} = 26$ and $d_\Lambda = 27$.
(2) If $(k_1, k_2) = (1, 1)$, then we have $\lambda = \left( -a_\Lambda \right) \cdot \frac{3}{2} = \frac{36}{3} \frac{3}{2} = 54$ and $d_\Lambda = 650$.
(3) If $(k_1, k_2) = \text{otherwise}$, then we have $\lambda = \left( -a_\Lambda \right) \cdot \frac{3}{2} \geq 56 > 54$.

Thus all eigenvalues $\lambda$ and their multiplicities of $\Delta_\Sigma$ between 0 and $2m = 2m = 12$ are determined as follows:

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$\alpha \in \mathcal{D}_\Sigma \cap [0, 2]$</th>
<th>$m_\Sigma(\alpha)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>26</td>
<td>1</td>
<td>54</td>
</tr>
<tr>
<td>54</td>
<td>2</td>
<td>650</td>
</tr>
</tbody>
</table>

Thus we obtain $N_\Sigma(2) = m_\Sigma(0) + m_\Sigma(1) + m_\Sigma(2)$, $m_\Sigma(0) = 1 = l^0(\Sigma)$, $m_\Sigma(1) = 54 = 2m$, $m_\Sigma(2) = 650 = m^2 - 1 - \dim G_\Sigma$. Hence we obtain $s\text{-ind}(C) = 0$ for $\Sigma = E_6/F_4$.

Getting together those results in each case, we conclude the following. Theorem 2.1 follows from Theorem 2.2.

**Theorem 2.2.** Let $\Sigma = SU(p), SU(p)/SO(p), SU(2p)/Sp(p)$ $(p \geq 3)$, $E_6/F_4$ (resp. $m = p^2, (p - 1)(p + 2)/2 + 1, (p - 1)(2p + 1) + 1, 27$) be an $(m - 1)$-dimensional minimal Legendrian submanifold embedded in $S^{2m-1}(1)$ in the above standard way and $C = C_\Sigma$ be the special Lagrangian cone in $\mathbb{C}^m$ over $\Sigma$. Then the rigidity, the Legendrian-index and the stability-index of $C$ are described as follows:

1. The equality $m_\Sigma(2) = m^2 - 1 - \dim(G_\Sigma)$ holds and hence each $C$ is rigid.
2. The Legendrian-index $l\text{-ind}(C)$ is equal to $l\text{-ind}(C) = s\text{-ind}(C) + 2m$.
3. The stability-index $s\text{-ind}(C)$ is given as in the following table:
SU(p)        SU(p)/SO(p)        SU(2p)/Sp(p)        E_6/F_4
\begin{tabular}{|c|c|c|c|c|}
\hline
p & SU(p) & SU(p)/SO(p) & SU(2p)/Sp(p) & E_6/F_4 \\
\hline
p \geq 8 & p^{2(p-1)^2} & 2 & 2 & - \\
\hline
p = 7 & 3332 & 1372 & 8008 & - \\
\hline
p = 6 & 850 & 385 & 1914 & - \\
\hline
p = 5 & 200 & 100 & 420 & - \\
\hline
p = 4 & 36 & 20 & 70 & - \\
\hline
p = 3 & 0 & 0 & 0 & 0 \\
\hline
\end{tabular}

Remark. In [1] it was shown that for each \( \Sigma = SU(p), SU(p)/SO(p), SU(2p)/Sp(p), E_6/F_4 \), the image \( \pi(\Sigma) = SU(p) / \mathbb{Z}_p, SU(p)/SO(p)\mathbb{Z}_p, SU(2p)/Sp(p)\mathbb{Z}_2p, E_6/F_4\mathbb{Z}_3 \) by the projection of the Hopf fibration is a Hamiltonian stable minimal Lagrangian submanifold embedded in a complex projective space.

And by using the formula (1.1) we also see the following.

**Theorem 2.3.** In each case \( \Sigma = SU(p), SU(p)/SO(p), SU(2p)/Sp(p) \) \( p \geq 4 \), \( C' = C\Sigma \setminus \{0\} \) has nonzero homogeneous harmonic function of order \( \alpha \) for some \( \alpha \) with \( 1 < \alpha < 2 \) and there is no nonzero homogeneous harmonic function on \( C' \) of order \( \alpha \) for any \( \alpha \) with \( 0 < \alpha < 1 \).

3. **Stability-index of a special Lagrangian cone in \( \mathbb{C}^4 \) over a minimal Legendrian \( SU(2) \)-orbit**

In this section we mention about the stability and the rigidity of a certain special Lagrangian cone over a minimal Legendrian \( SU(2) \)-orbit in \( \mathbb{C}^4 \). This example was also treated in [9, Example 5.7].

Let \( V_3 \) be the complex vector space of all complex homogeneous polynomials with two variables \( z_1, z_2 \) of degree 3. We equip \( V_3 \) with the standard Hermitian inner product such that

\[
\{v_k = \frac{1}{\sqrt{k!(3-k)!}}z_1^{3-k}z_2^k \mid k = 0, 1, 2, 3\}
\]

is a unitary basis of \( V_3 \cong \mathbb{C}^4 \cong \mathbb{R}^8 \). We know that \( V_3 \) is an irreducible unitary representation of \( SU(2) \). Now we consider the orbit of \( SU(2) \) through \( w = \frac{1}{\sqrt{2}}(v_0 + v_3) \). Then the orbit \( \Sigma = \rho_3(SU(2))w \subset S^7(1) \) is a 3-dimensional minimal Legendrian submanifold embedded in \( S^7(1) \).

**Theorem 3.1.** The special Lagrangian cone \( C \) in \( \mathbb{C}^4 \) over the minimal Legendrian orbit \( \Sigma = \rho_3(SU(2))(w) \) is not Legendrian stable, and hence not stable. Its stability-index and Legendrian-index of \( C \) are given by

\[
\begin{align*}
\text{s-ind}(C) &= 10 \\
\text{l-ind}(C) &= 11 (= 8 + 3).
\end{align*}
\]

Moreover, \( \Sigma \) satisfies

\[
m_{\Sigma}(2) = 19 > m^2 - 1 - \dim SU(2) = 12
\]

and hence \( C \) is not rigid.
We shall calculate all the eigenvalues and their multiplicities of the Laplacian of the \( SU(2) \)-orbit \( \Sigma = \rho_3(SU(2))(w) \) by the method used in [15].

Let \( G = SU(2) \) and \( g = su(2) \). Let \( \{ E_1, E_2, E_3 \} \) be a basis of \( g = su(2) \) defined by

\[
E_1 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\]

For each nonnegative integer \( n \), let \( (V_n, \rho_n) \) be an \((n + 1)\)-dimensional irreducible unitary representation of \( G = SU(2) \) as follows: Let \( V_n \) denote a complex vector space of all complex homogeneous polynomials with two variables \( z_1, z_2 \) of degree \( n \) and \( \rho_n : SU(2) \to U(V_n) \) is defined as

\[
(\rho_n \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix}) f(z_1, z_2) = f((z_1, z_2) \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix}).
\]

(3.1)

Here set

\[
v_k^{(n)} := \frac{1}{\sqrt{k!(n-k)!}} z_1^{n-k} z_2^k
\]

for each \( k = 0,1, \ldots, n \) and the standard Hermitian inner product \( \langle \cdot, \cdot \rangle \) of \( V_n \) invariant under \( \rho_n \) is defined such that \( \{ v_0^{(n)}, \ldots, v_n^{(n)} \} \) is a unitary basis of \( V^n \). Then the differential \( d\rho_n \) of the representation \( \rho_n \) is given by

\[
(d\rho_n (X) f)(z_1, z_2) = \left( \frac{\partial f}{\partial z_1}, \frac{\partial f}{\partial z_2} \right)^t X \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}.
\]

(3.2)

If we denote by \( \mathcal{D}(SU(2)) \) the complete set of all inequivalent irreducible unitary representations of \( SU(2) \), then we know

\[
\mathcal{D}(SU(2)) = \{(V_n, \rho_n) \mid n \in \mathbb{Z}, n \geq 0 \}.
\]

In \( V_3 \cong \mathbb{C}^4 \cong \mathbb{R}^8 (n = 3) \), we use the unitary basis \( v_0 = v_0^{(3)}, v_1 = v_1^{(3)}, v_2 = v_2^{(3)}, v_3 = v_3^{(3)} \). Then the orbit \( \Sigma = \rho_3(SU(2))(w) \) of \( SU(2) \) through a point \( w = \frac{1}{\sqrt{2}}(v_0 + v_3) \in S^7(1) = \{ v \in V_3 \mid \|v\| = 1 \} \) is a 3-dimensional compact minimal Legendrian submanifold embedded in \( S^7(1) \). We can see that it is a unique minimal Legendrian orbit on \( S^7(1) \) under \( \rho_3 \). Thus the minimal cone over \( \Sigma = \rho_3(SU(2))(w) \) is a special Lagrangian cone in \( \mathbb{C}^4 \). Then \( \{ \frac{1}{\sqrt{3}} E_1, \frac{1}{\sqrt{3}} E_2, \frac{1}{\sqrt{3}} E_3 \} \) is an orthonormal basis of \( g \) with respect to the induced metric from the orbit \( \rho_3(SU(2))v \subset \mathbb{C}^4 \). We denote by \( \Delta_\Sigma \) the Laplacian of \( \Sigma = G/K \) with respect to the induced metric acting smooth fuctions on \( G/K \). The isotropy subgroup

\[
K := \{ A \in G \mid \rho_3(A)w = w \}
\]

(3.3)
of $G = SU(2)$ at $w \in V_3$ is a cyclic subgroup $\mathbb{Z}_3$ of order 3 consisting of the following elements
\begin{align}
\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} e^{\sqrt{-1} \frac{2\pi}{3}} & 0 \\ 0 & e^{-\sqrt{-1} \frac{2\pi}{3}} \end{pmatrix}, \quad \begin{pmatrix} e^{-\sqrt{-1} \frac{2\pi}{3}} & 0 \\ 0 & e^{\sqrt{-1} \frac{2\pi}{3}} \end{pmatrix}
\end{align}
which is the fundamental group of $\Sigma \cong G/K$.

For each nonnegative integer $n$, we define a vector subspace $(V_n)_K$ of $V_n$ by
\begin{align}
(V_n)_K := \{ v \in V_n \mid \rho_n(A)v = v \text{ for each } A \in K \}.
\end{align}

Then by direct computations we have

**Lemma 3.1.**

(1) In case $n = 2\ell$ :
If we set $\ell = 3p + r$ for $p \in \mathbb{Z}$ with $p \geq 0$ and $r \in \mathbb{Z}$ with $0 \leq r < 3$, then $(V_n)_K$ is spanned by
\[ \{ v_k^{(n)} \mid k = \ell + 3j \quad (j = -p, \ldots, -1, 0, 1, \ldots, p) \}. \]

(2) In case $n = 2\ell + 1$ :
If $2\ell + 1 = 3p$ for $p \in \mathbb{Z}$, then $(V_n)_K$ is spanned by
\[ \{ v_k^{(n)} \mid k = 3j \quad (j = 0, 1, \ldots, p) \}. \]

If $2\ell + 1 = 3p + 1$ for $p \in \mathbb{Z}$, then $(V_n)_K$ is spanned by
\[ \{ v_k^{(n)} \mid k = 3j - 1 \quad (j = 1, 2, \ldots, p) \}. \]

If $2\ell + 1 = 3p + 2$ for $p \in \mathbb{Z}$, then $(V_n)_K$ is spanned by
\[ \{ v_k^{(n)} \mid k = 3j + 1 \quad (j = 0, 1, \ldots, p) \}. \]

By Peter-Weyl’s theorem we know
\begin{align}
C^\infty(G/K) = \bigoplus_{n \in \mathbb{Z}, n \geq 0} (V_n)_K^* \otimes V_n.
\end{align}

Here each $v \in (V_n)_K$ and each $u \in V_n$ corresponds to $f \in C^\infty(G/K)$ defined by
\[ f(aK) := \langle \langle \rho_n(a)v, u \rangle \rangle \quad (aK \in G/K). \]

Then we have
\begin{align}
(\Delta_{\Sigma}f)(aK)
= & \langle \langle \rho_n(a) \left( (d\rho_n(\frac{1}{3}E_1))^2 + (d\rho_n(\frac{1}{\sqrt{3}}E_2))^2 + (d\rho_n(\frac{1}{\sqrt{3}}E_3))^2 \right) v, u \rangle \rangle.
\end{align}

By direct computations we have the following lemmas.

**Lemma 3.2.**

\begin{align}
(\rho_n(a))^{(n)}(aK)
= & \left( (d\rho_n(\frac{1}{3}E_1))^2 + (d\rho_n(\frac{1}{\sqrt{3}}E_2))^2 + (d\rho_n(\frac{1}{\sqrt{3}}E_3))^2 \right) v_k^{(n)}

= & - \left\{ \frac{1}{9} (n-2k)^2 + \frac{2}{3} ((k+1)(n-k) + k(n-k+1)) \right\} v_k^{(n)}.
\end{align}
Lemma 3.3. All eigenvalues and their multiplicities of $\Delta_\Sigma$ are given as follows: Let $n \in \mathbb{Z}$ with $n \geq 0$.

1. In case $n = 2\ell$, if we set $\ell = 3p + r$ with nonnegative $p, r \in \mathbb{Z}$ and $0 \leq r < 3$, $\Delta_\Sigma$ has eigenvalues
   $$\frac{4}{3}\ell(\ell + 1) - 8j^2 \quad (j = -p, \cdots, -1, 0, 1, \cdots, p)$$
   and its multiplicity is $n + 1 = 2\ell + 1$.

2. In case $n = 2\ell + 1$, if $2\ell + 1 = 3p$ for an integer $p \geq 1$, then $\Delta_\Sigma$ has
   eigenvalues
   $$(p - 2j)^2 + 2((3j + 1)(p - j) + j(3p - 3j + 1)) \quad (j = 0, 1, \cdots, p)$$
   and its multiplicity is $n + 1 = 2\ell + 2$.

3. In case $n = 2\ell + 1$, if $2\ell + 1 = 3p + 1$ for an integer $p \geq 2$, then $\Delta_\Sigma$ has
   eigenvalues
   $$(p - 2j + 1)^2 + 2(j(3p - 3j + 2) + (3j - 1)(p - j + 1)) \quad (j = 1, \cdots, p)$$
   and its multiplicity is $n + 1 = 2\ell + 2$.

4. In case $n = 2\ell + 1$, if $2\ell + 1 = 3p + 2$ for an integer $p \geq 1$, then $\Delta_\Sigma$ has
   eigenvalues
   $$(p - 2j)^2 + \frac{2}{3}((3j + 2)(3p - 3j + 1) + (3j + 1)(3p - 3j + 2)) \quad (j = 0, 1, \cdots, p)$$
   and its multiplicity is $n + 1 = 2\ell + 2$.

From Lemma 3.3 all eigenvalues of $\Delta_\Sigma$ not greater than $\lambda \leq 2m = 8$ and their multiplicities are given as follows:

1. For $n = 2$, $\ell = 1$ and $j = 0$, the eigenvalue is $\frac{8}{3} (\alpha = \sqrt{33}/3 - 1)$ and its multiplicity is 3.
2. For $n = 3$, $\ell = 1, p = 1$ and $j = 0$, the eigenvalue is $3 (\alpha = 1)$ and its multiplicity is 4.
3. For $n = 3$, $\ell = 1, p = 1$ and $j = p$, the eigenvalue is $3 (\alpha = 1)$ and its multiplicity is 4.
4. For $n = 4$, $\ell = 2, p = 0$ and $j = 0$, the eigenvalue is $8 (\alpha = 2)$ and its multiplicity is 5.
5. For $n = 6$, $\ell = 3, p = 1$ and $j = -1$, the eigenvalue is $8 (\alpha = 2)$ and its multiplicity is 7.
6. For $n = 6$, $\ell = 3, p = 1$ and $j = 1$, the eigenvalue is $8 (\alpha = 2)$ and its multiplicity is 7.
7. Otherwise all other eigenvalues are greater than 8.

Thus we have
$$m_\Sigma(0) = 1, \quad m_\Sigma(\frac{\sqrt{33}}{3} - 1) = 3,$$
$$m_\Sigma(1) = 4 + 4 = 8, \quad m_\Sigma(2) = 5 + 7 + 7 = 19.$$
and

\[ N_Σ(2) = m_Σ(0) + m_Σ(\frac{\sqrt{33}}{3} - 1) + m_Σ(1) + m_Σ(2) = 31. \]

Therefore we obtain

\[ s-ind(C) = N_Σ(2) - b^0(Σ) - m^2 - 2m + 1 + \dim G_Σ = 10. \]

and

\[ l-ind(C) = m_Σ(\frac{\sqrt{33}}{3} - 1) + m_Σ(1) = 11 > 8. \]

and hence \( C \) is not Legendrian-stable. And we obtain

\[ m_Σ(2) = 19 > 12 = 4^2 - 1 - \dim SU(3) = m^2 - 1 - \dim G_Σ. \]

and hence \( C \) is not rigid. Therefore we obtain Theorem 3.1.

And by using the formula (1.1) we also see the following.

**Theorem 3.2.** For this minimal Legendirian orbit \( Σ = \rho_3(SU(2))w \), \( C' = C' \setminus \{0\} \) has nonzero homogeneous harmonic function of order \( α \) for some \( α \) with \( 0 < α < 1 \) and there is no nonzero homogeneous harmonic function on \( C' \) of order \( α \) for any \( α \) with \( 1 < α < 2 \).

Next we consider the Hopf fibration \( π : S^7(1) \to CP^3 \) from \( S^7(1) \subset V_3 \cong C^4 \) onto the 3-dimensional complex projective space \( CP^3 \) with the Fubini-Study metric of constant holomorphic sectional curvature 4. We denote also by \( ρ_3 \) the action of \( SU(2) \) on \( CP^3 \) induced by \( π \) from the representation \( ρ_3 \) of \( SU(2) \) on \( V_3 \cong C^4 \). By the projection of the minimal Legendrian orbit \( ρ_3(SU(2))w \), we obtain a minimal Lagrangian orbit \( L = ρ_3(SU(2))[w] \) on \( CP^3 \) through \( [w] = Cw \). It was also treated in [5] from the viewpoint of momentum maps. Then the isotropy subgroup

\[ K' := \{ A \in SU(2) \mid ρ_3(A)[w] = [w] \} \]

of \( SU(2) \) at \( [w] \in CP^3 \) is a finite subgroup of order 12 consisting of the following elements

\[ (3.10) \begin{pmatrix} a & 0 \\ 0 & \bar{a} \end{pmatrix}, \quad \begin{pmatrix} 0 & -\bar{b} \\ b & 0 \end{pmatrix} \]

where \( a, b \in \mathbb{C} \) with \( |a| = |b| = 1 \) and \( a^6 = 1, \ b^6 = -1 \). Let

\[ (3.11) \quad (V_n)_{K'} := \{ v \in V_m \mid ρ_m(A)v = v \ \text{for each} \ A \in K' \}. \]

Note that \( K \subset K' \) and thus \( (V_n)_{K'} \subset (V_n)_K \). Then by checking the results of Lemma 3.1 on \( (V_n)_K \) we can show

**Lemma 3.4.** (a) \((V_n)_{K'} \neq \{0\}\) if and only if \( n = 2\ell \) for some integer \( \ell \in \mathbb{Z} \)

satisfying the condition that \( \ell \) is odd with \( \ell \geq 3 \), or that \( \ell \) is even with \( \ell \geq 0 \),
(b) If $n = 2\ell$ and $\ell$ is odd with $\ell \geq 3$, setting $\ell = 3p + r$ for $0 \leq p \in \mathbb{Z}$ and $0 \leq r < 3$, then
\[
\left\{ v^{(2\ell)}_{\ell + 3j} - v^{(2\ell)}_{\ell - 3j} \mid j = 1, \cdots, p \right\}
\]
is a basis of $(V_n)_{K'}$.

(c) If $n = 2\ell$ and $\ell$ is even with $\ell \geq 0$, setting $\ell = 3p + r$ for $0 \leq p \in \mathbb{Z}$ and $0 \leq r < 3$, then
\[
\left\{ v^{(2\ell)}_{\ell + 3j} + v^{(2\ell)}_{\ell - 3j} \mid j = 0, 1, \cdots, p \right\}
\]
is a basis of $(V_n)_{K'}$.

Now by using Lemma 3.2 we can determine all eigenvalues for the Laplacian $\Delta'$ of $L$ on functions.

**Lemma 3.5.** All eigenvalues and their multiplicities of $\Delta'$ are given as follows: Let $n = 2\ell$ for $\ell \in \mathbb{Z}$ with $\ell \geq 0$.

1. In the case when $\ell$ is odd and $\ell \geq 3$, if we set $\ell = 3p + r$ with nonnegative $p, r \in \mathbb{Z}$ and $0 \leq r < 3$, $\Delta'$ has eigenvalues
\[
4 \frac{\ell(\ell + 1)}{3} - 8j^2 \quad (j = 1, \cdots, p)
\]
and its multiplicity is $n + 1 = 2\ell + 1$.

2. In the case when $\ell$ is even and $\ell \geq 0$, if we set $\ell = 3p + r$ with nonnegative $p, r \in \mathbb{Z}$ and $0 \leq r < 3$, $\Delta'$ has eigenvalues
\[
4 \frac{\ell(\ell + 1)}{3} - 8j^2 \quad (j = 0, 1, \cdots, p)
\]
and its multiplicity is $n + 1 = 2\ell + 1$.

By Lemma 3.5 we can determine the first eigenvalue of $\Delta'$ and its multiplicity as follows:

**Lemma 3.6.** (1) If $n = 4$, $\ell = 2$, $p = 0$ and $j = 0$, then the eigenvalue is 8 and its multiplicity is 5.

(2) If $n = 6$, $\ell = 3$, $p = 1$ and $j = 1$, then the eigenvalue is 8 and its multiplicity is 7.

(3) Otherwise all other eigenvalues are greater than 8.

Hence we obtain that the first eigenvalue of $\Delta'$ is 8 and its multiplicity is $12 = 4^2 - 1 - \dim(SU(2))$. Therefore we conclude

**Corollary 3.1.** $\pi(\Sigma) = \rho_3(SU(2))[w]$ is a 3-dimensional compact Hamiltonian stable minimal Lagrangian submanifold embedded in $\mathbb{C}P^3$ which does not have parallel second fundamental form. Moreover its null space is exactly the span of the normal projections of Killing vector fields on $\mathbb{C}P^3$.

**Remark.** This example gives a negative answer to the second problem in [1, p506]. Very recently it was also obtained independently by Lucio Bedulli and Anna Gori in their paper: A Hamiltonian stable minimal Lagrangian submanifolds of projective spaces with non-parallel second fundamental form.
REFERENCES


