Dynamics of p-brane system

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References:

(1) G.W. Gibbons, H. Lu, C.N. Pope; Phys.Rev.Lett.94:131602,2005

(2) D.Kastor & J. Traschen soltuion; Phys.Rev. D 47 (1993) 5370

(3) H. Kodama & K. Uzawa; JHEP 0507:061,2005

(4) H. Kodama & K. Uzawa; JHEP 0603:053,2006

(5) P. Binetruy, M. Sasaki, K. Uzawa, arXiv:0712.3615

[1] Introduction

▪Analysis of the early universe

Construction of the cosmological model that explains the observational facts on the basis of fundamental theory (String or SUGRA theory)

⇓

It is necessary to obtain the dynamical solution of the field equations (including the Einstein equations) , and to extract information of the cosmological behavior.

dynamics of 4d or internal space, symmetry breaking (SUSY, …)

Cosmological solution in the higher dimensional theory

Higher dimensional gravity
(e.g. Brane world model)

- Inflation model
- Stabilization of the scale of internal space

complementary

String theory

- Low energy SUGRA
- Dynamical solution of Einstein equation

cosmological model

Construction of the cosmological scenario

Overview

☆ **4 dimensional Gravity**

- Charged BH solution (RN)
  (Reissner & Nordstrøm)

- Dynamical solution in Einstein-Maxwell theory (Kastor & Traschen)

★ **Higher dimensional Gravity**

- p-brane solution of SUGRA
  (Horowitz & Strominger)

- Dynamical p-brane solution
  (Gibbons, Binetruy, Kodama, Sasaki, Uzawa)

- Intersecting brane
  (Guven, Papadopoulos & Townsend, Ohta)

- Dynamical solution of intersecting brane
  (Binetruy, Sasaki, Uzawa)

[2] Dynamical solution of 4-dim gravity
(D.Kastor & J. Traschen ; Phys.Rev. D 47 (1993) 5370)

・4-dim Einstein-Maxwell system + cosmological constant

◇ 4-dimensoinal action :

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\left(R - \frac{1}{4}F_{MN}F^{MN} - 2\Lambda\right)$$

☆ Solution of field equations :

$$ds^2 = -h^{-2}(t,y)dt^2 + h^2(t,y)\delta_{ij}dy^i dy^j$$

$$\begin{aligned} h(t,y) &= at + b + \sum_l \frac{M_l}{|\vec{y}-\vec{y}_l|}, \\ F_2 &= d(h^{-1}) \wedge dt, \\ a &= \pm\sqrt{\frac{\Lambda}{3}} \end{aligned}$$

⇒Time dependent solution

☆ Cosmological dynamics ⇒ linear function of time

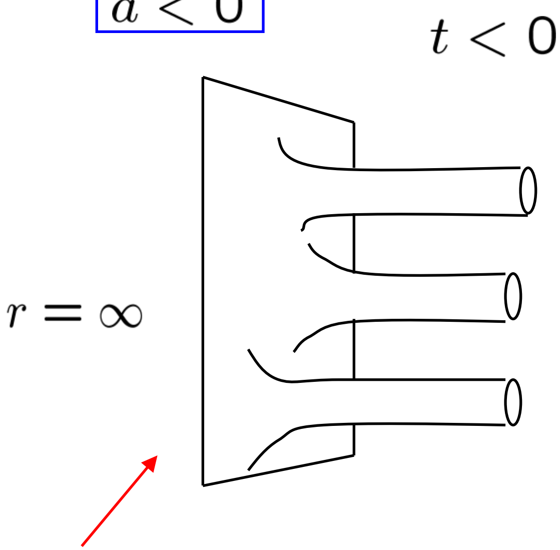
◆ Dynamics of the spatial background

$$ds^2(X_3) = (at + \sum_l M_l\,|\vec{y} - \vec{y}_l|^{-1})^2 \delta_{ij} dy^i dy^j$$

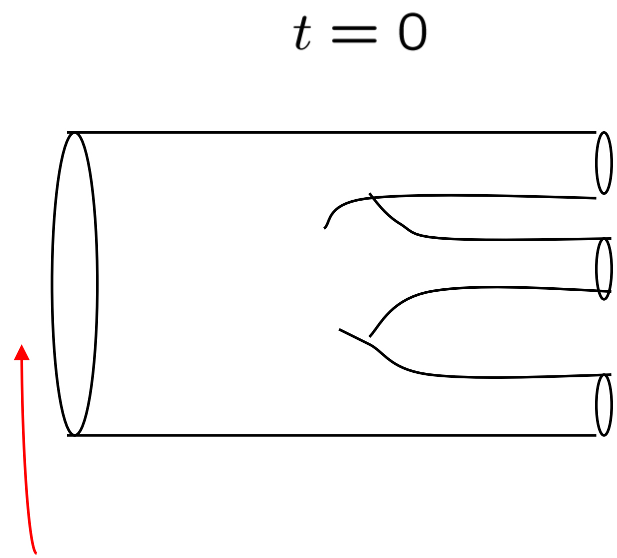
Near the $\vec{y} = \vec{y}_l$ the metric denotes the geometry for cylinder.

$$ds^2(X_3) \approx M^2 r^{-2} dr^2 + M^2 d\Omega_2^2$$

$a < 0$



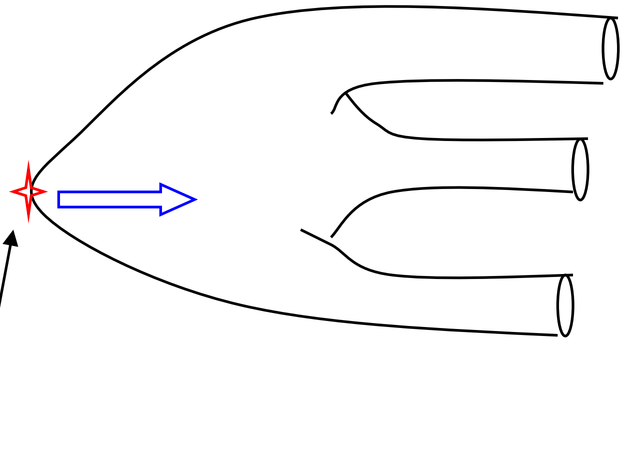
Spatial surface are asymptotically flat.



Spatial infinity asymptotically cylindrical.

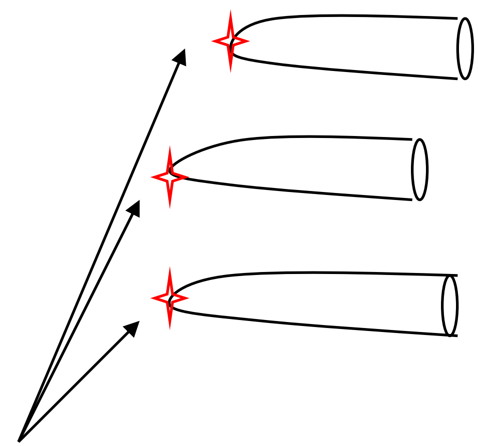
$$z = \ln r, \quad \Rightarrow \quad ds^2(X_3) = dw^2 + w^2 d\theta^2 + dz^2$$

$0 < t \ll -\sum_l \frac{M_l}{ac}$ $\quad c$: typical coordinate distance from the center



Singularity moves in from spatial infinity.

$t \gg -\sum_l \frac{M_l}{ac}$



Singularity splits and eventually surrounds each of the throats individually.

[3] Dynamical solutoin of p-brane system

(G.W. Gibbons, H. Lu, C.N. Pope Phys.Rev.Lett.94:131602,2005)
(P. Binetruy, M. Sasaki, K. Uzawa, arXiv:0712.3615)

Let us consider the case of an arbitrary p-brane background

$$S = \frac{1}{2\kappa^2}\int\left(R*\mathbf{1}_D - \frac{1}{2}d\phi\wedge *d\phi - \frac{1}{2}e^{-c\phi}F_{(p+2)}\wedge *F_{(p+2)}\right),$$
$$c^2 = 4 - \frac{2(p+1)(D-p-3)}{D-2}.$$

The dynamical background of the p-brane can be written by

$$ds^2 = h^{-(D-p-3)/(D-2)}q_{\mu\nu}dx^\mu dx^\nu + h^{(p+1)/(D-2)}u_{ij}dy^i dy^j,$$
$$e^\phi = h^{-c/2}, \qquad h(x,y) = h_0(x) + h_1(y),$$
$$F_{(p+2)} = d(h^{-1})\wedge\Omega(\mathrm{X}), \qquad \Omega(\mathrm{X}) = \sqrt{-q}\,dx^0\wedge dx^1\wedge\cdots\wedge dx^p$$

In the $c \neq 0$ case, the field equations are reduced to

$$R_{\mu\nu}(\mathrm{X}) = 0, \qquad R_{ij}(\mathrm{Y}) = 0,$$
$$D_\mu D_\nu h_0 = 0, \qquad \triangle_{\mathrm{Y}} h_1 = 0$$

	0	1	$\cdots$	p	$p+1$	$\cdots$	D
p-brane	○	○	○	○			

- Internal and external space are Ricci flat.

From the assumption, the Einstein equations are

$$R_{\mu\nu}(\mathrm{X}) - h^{-1}D_\mu D_\nu h - \frac{a}{2}h^{-1}q_{\mu\nu}\left(\triangle_{\mathrm{X}}h + h^{-1}\triangle_{\mathrm{Y}}h\right) = 0,$$
$$R_{ij}(\mathrm{Y}) - \frac{b}{2}u_{ij}\left(\triangle_{\mathrm{X}}h + h^{-1}\triangle_{\mathrm{Y}}h\right) = 0,$$
$$\partial_\mu\partial_i h = 0,$$

Last equation leads

$$\partial_\mu\partial_i h = 0, \quad \Rightarrow \quad h(x,y) = h_0(x) + h_1(y).$$

The gauge field equation yields

$$d\left[e^{-c\phi} * F_{(p+2)}\right] = -d\left[\partial_i h(*_{\mathrm{Y}}dy^i)\right] = 0,$$

Under the assumptions, the scalar field equation reads

$$\frac{c}{2}h^{-b}\left(\triangle_{\mathrm{X}}h_0 + h^{-1}\triangle_{\mathrm{Y}}h_1\right) = 0,$$

Thus, unless the parameter c is zero, the warp factor should satisfy the equations

$$\triangle_{\mathrm{X}}h_0 = 0, \quad \triangle_{\mathrm{Y}}h_1 = 0$$

In the $c=0$ case, the field equations are reduced to

$$R_{\mu\nu}(\mathrm{X})=0, \qquad R_{ij}(\mathrm{Y})=\frac{b}{2}\lambda u_{ij}(\mathrm{Y}),$$
$$D_\mu D_\nu h_0=\lambda q_{\mu\nu}(\mathrm{X}), \qquad \triangle_{\mathrm{Y}} h_1=0.$$

◆ For example, in the case of $q_{\mu\nu}=\eta_{\mu\nu}, \quad u_{ij}=\delta_{ij}$
the solution is

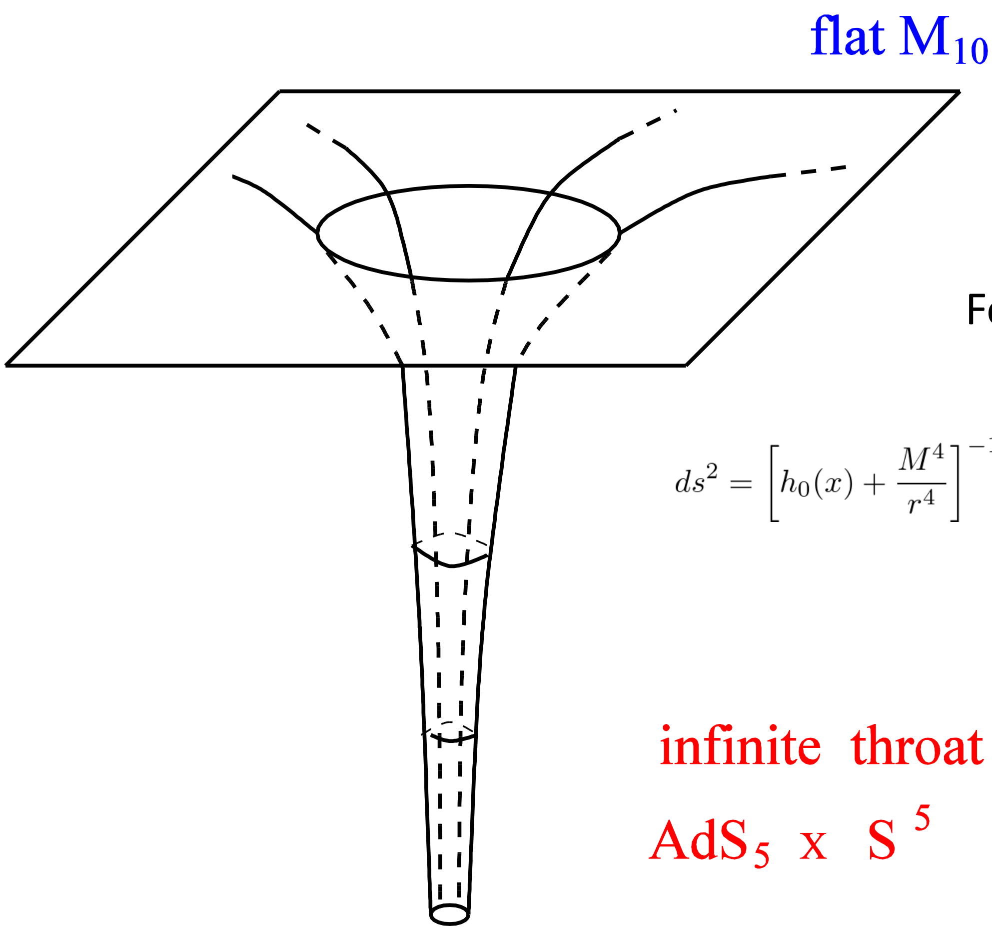
$$(1)\ c\neq 0 \quad : \qquad h_0(x)=c_\mu x^\mu+\tilde{c}, \quad h_1(y)=\sum_l \frac{M_l}{|y^i-y^i_l|^{D-p-3}}$$

(G.W. Gibbons, H. Lu, C.N. Pope; Phys.Rev.Lett.94:131602,2005）

$$(2)\ c=0 \quad : \qquad h_0(x)=\frac{\lambda}{2}x^\mu x_\mu+c_\mu x^\mu+\tilde{c}, \quad h_1(y)=\sum_l \frac{M_l}{|y^i-y^i_l|^{D-p-3}}$$

$c_\mu, \quad \tilde{c}$: constant parameters

$$
\begin{aligned}
ds^2 &= h^{-(D-p-3)/(D-2)}\eta_{\mu\nu}dx^\mu dx^\nu + h^{(p+1)/(D-2)}(dr^2 + r^2 d\Omega^2), \\
h(t,r) &= c_1 t + c_2 + M r^{-D+p+3}
\end{aligned}
$$



For example, $D = 10, \quad p = 3$

$$
ds^2 = \left[h_0(x) + \frac{M^4}{r^4}\right]^{-1/2} q_{\mu\nu}dx^\mu dx^\nu + \left[h_0(x) + \frac{M^4}{r^4}\right]^{1/2} \left(dr^2 + r^2 d\Omega^2\right)
$$

infinite throat

$AdS_5 \times S^5$

★ Kaluza-Klein compactification

- "product type" ansatz

$$
\begin{aligned}
ds^2 &= a(x,y)q_{\mu\nu}dx^\mu dx^\nu + b(x,y)u_{ij}dy^i dy^j , \\
a(x,y) &= a_0(x)a_1(y), \qquad b(x,y) = b_0(x)b_1(y)
\end{aligned}
$$

- Kastor & Traschen
- Dynamical p-brane

$$
\begin{aligned}
ds^2 &= h^a(x,y)q_{\mu\nu}dx^\mu dx^\nu + h^b(x,y)u_{ij}dy^i dy^j , \\
h(x,y) &= h_0(x) + h_1(y)
\end{aligned}
$$

★ Dynamics of 4-dimensional spacetime (scalar field: constant)

◇ Let us consider the case $q_{\mu\nu} = \eta_{\mu\nu}, \; u_{ij} = \delta_{ij}$ in more detail.

In this case, the solution for the warp factor h can be obtained explicitly as

$$h(x,y) = \frac{\lambda}{2} x^\mu x_\mu + a_\mu x^\mu + b + \sum_l \frac{M_l}{|\vec{y}^i - \vec{y}^i_l|^4}$$

In the following, we consider the simple case $\lambda = 0, \; h_0 = h_0(t)$

If we introduce a new time coordinate τ by

$$\frac{\tau}{\tau_0} = (at)^{3/4}, \qquad \tau_0 = \frac{4}{3a}$$

■ metric of ten-dimensional spacetime :

$$\begin{aligned} ds^2 \;&=\; \left[1 + \left(\frac{\tau}{\tau_0}\right)^{-4/3} h_1(y)\right]^{-1/2} \left[-d\tau^2 + \left(\frac{\tau}{\tau_0}\right)^{-2/3} \delta_{ab} dx^a dx^b\right] \\ &\quad + \left[1 + \left(\frac{\tau}{\tau_0}\right)^{-4/3} h_1(y)\right]^{1/2} \left(\frac{\tau}{\tau_0}\right)^{2/3} \delta_{ij} dy^i dy^j \end{aligned}$$

$h_1(y) = 0$; 4d scale factor is proportional to $\tau^{-2/3}$

☆ Constants M_l are nonzero ;

★metric of 3-brane in ten dimension

$$
\begin{aligned}
ds^2 &= (-at)^{-1/2}[(-at)^{-1}\Phi(y)-1]^{-1/2}\eta dx^\mu dx^\nu \\
&+ (-at)^{1/2}[(-at)^{-1}\Phi(y)-1]^{1/2}\delta_{ij}dy^i dy^j, \\
\Phi(y) &= \sum_l M_l|\vec{y}-\vec{y}_l|^{-4}
\end{aligned}
$$

For $t > 0$, the metric exists inside a domain D_t

D_t ; bounded by the level set $\Phi(y) = -at$

$$(-at)^{-1}\Phi(y) - 1 > 0 \quad \rightarrow \quad \Phi(y) + at > 0, \quad (a < 0)$$

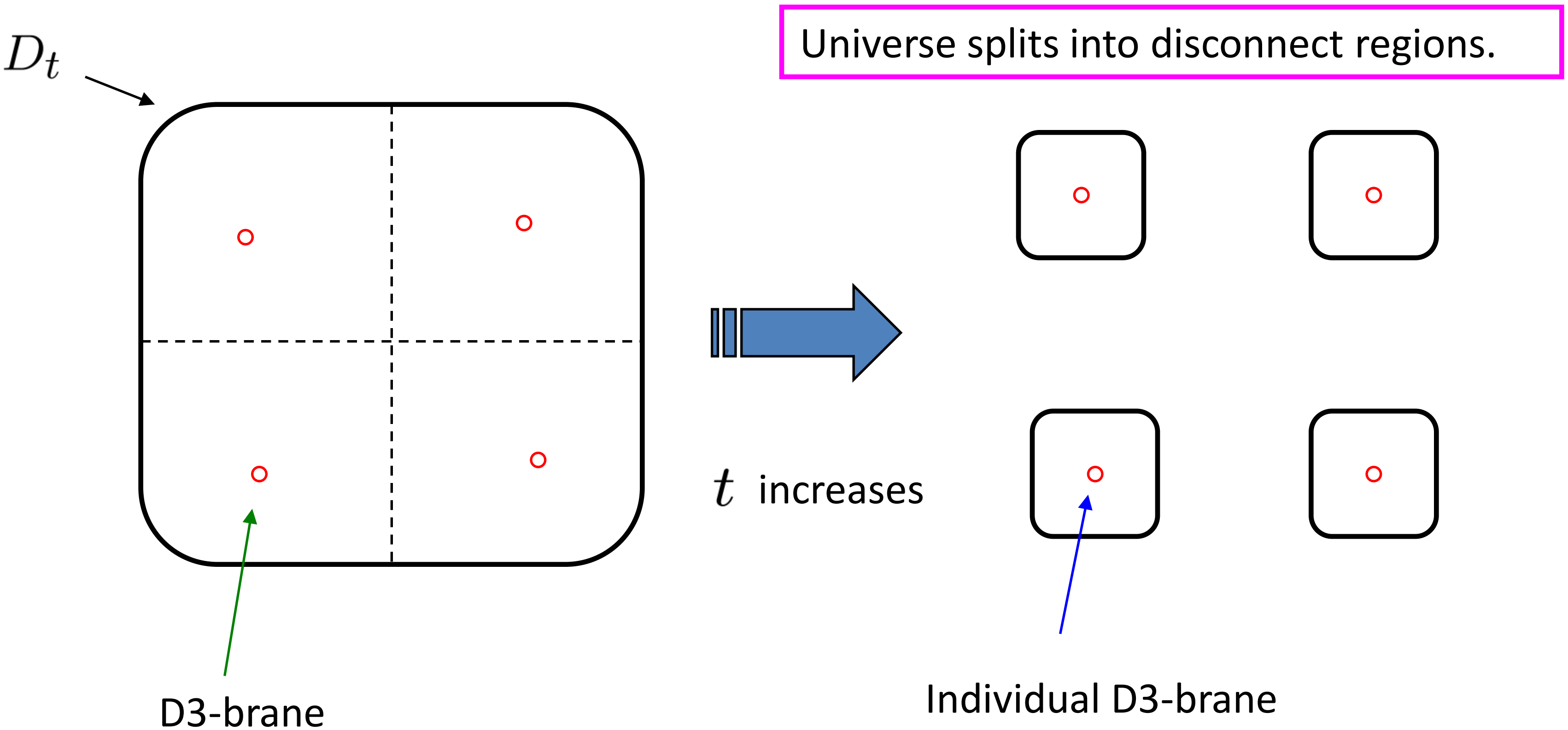
➡ Small positive t, $|\vec{y}-\vec{y}_l|$ is large.

Large positive t $|\vec{y}-\vec{y}_l|$ is small.

➡ t increases, domain D_t shrinks

Small positive t $|\vec{y} - \vec{y}_l|$ Is large.

Large positive t $|\vec{y} - \vec{y}_l|$ is small.



★(p+1)-dimensional effective theory (No flux case)

$$S = \frac{1}{2\kappa^2}\int \left(R * \mathbf{1}_D - \frac{1}{2} d\phi \wedge * d\phi\right)$$

★ Ansatz for background

$$\begin{aligned} ds^2 &= h_0^a(x)\, ds^2(\mathrm{X}) + h_0^b(x)\, ds^2(\mathrm{Y}), \\ e^{\phi} &= h_0^{-c/2} \end{aligned}$$

$$a = -\frac{D-p-3}{D-2}, \qquad b = \frac{p+1}{D-2}$$

★ Field equations are reduced to

$$\boxed{\begin{aligned} &R_{\mu\nu}(\mathrm{X}) - h_0^{-1} D_\mu D_\nu h_0 = 0, \\ &R_{ij}(\mathrm{Y}) = 0 \end{aligned}}$$

□ lower-dimensional effective action

- No flux and internal space is Ricci flat space
- Scalar field satisfies the equation of motion.

★ Ansatz for background

$$
\begin{aligned}
ds^2 &= h_0^a(x)\, ds^2(\mathrm{X}) + h_0^b(x)\, ds^2(\mathrm{Y}), \\
e^{\phi} &= h_0^{-c/2}
\end{aligned}
$$

$$
\begin{aligned}
S &= \frac{1}{2\tilde{\kappa}^2} \int_{\mathrm{X}} h_0(x) R(\mathrm{X}) *_{\mathrm{X}} \mathbf{1}_{(p+1)}, \\
\tilde{\kappa} &= \left[V_{(D-p-1)}\right]^{-1/2} \kappa, \qquad V_{(D-p-1)} = \int * \mathbf{1}_{(D-p-1)}
\end{aligned}
$$

□ (p+1)-dimensional field equations

$$
\begin{aligned}
& R_{\mu\nu}(\mathrm{X}) = h_0^{-1} D_\mu D_\nu h_0, \\
& \triangle_{\mathrm{X}} h_0 = 0
\end{aligned}
$$

★(p+1)-dimensional effective theory with Flux

◇ D-dimensional model

● Ansatz for background

$$
\begin{aligned}
ds^2 &= h^{-(D-p-3)/(D-2)}(x,y)q_{\mu\nu}(\mathrm{X})dx^\mu dx^\nu + h^{(p+1)/(D-2)}(x,y)u_{ij}(\mathrm{Y})dy^i dy^j, \\
e^\phi &= h^{c/2}, \qquad h(x,y) = h_0(x) + h_1(y), \\
F_{(p+2)} &= d(h^{-1}) \wedge \Omega(\mathrm{X}_{p+1}), \qquad \Omega(\mathrm{X}_{p+1}) = \sqrt{-q}dx^0 \wedge dx^1 \wedge \cdots \wedge dx^p
\end{aligned}
$$

- Internal space is Einstein space

- Gauge fields satisfy field equations.

◎ D-dimensional action

$$
\begin{aligned}
S &= \frac{1}{2\kappa^2}\int \left(R * \mathbf{1}_D - \frac{1}{2}d\phi \wedge * d\phi - \frac{1}{2}e^{-c\phi}F_{(p+2)} \wedge * F_{(p+2)} \right), \\
c^2 &= 4 - \frac{2(p+1)(D-p-3)}{D-2}.
\end{aligned}
$$

□ (p+1)-dimensional effective action $(c \neq 0)$

- No flux and internal space is Ricci flat space

$$
\begin{aligned}
S &= \frac{1}{2\tilde{\kappa}^2}\int_{\mathrm{X}} H(x)R(\mathrm{X}) *_{\mathrm{X}} \mathbf{1}_{(p+1)}, \\
H(x) &= h_0(x) + \bar{c}; \quad \bar{c} := V_{d_2}^{-1}\int_{\mathrm{Y}} h_1 *_{\mathrm{Y}} \mathbf{1}_{(D-p-1)}, \\
\tilde{\kappa} &= \left[V_{(D-p-1)}\right]^{-1/2}\kappa, \quad V_{(D-p-1)} = \int_{\mathrm{Y}} *_{\mathrm{Y}}\mathbf{1}_{(D-p-1)}
\end{aligned}
$$

Conformal transformation : $ds^2(\mathrm{X}) = H^{2/(1-p)}ds^2(\bar{\mathrm{X}})$

$$
S = \frac{1}{2\tilde{\kappa}^2}\int_{\bar{\mathrm{X}}}\left[R(\bar{\mathrm{X}}) *_{\bar{\mathrm{X}}} \mathbf{1}_{(p+1)} - \frac{p}{(p-1)} d\ln H \wedge *_{\bar{\mathrm{X}}} d\ln H\right]
$$

★ Field equations

$$
\begin{aligned}
R_{\mu\nu}(\mathrm{X}) &= H^{-1}D_\mu D_\nu H, \\
\triangle_{\mathrm{X}} H &= 0
\end{aligned}
$$

$$
\begin{aligned}
&R_{\mu\nu}(\bar{\mathrm{X}}) = \frac{(d_1-1)}{(d_1-2)}\bar{D}_\mu \ln H\, \bar{D}_\nu \ln H, \\
&\triangle_{\bar{\mathrm{X}}} \ln H = 0
\end{aligned}
$$

□ (p+1)-dimensional effective action $(c=0)$

- No flux and internal space is Einstein space

$$S = \frac{1}{2\tilde{\kappa}^2}\int_{\mathrm{X}}\left[H(x)R(\mathrm{X}) + \frac{\lambda}{2}b(p+1)(D-p-1)\right] *_{\mathrm{X}} \mathbf{1}_{(p+1)},$$
$$H(x) = h_0(x) + \bar{c}; \quad \bar{c} := V^{-1}_{(D-p-1)}\int_{\mathrm{Y}} h_1 *_{\mathrm{Y}} \mathbf{1}_{(D-p-1)},$$
$$\tilde{\kappa} = \left[V_{(D-p-1)}\right]^{-1/2}\kappa, \quad V_{(D-p-1)} = \int_{\mathrm{Y}} *_{\mathrm{Y}}\mathbf{1}_{(D-p-1)}$$

Conformal transformation : $ds^2(\mathrm{X}) = H^{2/(1-p)}ds^2(\bar{\mathrm{X}})$

$$S = \frac{1}{2\tilde{\kappa}^2}\int_{\bar{\mathrm{X}}}\left[R(\bar{\mathrm{X}}) *_{\bar{\mathrm{X}}} \mathbf{1}_{(p+1)} - \frac{p}{(p-1)}d\ln H \wedge *_{\bar{\mathrm{X}}} d\ln H + \frac{\lambda}{2}(p+1)(D-p-1)bH^{(1+p)/(1-p)}\right]$$

★ Field equations

$$R_{\mu\nu}(\mathrm{X}) = H^{-1}\left[D_\mu D_\nu H - \frac{\lambda}{4p}(p+1)(D-p-1)bq_{\mu\nu}(\mathrm{X})\right],$$
$$\triangle_{\mathrm{X}} H = \frac{(p+1)^2}{4p}(D-p-1)\lambda b.$$

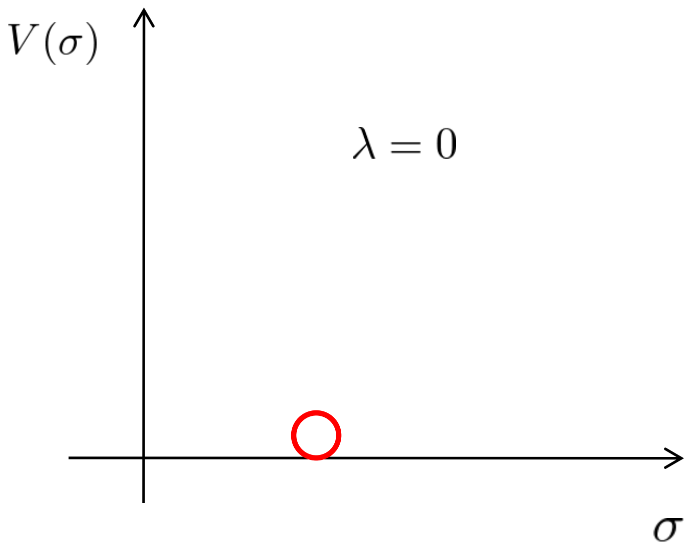
$$R_{\mu\nu}(\bar{\mathrm{X}}) = \frac{p}{(p-1)}\bar{D}_\mu \ln H\, \bar{D}_\nu \ln H - \frac{\lambda}{2(p-1)}(p+1)(D-p-1)bH^{(1+p)/(1-p)}q_{\mu\nu}(\bar{\mathrm{X}}),$$
$$\triangle_{\bar{\mathrm{X}}} \ln H = \frac{(p+1)^2}{4p}(D-p-1)\lambda b H^{(1+p)/(1-p)}$$

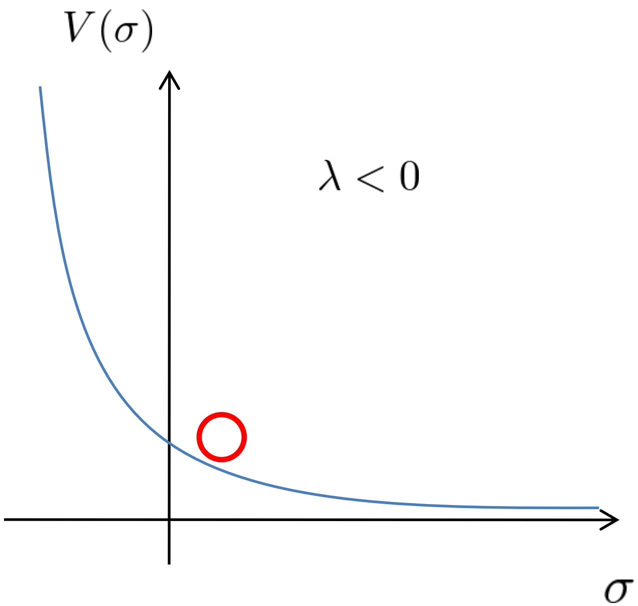
☆ p=3, D=10 case, ⇒ 4-dimensional moduli potential (Einstein frame)

$$
\begin{aligned}
V(\sigma) &= -6\lambda e^{-2\sigma/\sqrt{3}}, \\
\sigma &= \sqrt{3}\ln\left[h_0(x) + V_6^{-1}\int_{\mathrm{Y}} h_1 *_{\mathrm{Y}} \mathbf{1}\right]
\end{aligned}
$$

$\lambda = 0 \;\cdots\;$ flat potential,

$\lambda \neq 0 \;\cdots\;$ run away potential





[4] Summary :

★ The solutions we found have the property that they are genuinely higher-dimensional in the sense that one can never neglect the dependence on y^i say of h .

・The same results hold for other intersecting brane model (D2-D6 brane, D2-D4-D8 brane, D3-D7,brane, M2-M5 brane system).

☆ Warped structure :

linear combination of the $h_0(x)$ and $h_1(y)$

→ 10-dimensional IIA, IIB and 11-dimensional supergravity

→ metric ansatz or supersymmetry

★ Further calculations of the cosmological dynamics :

・Construction of the special solution of Einstein equation

→ other black intersecting brane solution

→ stabilization and application to cosmology