Spherical dust collapse from the perspective of loop quantum gravity

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Contents

Contents

Quantum gravity

We have yet no completed theory of quantum gravity. Quantum gravity should answer the following questions.

- Spacetime singularities (big bang and gravitational collapse)
- Black hole thermodynamics, Hawking evaporation, information loss problem
- How does the law of gravity change from low-energy scale to the Planck scale?

We here take loop quantum gravity. Loop quantum gravity is

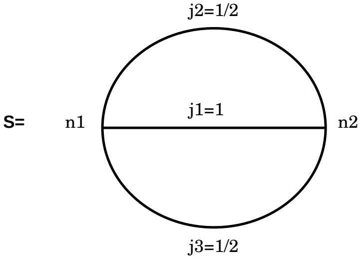
- Nonperturbative
- Background independent
- Canonical quantisation
- No ultraviolet divergence
- Not intended to be a unified theory

Spacetime singularity

- Spacetime singularity: a place where quantities which are used to measure the gravitational field become infinite More accurately, a point which is in the boundary of the maximally extended spacetime manifold and is not infinity.
- Classical general relativity breaks down at the singularity.
- Cosmic censorship (Penrose 1969) states that singularities are hidden behind horizons in physical spacetimes. But we have many naked singularities...
- Censored or not, spacetime singularities should be dealt with in a regular manner.

What is loop quantum gravity?

- 3 dimensional space + time
- Einstein-Hilbert action
- Canonical conjugate pair: tetrad E_i^a and connection A_a^i
- Hamiltonian H expressed in terms of curvature F_{ab}^i and volume V
- Quantum state space constructed by spin-network states
- holonomy $\hat{h}$, area operator $\hat{S}$, volume operator $\hat{V}$
- Hamiltonian operator $\hat{H}$ quantised in terms of $\hat{h}$ and $\hat{V}$



Contents

Triad

- Tetrad:

$$\begin{aligned} e^I &= e^I_\mu dx^\mu; \\ \eta^{IJ} &= e^I_\mu e^J_\nu g^{\mu\nu}; \\ g_{\mu\nu} &= e^I_\mu e^J_\nu \eta_{IJ}. \end{aligned}$$

- We take $e^0_a = 0$ gauge. $q_{ab} = g_{ab}$ $(a, b = 1, 2, 3)$
- Densitized triad:

$$E^a_i = \frac{1}{2}\epsilon_{ijk}\epsilon^{abc}e^j_b e^k_c$$

or

$$E^a_i E^b_j \delta^{ij} = (\det q_{cd}) q^{ab},$$

where $i, j = 1, 2, 3$.
- Notation for an antisymmetric tensor A_{ij}

$$A^i = \epsilon^{ijk} A_{jk}$$

Connection

- Spin connection:

$$
\begin{aligned}
de^I + \omega^I_J \wedge e^J &= 0, \\
de^i + \Gamma^i_j \wedge e^j &= 0.
\end{aligned}
$$

- Ashtekar (Barbero) connection:

$$A^i = \omega^i + \gamma \omega^{0i},$$

where γ is the only one unspecified parameter in the theory. This is called the Immirzi parameter.

- Extrinsic curvature

$$K^i \equiv A^i - \Gamma^i$$

- Poisson bracket

$$\{A^i_a(x), E^b_j(y)\} = 8\pi\gamma G \delta^b_a \delta^i_j \delta^3(x,y)$$

Hamiltonian

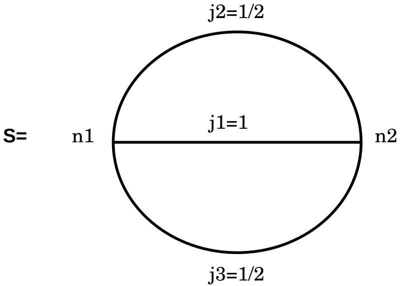
- Curvature: $F^i_{ab} = \partial_a A^i_b - \partial_b A^i_a + \epsilon^i_{jk} A^j_a A^k_b$
- Hamiltonian (unsmeared form):
  $H = (F^{ij}_{ab} + (\gamma^2 + 1) K^i_{[a} K^j_{b]}) E^a_i E^b_j / (\sqrt{\det E})$
- Volume: $\mathrm{V} = \int d^3x \sqrt{\det E}$
- Hamiltonian (K^2 term neglected):

$$H[N] = \int N \mathrm{tr}(F \wedge \{\mathrm{V}, A\}),$$

where N is the lapse function.

Spin network

- Spin network (Γ, j_l, i_n):



- Spin network state

$$\Psi_S[A] = \left(\bigotimes_l R^{j_l}(H[A,\gamma_l]) \right) \cdot \left(\bigotimes_n i_n \right),$$

where $H[A,\gamma_l]$ is a holonomy.

Smeared operators

- $A^i_a(x)$ and $E^a_i(x) = -i\hbar\frac{\delta}{\delta A^i_a(x)}$ are not well defined.
- Holonomy $H(A,\gamma)$ and Flux $E_i(\mathcal{S})$

$$\begin{aligned} H(A,\gamma) &= \mathcal{P}\exp\int_\gamma A, \\ E_i(\mathcal{S}) &\equiv -i\hbar\int_{\mathcal{S}} dS^2 n_a \frac{\delta}{\delta A^i_a(x(\sigma))} \end{aligned}$$

- Area operator and volume operator

$$\mathrm{A}(\mathcal{S})|\mathrm{S}\rangle = \hbar\sum_i \sqrt{j_i(j_i+1)}|\mathrm{S}\rangle$$

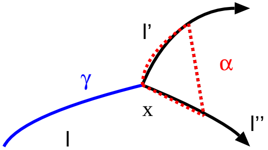
$$\mathrm{V}(\mathcal{R})|\mathrm{S}\rangle = \lim_{\epsilon\to 0}\sum_{n\in S\cap\mathcal{R}}\sqrt{|W_{I^n_\epsilon}|}|\mathrm{S}\rangle$$

Hamiltonian

- Discretized Hamiltonian

$$
\begin{aligned}
H|\mathrm{S}\rangle &= -\frac{i}{\hbar}\sum_{n\in S} N_n \sum_{l,l',l''} \epsilon_{ll'l''} \\
&\times \ \mathrm{tr}\left(h_{\gamma^{-1}_{x_n,l}} h_{\alpha_{x_n,l',l''}} [\mathrm{V}(\mathcal{R}_n), h_{\gamma_{x_n,l}})]\right) |\mathrm{S}\rangle,
\end{aligned}
$$

where N_n is the lapse, h denotes a holonomy.



Contents

Contents

Spherially symmetric spacetime

- Metric $\mathrm{d}\Omega^2 = \mathrm{d}\vartheta^2 + \sin^2\vartheta \mathrm{d}\varphi^2$

$$\mathrm{d}s^2 = -N^2\mathrm{d}t^2 + L^2(\mathrm{d}x + N^x\mathrm{d}t)^2 + R^2\mathrm{d}\Omega^2$$

- Spatial metric and densitised triad

$$\mathrm{d}s^2 = \frac{E^{\varphi 2}}{|E^x|}dx^2 + |E^x|\mathrm{d}\Omega^2$$

- Canonical pairs (γ: Immirzi parameter)

$$\begin{aligned} \{A_x(x), E^x(y)\} &= 2\gamma G\delta(x,y), \\ \{\gamma K_\varphi(x), E^\varphi(y)\} &= \gamma G\delta(x,y) \end{aligned}$$

The LTB system interms of triad and connection

- Classical LTB metric (marginally bound)

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + R'^2\mathrm{d}x^2 + R^2\mathrm{d}\Omega^2$$

- "LTB relation"

$$E^\varphi(x) = \frac{1}{2}|E^x(x)|' \quad \text{and} \quad K'_\varphi = K_x \mathrm{sgn} E^x$$

- Hamiltonian (unsmeared)

$$\begin{aligned} H_{\mathrm{grav}} &= -\frac{1}{2G}\left(\frac{K_\varphi^2 E^\varphi}{\sqrt{|E^x|}} + 2K_\varphi K_x\sqrt{|E^x|}\right), \\ H_{\mathrm{dust}} &= \frac{F'(x)}{2G}, \end{aligned}$$

where $F(x)/2$ is the conserved dust mass.

Contents

Spherically symmetric loop quantum gravity

- $T_{g,k,\mu}$: An orthonormal basis of gauge-invariant states

$$T_{g,k,\mu} = \prod_{e\in g} \exp\left(\frac{1}{2} i k_e \int_e (A_x + \eta')\mathrm{d}x\right) \prod_{v\in g} \exp(i\mu_v \gamma K_\varphi(v))$$

with integer labels k_e and positive real labels μ_v on edges e and vertices v, respectively, forming a finite graph g in the 1-dimensional radial line.

- Eigenvalues

$$\begin{aligned} \hat{E}^x(x) T_{g,k,\mu} &= \gamma \ell_\mathrm{P}^2 \frac{k_{e^+(x)} + k_{e^-(x)}}{2} T_{g,k,\mu}, \\ \int_{\mathcal{I}} \hat{E}^\varphi T_{g,k,\mu} &= \gamma \ell_\mathrm{P}^2 \sum_{v\in\mathcal{I}} \mu_v T_{g,k,\mu}, \end{aligned}$$

where where $\ell_\mathrm{P}^2 = G\hbar$ is the Planck length squared and $e^\pm(x)$ denote the neighboring edges to a point x.

Inverse triad effects

- Inverse triad can be replaced as

$$\int_{\mathcal{I}} \frac{\widehat{E^{\varphi}\mathrm{sgn}(E^x)}}{\sqrt{|E^x|}} = \frac{-i}{2\pi\gamma G\hbar}\mathrm{tr}(\tau_3 h_x[h_x^{-1}, \hat{V}])$$

- Eigenvalues

$$\sim \left(\sqrt{|k_{e^+} + k_{e^-} + 1|} - \sqrt{|k_{e^+} + k_{e^-} - 1|}\right)$$

Inverse triad correction

- Correction function

$$\alpha(E^x) :== 2\frac{\sqrt{|E^x+\gamma\ell_{\rm P}^2/2|}-\sqrt{|E^x-\gamma\ell_{\rm P}^2/2|}}{\gamma\ell_{\rm P}^2}\sqrt{|E^x|}$$

- Corrected Hamiltonian

$$H^{(I)}_{\rm grav} = -\frac{1}{2G}\left(\frac{\alpha(E^x)}{\sqrt{|E^x|}}K_\varphi^2E^\varphi+2K_\varphi K_x\sqrt{|E^x|}\right).$$

$$H^{(II)}_{\rm grav} = -\frac{1}{2G}\frac{\alpha(E^x)}{\sqrt{|E^x|}}(K_\varphi^2E^\varphi+2K_\varphi K_xE^x).$$

Consistent formulation ver. 1

- Corrected LTB relation:

$$(E^x)' = 2f(E^x)E^\varphi \quad , \quad K_\varphi' = f(E^x)K_x$$

- Hamiltonian constraint and evolution

$$\dot{R}^2 R'(\alpha(R) - 1) + 2R\dot{R}\dot{R}' + \dot{R}^2 R' = f(R)F'$$
$$2R\ddot{R} + \dot{R}^2 + (\alpha(R) - 1)\dot{R}^2 = 0\,.$$

- Correction functions: $\alpha(R)$ and $f(R)$

$$\frac{df(R)}{dR} = (1 - \alpha(R))\frac{f}{R}$$

Correction functions

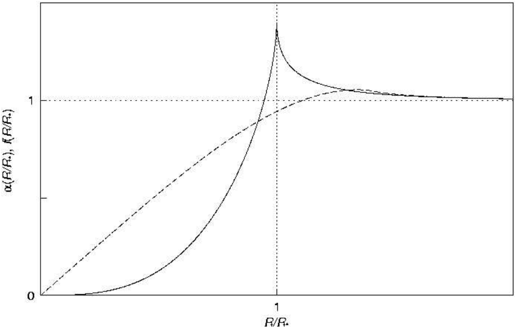


Figure: $\alpha(R)$ (solid) and $f(R)$ (dashed), where $R_* := \sqrt{\gamma/2}\ell_{\rm P}$.

$$\alpha(R) = 2\frac{\sqrt{|R^2+\gamma\ell_{\rm P}^2/2|}-\sqrt{|R^2-\gamma\ell_{\rm P}^2/2|}}{\gamma\ell_{\rm P}^2}R$$

Consistent formulation ver. 2

- Corrected LTB relation:

$$E^{\varphi} = \frac{1}{2\alpha}(E^{x})' \quad , \quad \alpha K_{x} = K_{\varphi}'$$

- Hamiltonian constraint

$$\left(\frac{\dot{R}^2 R}{\alpha^2}\right)' - F' = 0$$

- Evolution equation

$$2R\ddot{R} + \dot{R}^2 = 2\frac{\mathrm{d}\log\alpha}{\mathrm{d}\log R}\dot{R}^2$$

Contents

holonomy effect

- Holonomy is a periodic funtion of the connection. The effects may be included by the replacements:

$$K_\varphi \to (\gamma\delta)^{-1}\sin(\gamma\delta K_\varphi), \quad K_x \to (\gamma\ell_0)^{-1}\sin(\gamma K_x \ell_0),$$

where ℓ_0 and δ parametrize the discreteness of the state along the radial direction and on the sphere, respectively.

- Corrected Hamiltonian

$$H_{\rm grav}^{(III)} = -\frac{1}{2G}\left(\frac{\sin^2(\gamma\delta K_\varphi)}{\gamma^2\delta^2}\frac{E^\varphi}{\sqrt{|E^x|}} + 2\frac{\sin(\gamma\delta K_\varphi)}{\gamma\delta}\frac{\sin(\gamma K_x\ell_0)}{\gamma\ell_0}\sqrt{|E^x|}\right).$$

Consistent formulation ver. 3

- Corrected LTB relation

$$(E^x)' = 2g(K_\varphi)E^\varphi \quad , \quad K_\varphi' = g(K_\varphi)K_x,$$

where $g(K_\varphi) = \cos^4(\gamma\delta K_\varphi/2)$ is required.

- Hamiltonian constraint

$$\begin{aligned} & 4\dot{R}^2 R'\sqrt{1-\gamma^2\delta^2\dot{R}^2} + 8R\dot{R}\dot{R}' \\ = \;\; & F'\left(1+\sqrt{1-\gamma^2\delta^2\dot{R}^2}\right)^2\sqrt{1-\gamma^2\delta^2\dot{R}^2}\,. \end{aligned}$$

- Evolution equation

$$2R\ddot{R} + \dot{R}^2\sqrt{1-\gamma^2\delta^2\dot{R}^2} = 0\,.$$

Contents

Contents

Results (1)

- A bounce does not occur because $\dot{R} = 0$ is impossible.
- The vacuum static solution is nontrivial.

$$\mathrm{d}s^{2(I)} = -\mathrm{d}t^2 + \frac{1}{f(R)^2}\mathrm{d}R^2 + R^2\mathrm{d}\Omega^2$$

- No Friedmann or Oppenheimer-Snyder solution because the homogenous expansion $R(t,x) = a(t)x$ is not a solution.
- No self-similar (or homothetic) solution is expected because of ℓ_{P} in the equation.

Results (2)

- Effective mass

$$m^{(I)} = \frac{1}{2}R(1-f^2+\dot{R}^2) = m_{\rm class} - \frac{1}{2}R(f^2-1)$$

- Effective density

$$\epsilon^{(I)} = \frac{m^{(I)\prime}}{4\pi G R^2 R'}$$

- Small x expansion:
  - If $F(x) = F_3 x^3 + \cdots$, $R(t,x) = R_1(t)x + \cdots$, we have $\dot{R}_1 = 0$. The central density is constant as long as expanded regularly.
  - We might allow for $F(x) = F_2 x^2 + \cdots$ for a finite effective density. Then, we have a central singularity in a finite time.

Contents

Classical collapse

- Uniform model

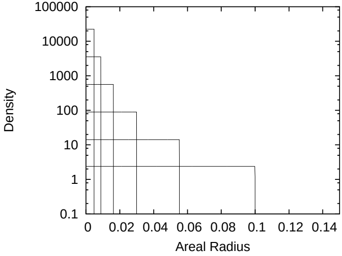


Figure: Oppenheimer-Snyder

- Quadratic model

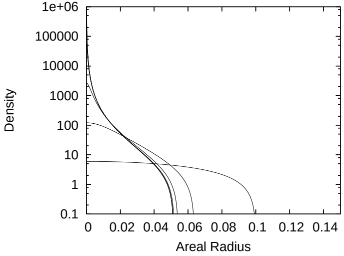


Figure: LTB: naked singularity

Inverse triad correction (1)

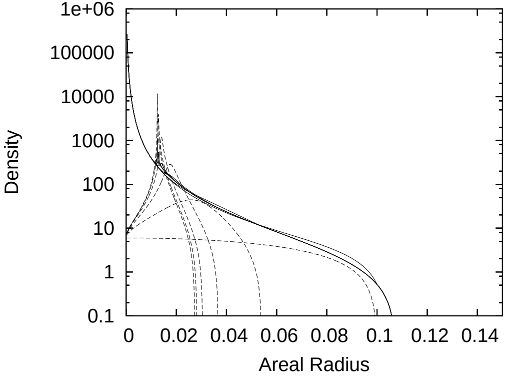


Figure: Conserved density (dashed) and effective density (solid): Quadratic model with $R_s = 0.1, M = 0.01$ ($R_* = \sqrt{\gamma/2}\ell_P \sim 0.25$)

Inverse triad correction (2)

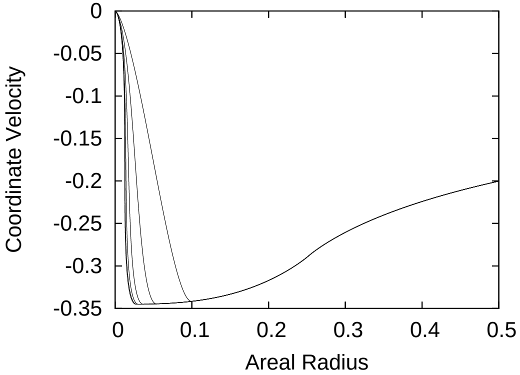


Figure: Velocity profile: Quadratic model with $R_s = 0.1, M = 0.01$ $(R_* = \sqrt{\gamma/2}\ell_{\mathrm{P}} \sim 0.25)$

Summary of numerical results

- The gravitational collapse is strongly slowed down for $R \lesssim R_*$.
- There appears a density spike for $0 < R \lesssim R_*$. This will correspond to shell-crossing singularity and hence gravitationally weak.
- The slow-down implies repulsive effects of quantum gravity but this is not so strong that the collapse turns to bounce.
- The qualitative feature does not change for the second version formulation or even if we allow for $F(x) = F_2 x^2 + \cdots$.

Contents

Summary

- Loop quantum gravity is a promising candidate for successful quantum gravity.
- Inverse triad correction and honolomy correction are taken into account in the marginally bound LTB spacetime.
- Formulations are obtained for both effects, where the Hamiltonian constraint is consistent with the evolution equation.
- The dynamics with the inverse triad correction is studied.
  - No bounce, no homogenous soloution, nontrivial vacuum solution and no self-similar solution
  - The centre will be frozen or collapse to singularity depending on whether the conserved or effective density is physical.
  - The collapse only near the centre is slowed down, which results in shell-crossing.
  - This case gives a caution to loop quantum cosmology.