Innermost stable circular orbits around squashed Kaluza-Klein black holes

Ken Matsuno & Hideki Ishihara
(Osaka City University)

1. Introduction

- Five-dimensional black objects
- Circular orbits around squashed Kaluza-Klein black holes

2. Innermost stable circular orbits around squashed Kaluza-Klein black holes

- (Charged) static squashed black holes
- Charged rotating squashed black holes

1. Introduction

Motivations

- 我々は 4次元時空 に住んでいる

  空間 3次元
  時間 1次元

- 量子論と矛盾なく, 4種類の力を統一的に議論する

  弦理論
  超重力理論
  → 高次元時空 上の理論

- 余剰次元 の効果が顕著

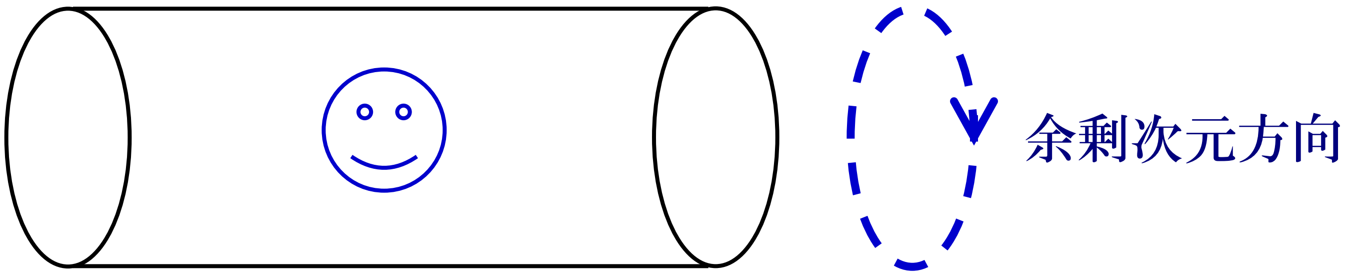
  高エネルギー現象
  強重力場

高次元ブラックホール (BH) に注目

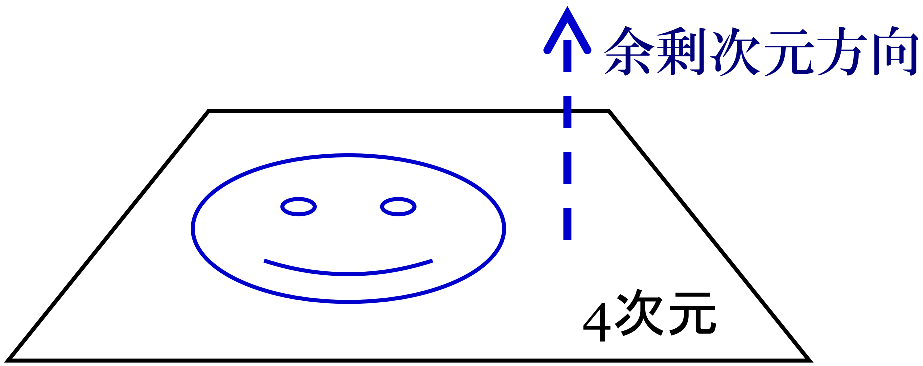
Dimensional reductions

高次元時空 ⇒ 有効的に4次元時空

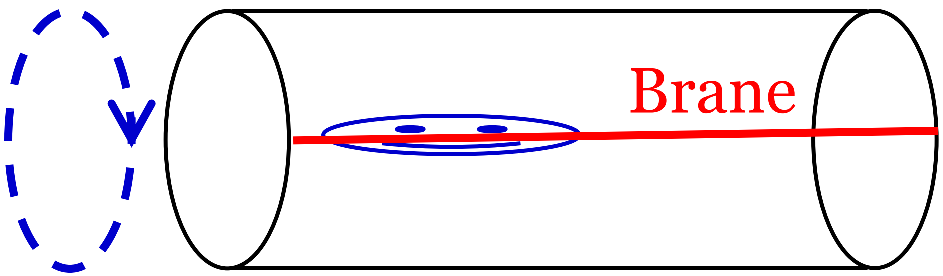
a. Kaluza-Klein model “ とても小さく丸められていて見えない ” (針金)

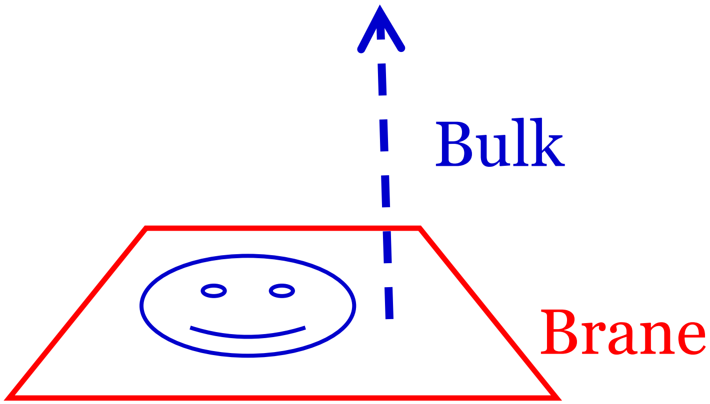


b. Brane world model “行くことが出来ないため見えない”



“ Hybrid ” Brane world model





Brane (4次元時空) : 物質 と 重力以外の力 が束縛
Bulk (高次元時空) : 重力のみ伝播

重力の逆2乗則から制限 ⇒ (余剰次元) ≦ 0.1 mm

加速器内で ミニ・ブラックホール 生成 ?
(高次元時空の実験的検証)

5-dim. Black Objects

［ 以降、5次元時空に注目 ］

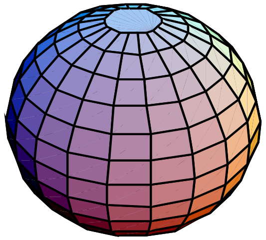
- 4次元 ： 漸近平坦 , 真空 , 定常 , 地平線の上と外に特異点なし

  ⇒ Kerr BH with S^2 horizon only

- 5次元 ： For above conditions

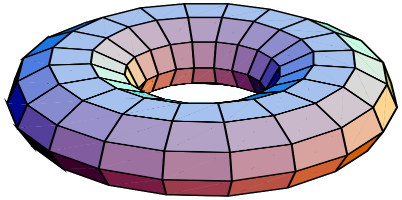
  ⇒ Variety of Horizon Topologies

$$\begin{cases} S^3 & : \text{Three - sphere} \\ S^3/\mathbb{Z}_n & : \text{Lens Space} \\ S^2 \times S^1 & : \text{Black Ring} \end{cases}$$



Black Holes

(S^3)



Black Rings

($S^2 \times S^1$)

Asymptotic Structures of Black Holes

- 4D Black Holes : Asymptically Flat $\left(R^1 \times R^1 \times S^2 \right)$ (time) (radial) (angular)

漸近平坦

- 5D Black Holes : Variety of Asymptotic Structures

Asymptotically Flat : $R^1 \times R^1 \times S^3$: 5D Minkowski

Asymptotically Locally Flat :
- $R^1 \times R^1 \times S^3/\mathbb{Z}_n$: Lens Space
- $R^1 \times R^1 \times S^2 \times S^1$: 4D Minkowski + a compact dim.
- . . .

Kaluza-Klein Black Holes

Squashed Kaluza-Klein Black Holes

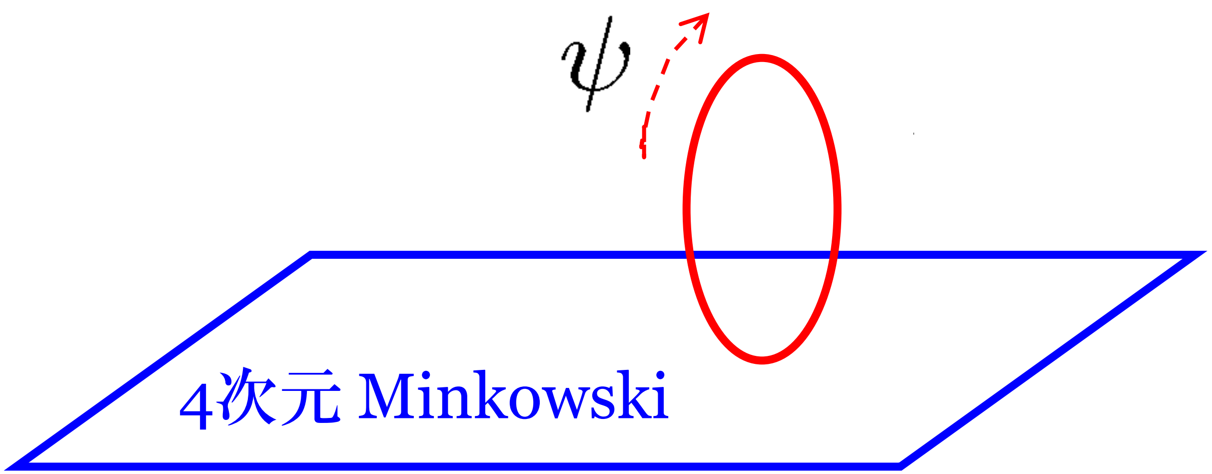
Kaluza-Klein Black Holes

- 無限遠における計量の漸近形

$$ds^2 \simeq -dt^2 + dr^2 + r^2 d\Omega_{S^2}^2 + L^2 d\psi^2$$

4次元 Minkowski　　Compact S¹

[4次元 Minkowski と Compact S¹ の直積]



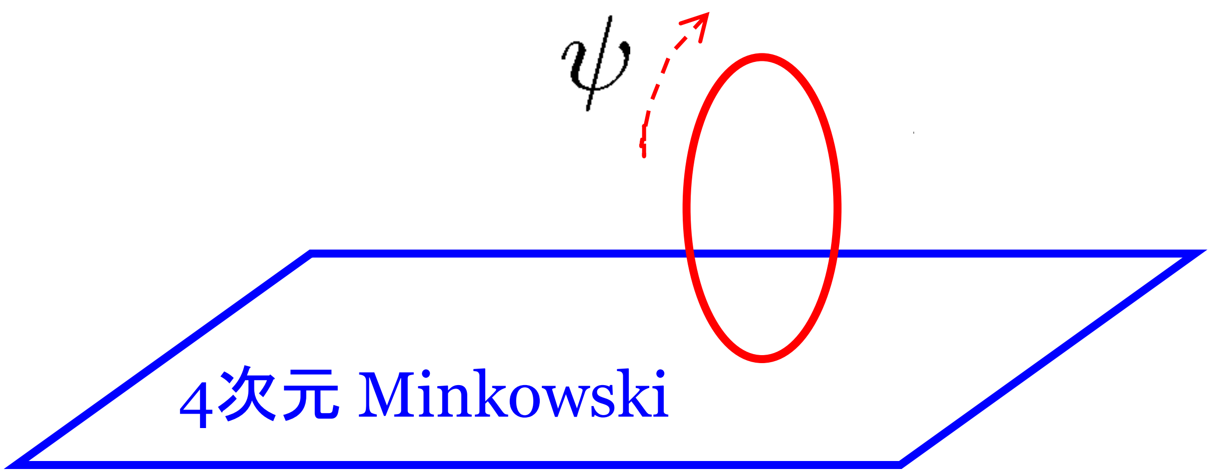
Squashed Kaluza-Klein Black Holes

- 無限遠における計量の漸近形　$d\Omega_{S^2}^2 := d\theta^2 + \sin^2\theta d\phi^2$

$$ds^2 \simeq -dt^2 + dr^2 + r^2 d\Omega_{S^2}^2$$
$$+ L^2 (d\psi + \cos\theta d\phi)^2$$

Twisted S^1

[4次元Minkowski上に 小さなサイズのツイストされた余剰次元 S^1]



Vacuum Squashed Kaluza-Klein Black Holes

- metric

$$ds^2 = -V(\rho)dt^2 + U(\rho)\left[\frac{d\rho^2}{V(\rho)} + \rho^2 d\Omega_{S^2}^2\right] + \frac{r_\infty^2}{4U(\rho)}(d\psi + \cos\theta d\phi)^2$$

$$V(\rho) = 1 - \frac{\rho_g}{\rho}, \quad U(\rho) = 1 + \frac{\rho_0}{\rho}, \qquad d\Omega_{S^2}^2 := d\theta^2 + \sin^2\theta d\phi^2$$

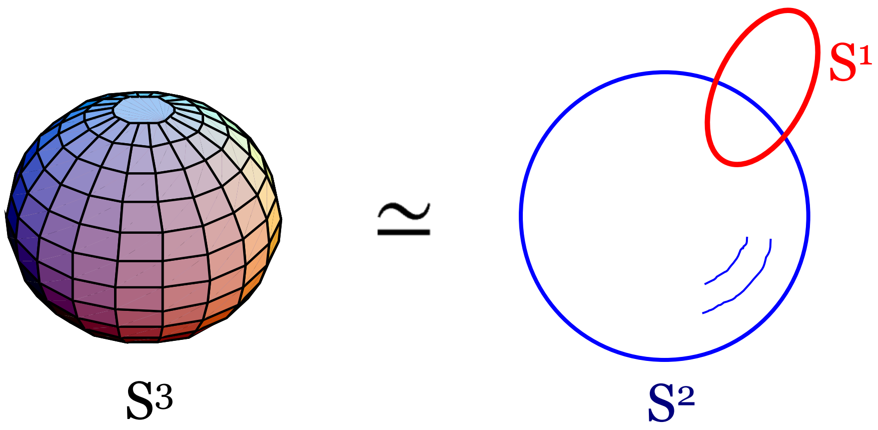
$$-\infty < t < \infty,\ 0 < \rho < \infty,\ 0 \le \theta \le \pi,\ 0 \le \phi \le 2\pi, \text{ and } 0 \le \psi \le 4\pi.$$

- Killing vector fields : $\partial/\partial t$, $\partial/\partial\varphi$, $\partial/\partial\psi$
- Parameters : $\rho_g > 0$ and $\rho_0 > 0$

$$r_\infty^2 = 4\rho_0(\rho_g + \rho_0).$$

Three-sphere S^3

- S^3 : Hopfバンドル構造 (S^2上のツイストされたS^1ファイバー)



$$d\Omega^2_{S^3} = \frac{1}{4}\left[d\Omega^2_{S^2} + (d\psi + \cos\theta d\phi)^2\right]$$

(S^2 base)　　(twisted S^1 fiber)

丸いS^3 = S^2 と S^1 のサイズ比が 1 : 1

Induced metric on black hole horizon ρ_g

$$ds^2 = -V(\rho)dt^2 + U(\rho)\left[\frac{d\rho^2}{V(\rho)} + \rho^2 d\Omega^2_{S^2}\right] + \frac{r^2_\infty}{4U(\rho)}(d\psi + \cos\theta d\phi)^2$$

$$V(\rho) = 1 - \frac{\rho_g}{\rho}, \quad U(\rho) = 1 + \frac{\rho_0}{\rho},$$

The black hole horizon is located at $\rho = \rho_g$

$$ds^2\big|_{\rho=\rho_g,\ t=const.} = \rho_g\rho_0\left[\left(1 + \frac{\rho_g}{\rho_0}\right)d\Omega^2_{S^2} + (d\psi + \cos\theta d\phi)^2\right]$$

the radius of the S^2 base is larger than that of the S^1 fiber.

Asymptotic structure of squashed black hole

$$ds^2 = -V(\rho)dt^2 + U(\rho)\left[\frac{d\rho^2}{V(\rho)} + \rho^2 d\Omega_{S^2}^2\right] + \frac{r_\infty^2}{4U(\rho)}(d\psi + \cos\theta d\phi)^2$$

$$V(\rho) = 1 - \frac{\rho_g}{\rho}, \quad U(\rho) = 1 + \frac{\rho_0}{\rho},$$

At the infinity, $\rho = \infty$,

$$ds^2 \simeq -dt^2 + d\rho^2 + \rho^2 d\Omega_{S^2}^2 + \frac{r_\infty^2}{4}(d\psi + \cos\theta d\phi)^2$$

A twisted S^1 fiber bundle over 4D Minkowski spacetime

余剰次元サイズ： $r_\infty^2 = 4\rho_0(\rho_g + \rho_0).$

“天体的ブラックホール” $\Leftrightarrow$ $r_\infty << \rho_g$

Physical Quantities

$$ds^2 = -V(\rho)dt^2 + U(\rho)\left[\frac{d\rho^2}{V(\rho)} + \rho^2 d\Omega_{S^2}^2\right] + \frac{r_\infty^2}{4U(\rho)}(d\psi + \cos\theta d\phi)^2$$

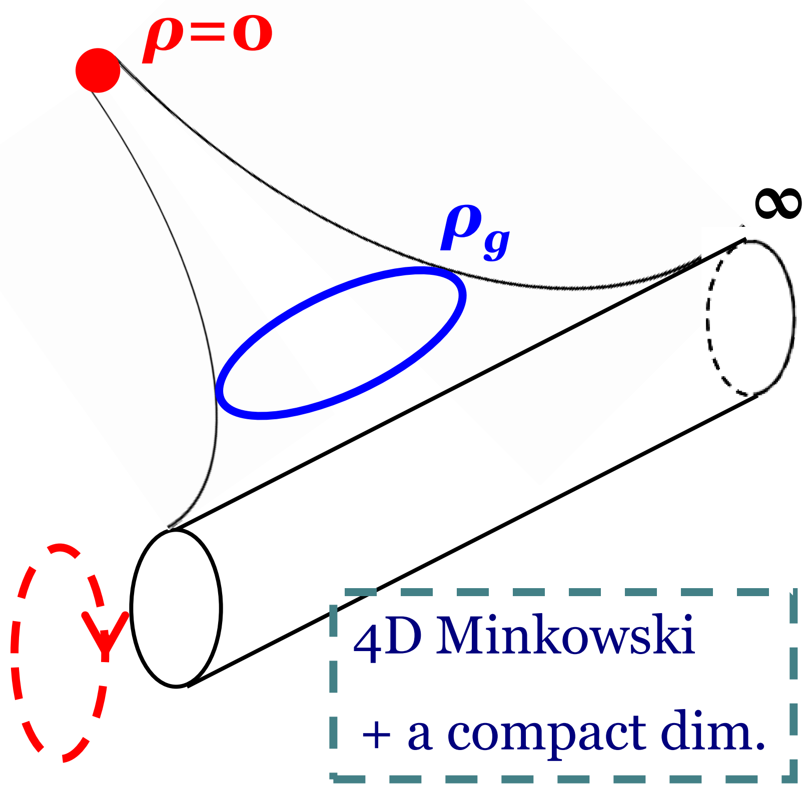
$$V(\rho) = 1 - \frac{\rho_g}{\rho}, \quad U(\rho) = 1 + \frac{\rho_0}{\rho},$$

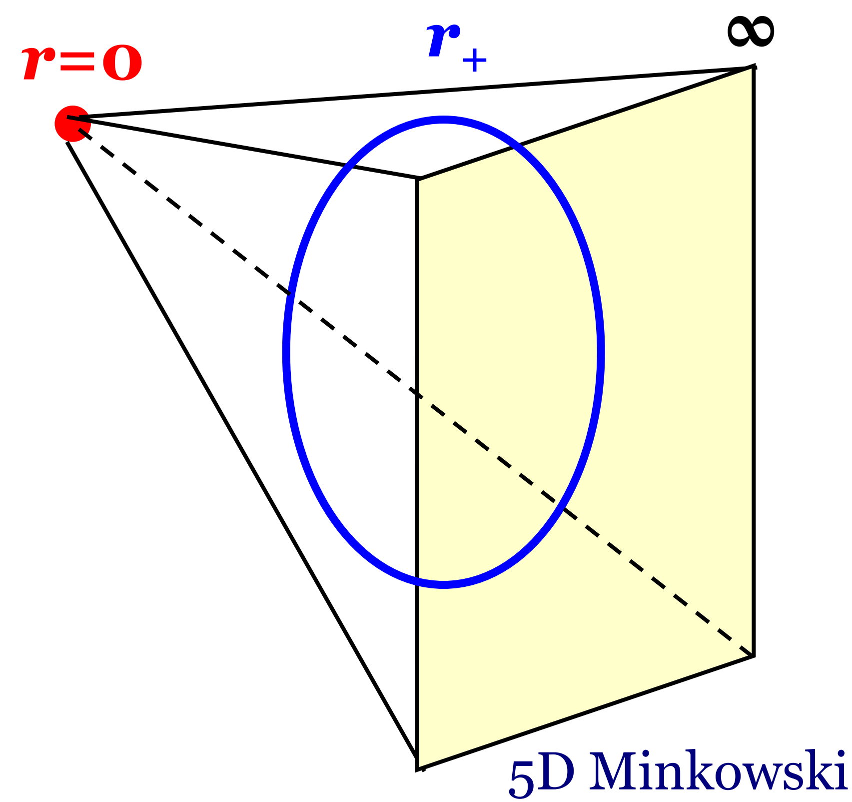
- Killing vector fields : $\partial/\partial t$, $\partial/\partial \varphi$, $\partial/\partial \psi$
- Komar mass : $M = \frac{\pi r_\infty \rho_g}{G_5}$

異なる漸近構造を持つ5次元ブラックホール

5D squashed Kaluza-Klein BH

5D 漸近平坦 BH





近傍 ： 5次元的
遠方 ： 4次元的

至る所、5次元的

Squashed KK BH解の一般化と応用

- Squashed Kaluza-Klein ブラックホール解の一般化
  - 回転パラメータを含むブラックホール解
  - 多体BPSブラックホール解
  - Dilaton場や非可換ゲージ場を含む重力理論におけるブラックホール解

- 厳密解が得られたことにより…
  - 安定性などの摂動的研究
  - 熱力学
  - ホーキング輻射
  - ブラックホールの周りの試験粒子の運動の研究
    (geodetic precessions , 重力レンズ , …)

ブラックホール時空を用いた余剰次元の検証に向けた研究の端緒

Timelike Geodesics in Vacuum Squashed Kaluza-Klein Black Hole Spacetimes

Phys. Rev. D 80, 104037 (2009)

Lagrangian and constants of motion

- metric

$$ds^2 = -V(\rho)dt^2 + U(\rho)\left[\frac{d\rho^2}{V(\rho)} + \rho^2 d\Omega_{S^2}^2\right] + \frac{r_\infty^2}{4U(\rho)}(d\psi + \cos\theta d\phi)^2$$

$$V(\rho) = 1 - \frac{\rho_g}{\rho}, \quad U(\rho) = 1 + \frac{\rho_0}{\rho}, \qquad d\Omega_{S^2}^2 := d\theta^2 + \sin^2\theta d\phi^2$$

- Lagrangian : $2\mathcal{L} = -1$

$$\mathcal{L} = \frac{1}{2}\left[-V\dot{t}^2 + \frac{U}{V}\dot{\rho}^2 + U\rho^2\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right) + \frac{r_\infty^2}{4U}\left(\dot{\psi} + \cos\theta\dot{\phi}\right)^2\right]$$

- Constants of motion :

$$E := V\dot{t},$$

$$L := U\rho^2\sin^2\theta\dot{\phi} + \frac{r_\infty^2\cos\theta}{4U}\left(\dot{\psi} + \cos\theta\dot{\phi}\right),$$

$$p_\psi := \frac{r_\infty^2}{4U}\left(\dot{\psi} + \cos\theta\dot{\phi}\right).$$

Timelike geodesics without extra dimensional direction

- Lagrangian : $2\mathcal{L} = -1$

$$\mathcal{L} = \frac{1}{2}\left[-V\dot{t}^2 + \frac{U}{V}\dot{\rho}^2 + U\rho^2\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right) + \frac{r_\infty^2}{4U}\left(\dot{\psi} + \cos\theta\dot{\phi}\right)^2\right]$$

- Assumption : Particle has no momentum in extra direction

$$p_\psi = \frac{r_\infty^2}{4U}\left(\dot{\psi} + \cos\theta\dot{\phi}\right) = 0$$

- Effective Lagrangian

$$\mathcal{L}_{\text{eff}} = \frac{1}{2}\left[-V\dot{t}^2 + \frac{U}{V}\dot{\rho}^2 + U\rho^2\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right)\right] \qquad 2\mathcal{L}_{\text{eff}} = -1$$

We can concentrate on orbits with $\theta=\pi/2$ on assumption of $p_\psi=0$

Timelike Geodesics

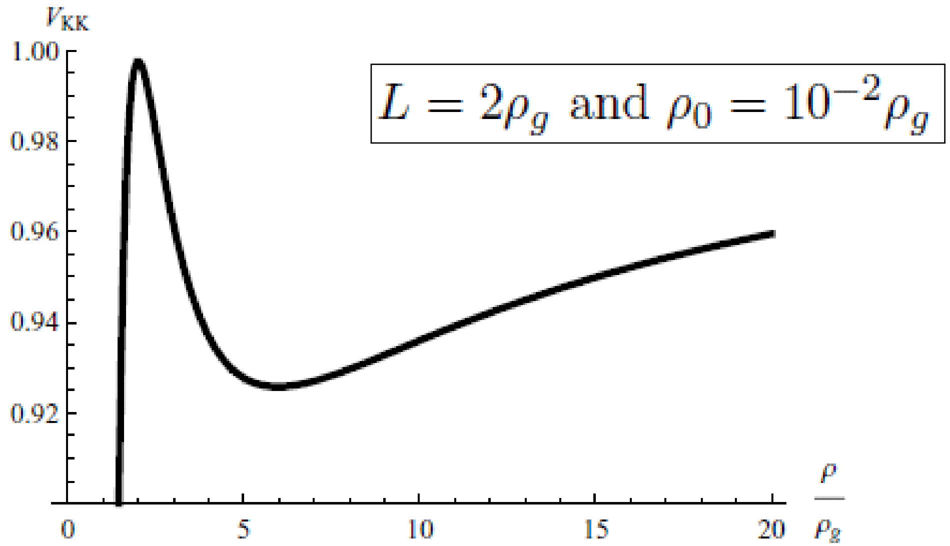
- Energy conservation equation

$$\left(1+\frac{\rho_0}{\rho}\right)\left(\frac{d\rho}{d\tau}\right)^2 + V_{\mathrm{KK}}(\rho) = E^2$$

- Effective potential

$$V_{\mathrm{KK}}(\rho) = \left(1-\frac{\rho_g}{\rho}\right)\left(1+\frac{L^2}{\rho\left(\rho+\rho_0\right)}\right)$$

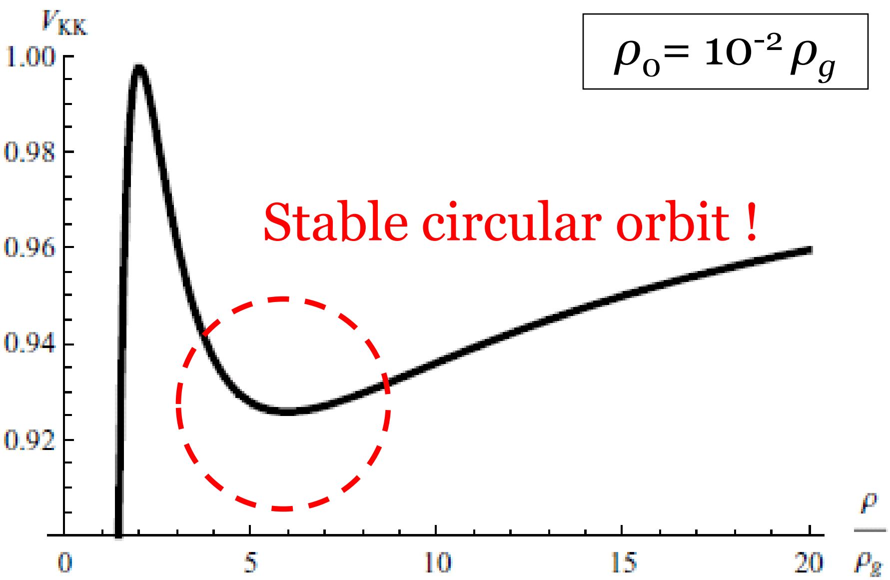
$\rho_0 \Rightarrow 0$: Effective potential for 4D Schwarzschild black holes



Comparison of Timelike Geodesics

5D squashed Kaluza-Klein BH

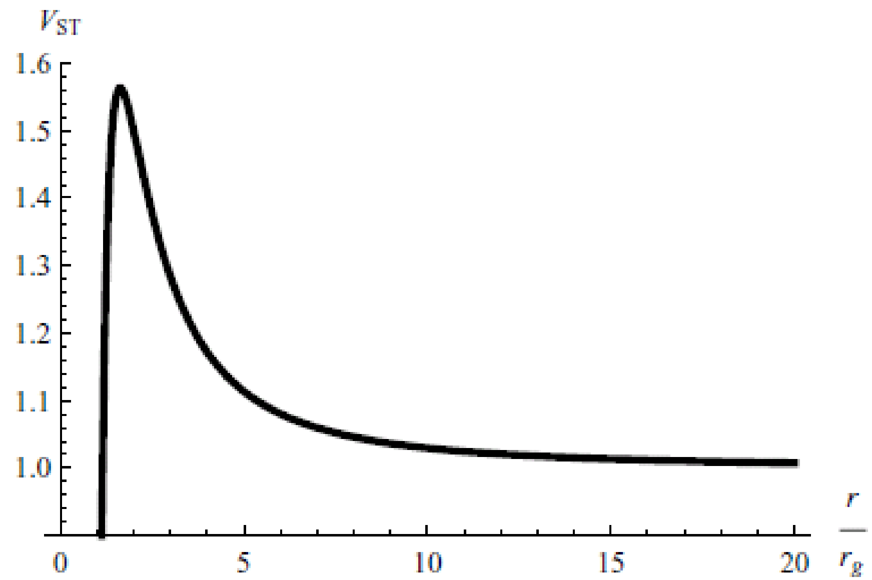
$$V_{\rm KK}(\rho) = \left(1 - \frac{\rho_g}{\rho}\right)\left(1 + \frac{L^2}{\rho(\rho+\rho_0)}\right)$$



Describing geodesics around compact objects

5D 漸近平坦 BH

$$V_{\rm ST}(r) = \left(1 - \frac{r_g^2}{r^2}\right)\left(1 + \frac{L_{\rm ST}^2}{r^2}\right)$$



Stable circular motions

circular motion $\rho = R =$ const. with $p_\psi = 0$ and $\theta = \pi/2$.

From $V_{\mathrm{KK}} = E^2$ and $dV_{\mathrm{KK}}/d\rho = 0$,

$$u^\rho = 0, \quad u^\theta = 0, \quad \text{and} \quad u^\psi = 0,$$

$$u^t = \sqrt{\frac{R(2R+\rho_0)}{R(2R-3\rho_g)+\rho_0(R-2\rho_g)}},$$

$$u^\phi = \sqrt{\frac{\rho_g}{R^2(2R-3\rho_g)+\rho_0 R(R-2\rho_g)}}.$$

Kepler's third law

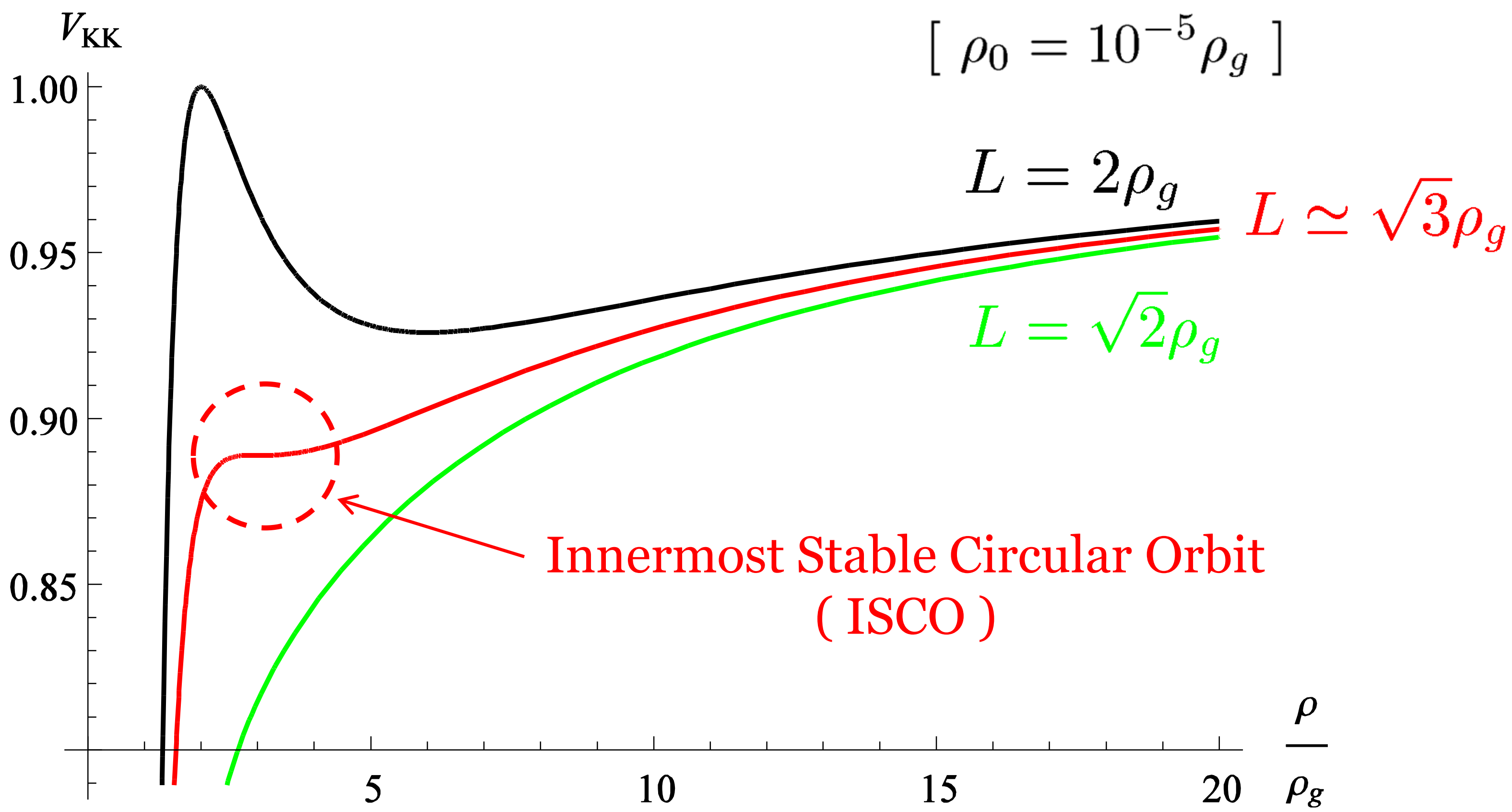
$$T^2 = \frac{4\pi^2}{G_4 M} R^3 \left(1 + \frac{1}{2}\frac{\rho_0}{R}\right)$$

2. Innermost Stable Circular Orbits around Squashed Kaluza-Klein Black Holes

Vacuum static squashed Kaluza-Klein black hole case

Types of timelike geodesics in vacuum squashed KK BHs

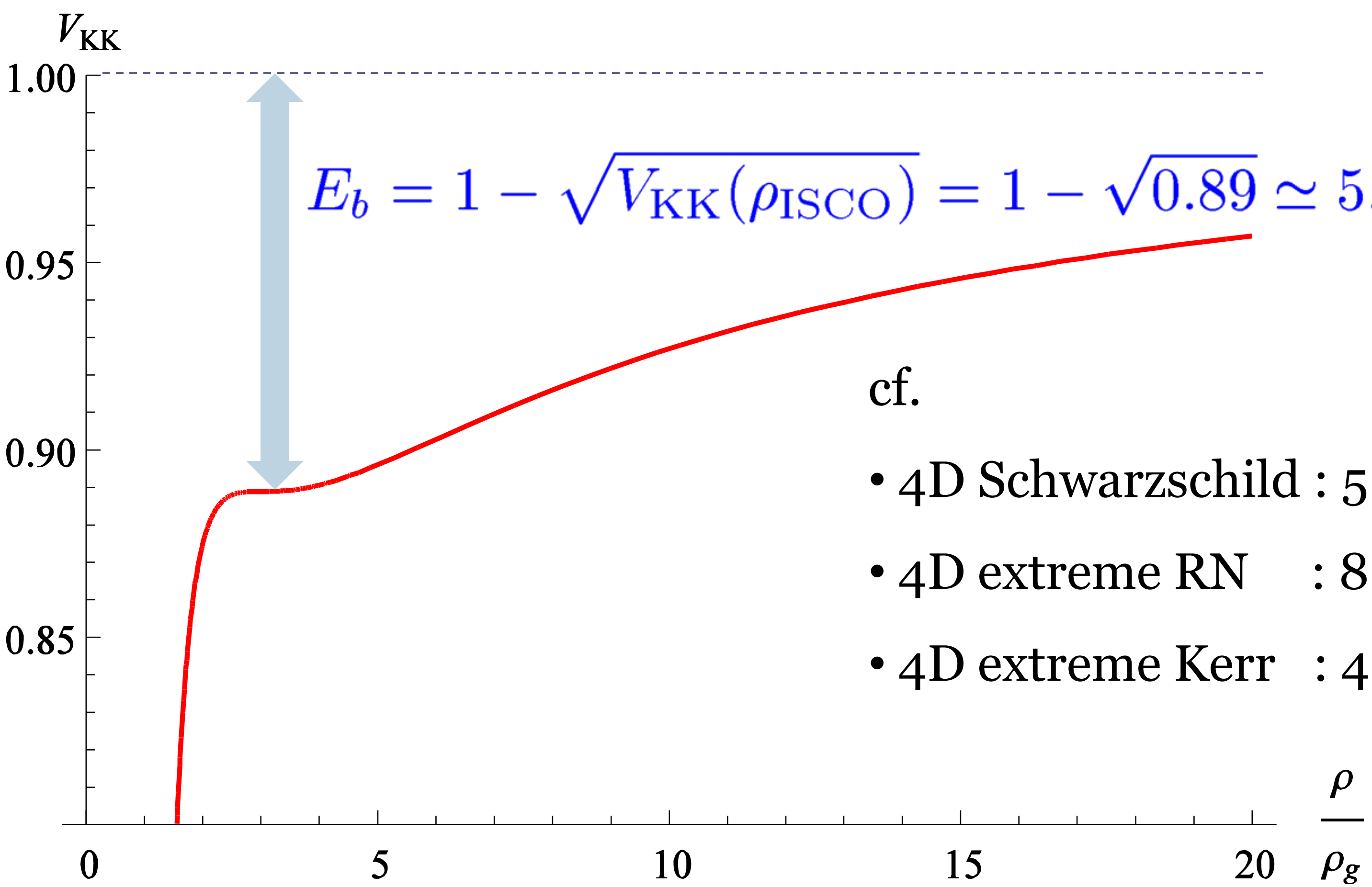
$$V_{\rm KK}(\rho) = \left(1 - \frac{\rho_g}{\rho}\right)\left(1 + \frac{L^2}{\rho(\rho + \rho_0)}\right)$$



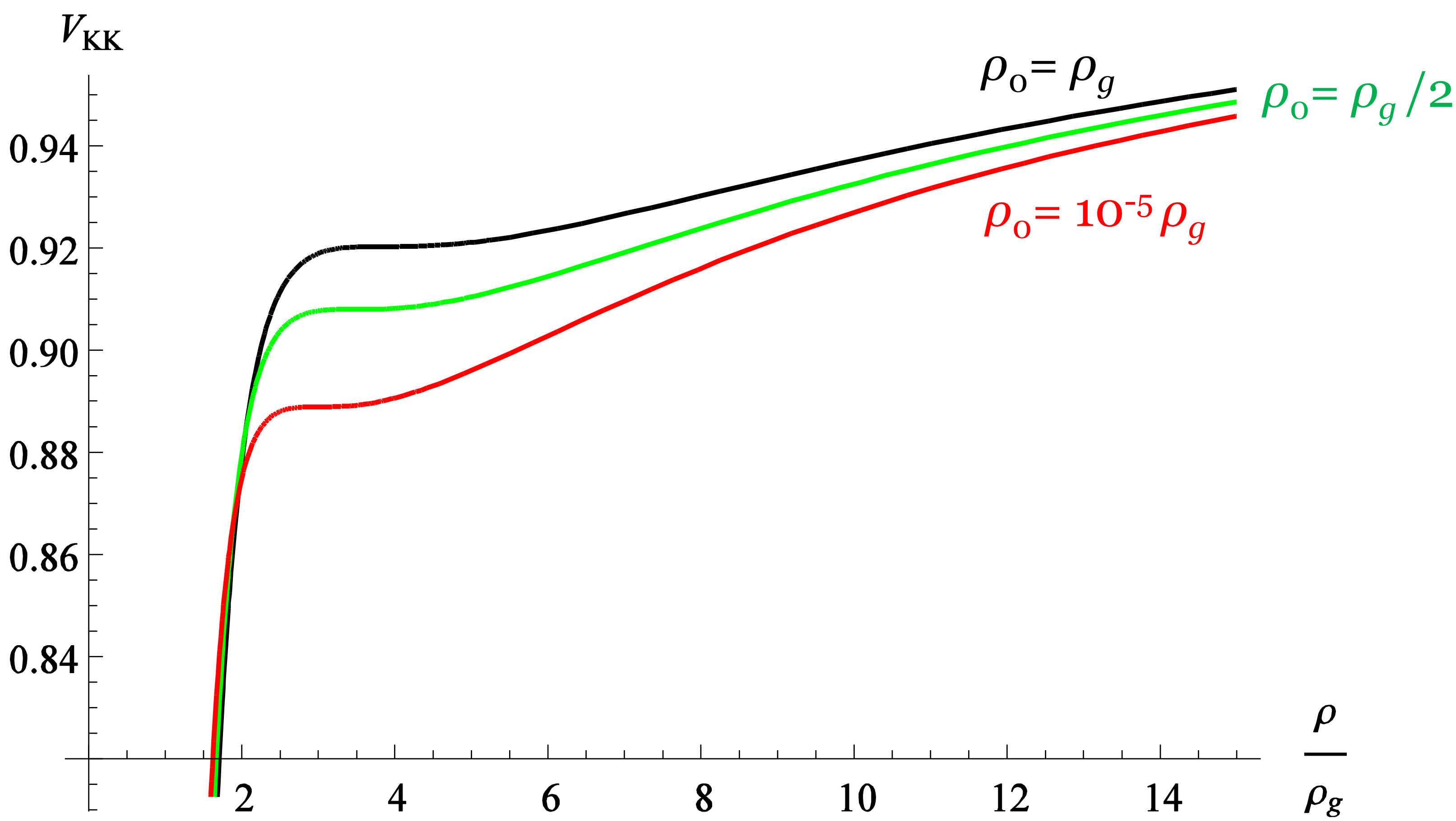
4D Schwarzschild like behaviors

Innermost Stable Circular Orbit and Binding Energy E_b

- Energy conservation equation : $\left(1+\frac{\rho_0}{\rho}\right)\left(\frac{d\rho}{d\tau}\right)^2 + V_{KK}(\rho) = E^2$
- Binding energy : 落下物がISCOに落ち着くまでに放出するエネルギー



Innermost stable circular orbits in vacuum squashed BH spacetimes



余剰次元サイズ： $r_\infty^2 = 4\rho_0(\rho_g + \rho_0).$

"天体的ブラックホール" 程、Binding Energy ⇒ 大

Charged static squashed Kaluza-Klein black hole case

Timelike geodesics of neutral test particles

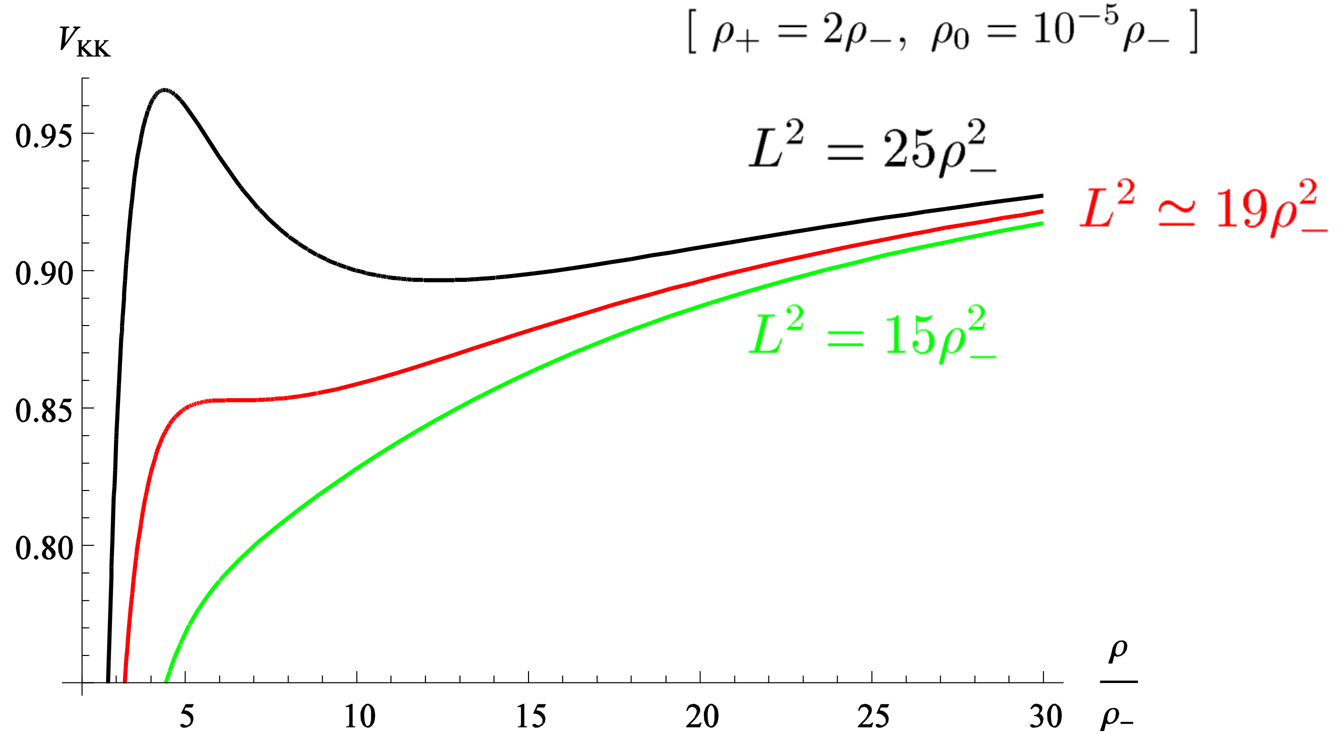
- Energy conservation equation

$$\left(1+\frac{\rho_0}{\rho}\right)\left(\frac{d\rho}{d\tau}\right)^2 + V_{\rm KK}(\rho) = E^2$$

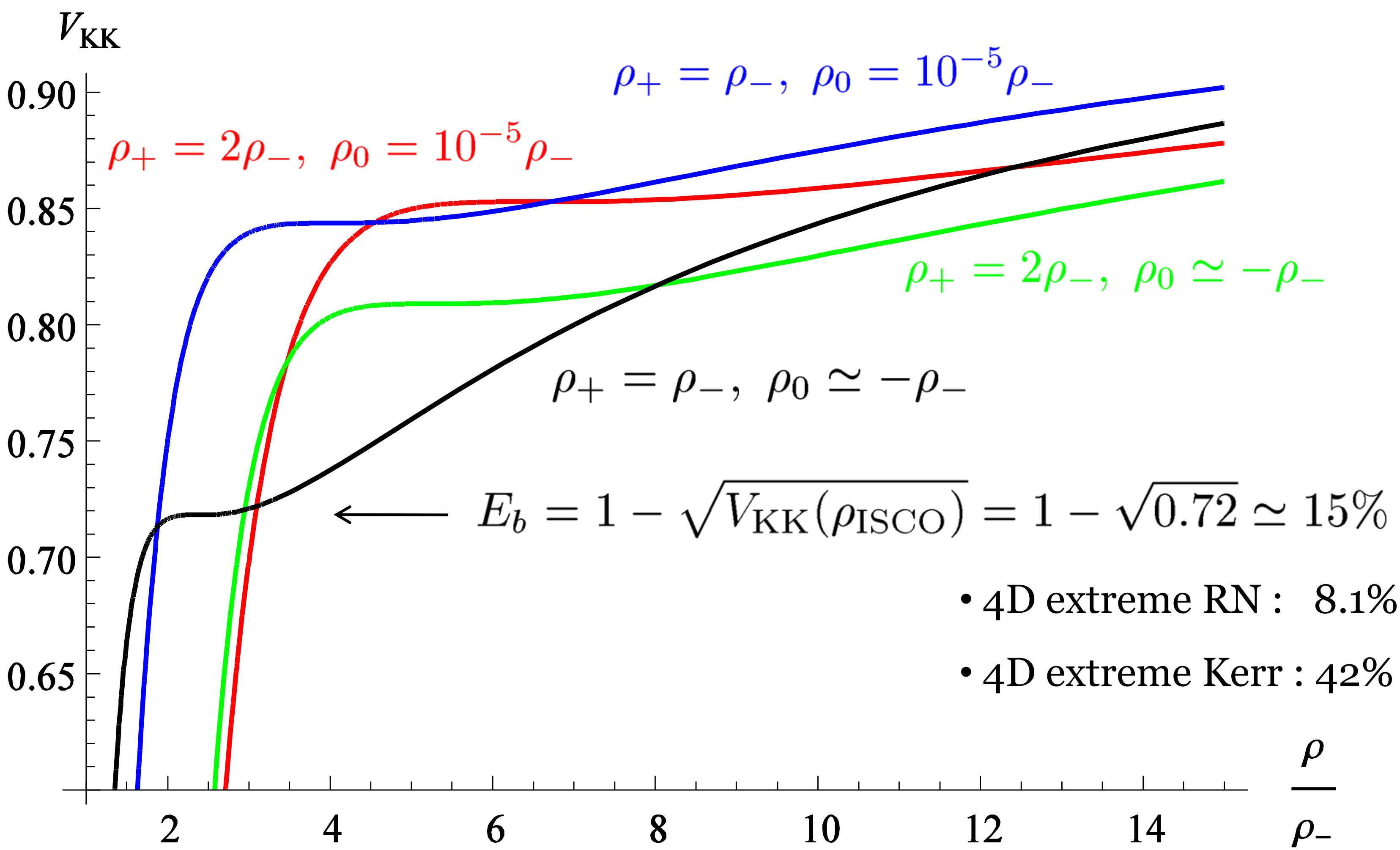
- Effective potential

$$V_{\rm KK}(\rho) = \left(1-\frac{\rho_+}{\rho}\right)\left(1-\frac{\rho_-}{\rho}\right)\left(1+\frac{L^2}{\rho(\rho+\rho_0)}\right)$$

$\rho_0 \Rightarrow 0$: Effective potential for 4D Reissner-Nordstrom black holes



Innermost stable circular orbits in charged squashed BH spacetimes



余剰次元サイズ： $r_\infty^2 = 4(\rho_+ + \rho_0)(\rho_- + \rho_0)$

“天体的ブラックホール” 程、Binding Energy ⇒ 大

Charged Rotating Squashed Kaluza-Klein Black Holes

Phys. Rev. D 77, 044040 (2008)

5D Einstein-Maxwell system with Chern-Simons term

- Action

$$S = \frac{1}{16\pi G_5}\int d^5x\sqrt{-g}\Big[R - F_{\mu\nu}F^{\mu\nu} - \frac{2}{3\sqrt{3}}(\sqrt{-g})^{-1}\epsilon^{\mu\nu\rho\sigma\lambda}A_\mu F_{\nu\rho}F_{\sigma\lambda}\Big]$$

- Equations of motion

$$R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu} = 2(F_{\mu\lambda}F_\nu{}^\lambda - \tfrac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma})$$

$$F^{\mu\nu}{}_{;\nu} + \frac{1}{2\sqrt{3}\sqrt{-g}}\epsilon^{\mu\nu\rho\sigma\lambda}F_{\nu\rho}F_{\sigma\lambda} = 0.$$

Charged Rotating Squashed Kaluza-Klein Black Holes

- Metric

$$ds^2 = -F(\rho)dt^2 + U(\rho)\left[\frac{d\rho^2}{V(\rho)} + \rho^2 d\Omega_{S^2}^2\right] + W(\rho)\left(d\psi + \cos\theta d\phi\right)^2 + 2K(\rho)dt\left(d\psi + \cos\theta d\phi\right)$$

- Gauge potential

$$\boldsymbol{A} = \frac{\sqrt{3}q}{2r_\infty^2}\left(1 + \frac{\rho_0}{\rho}\right)\left[\frac{r_\infty^4 + a^2 q}{4r_\infty^2 L\rho_0}dt - \frac{a}{2}\left(d\psi + \cos\theta d\phi\right)\right]$$

- Killing vector fields : $\partial/\partial t$, $\partial/\partial\varphi$, $\partial/\partial\psi$
- Parameters : $m,\ q,\ a,\ r_\infty,\ L$ and ρ_0

$$\rho_0^2 = \frac{r_\infty^4 - 2mr_\infty^2 + q^2 + 2a^2(m+q)}{4r_\infty^2}$$

$$L^2 = \frac{r_\infty^6 - a^2(q^2 - 2(m+q)r_\infty^2)}{4r_\infty^4}$$

Metric functions

$$U(\rho) = 1 + \frac{\rho_0}{\rho},$$

$$F(\rho) = \Big[16L^2\left(r_\infty^4 - 2mU(\rho)r_\infty^2 + q^2U(\rho)^2\right)r_\infty^8$$
$$+ 8a^2\left(q^2 - (2m+q)r_\infty^2\right)U(\rho)\left(q^2U(\rho) - (2m+q)r_\infty^2\right)r_\infty^4$$
$$+ \frac{a^2\left(q^2 - (2m+q)r_\infty^2\right)^2\left(-r_\infty^6 - 2a^2(m+q)U(\rho)^2r_\infty^2 + a^2q^2U(\rho)^3\right)}{L^2U(\rho)}\Big] / 16\rho_0^2r_\infty^{12}$$

$$V(\rho) = \frac{r_\infty^4 - 2mU(\rho)r_\infty^2 + \left(2(m+q)a^2 + q^2\right)U(\rho)^2}{4\rho_0^2r_\infty^2},$$

$$W(\rho) = \frac{r_\infty^6 + 2a^2(m+q)U(\rho)^2r_\infty^2 - a^2q^2U(\rho)^3}{4r_\infty^4U(\rho)},$$

$$K(\rho) = \Big[a\Big((2m+q)r_\infty^8 - \left(q^2 + 4L^2(2m+q)U(\rho)^2\right)r_\infty^6$$
$$+ 2U(\rho)^2\left((m+q)(2m+q)a^2 + 2L^2q^2U(\rho)\right)r_\infty^4$$
$$-a^2q^2U(\rho)^2(2(m+q) + (2m+q)U(\rho))r_\infty^2 + a^2q^4U(\rho)^3\Big)\Big] / 8L\rho_0r_\infty^8U(\rho)$$

Region of Parameters (m, q, a, r_∞)

- No naked singularity and closed timelike curve on and outside BH horizon

$$m > 0,$$

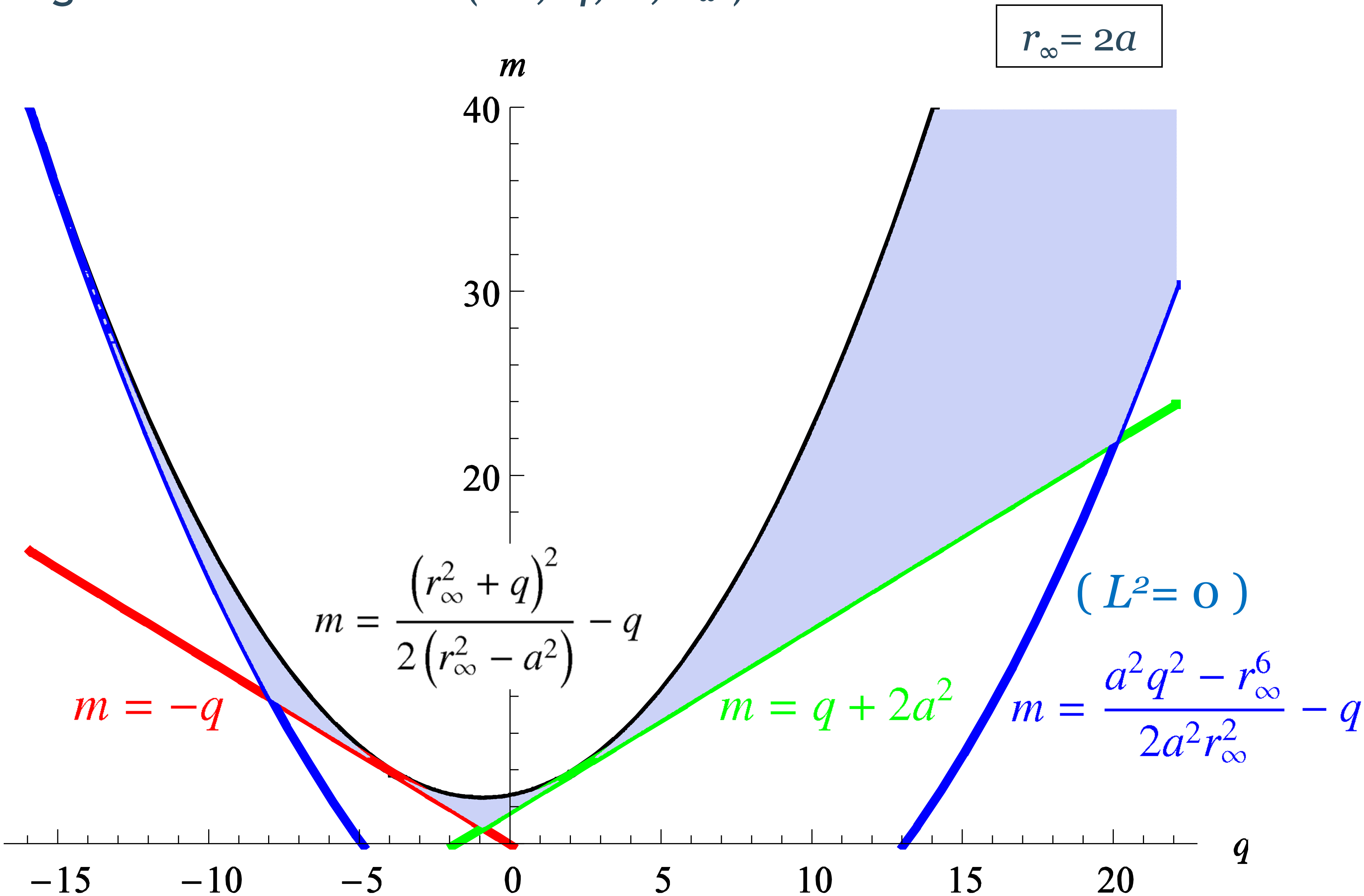
$$m > -q,$$

$$m > q + 2a^2,$$

$$m < \frac{(r_\infty^2 + q)^2}{2(r_\infty^2 - a^2)} - q,$$

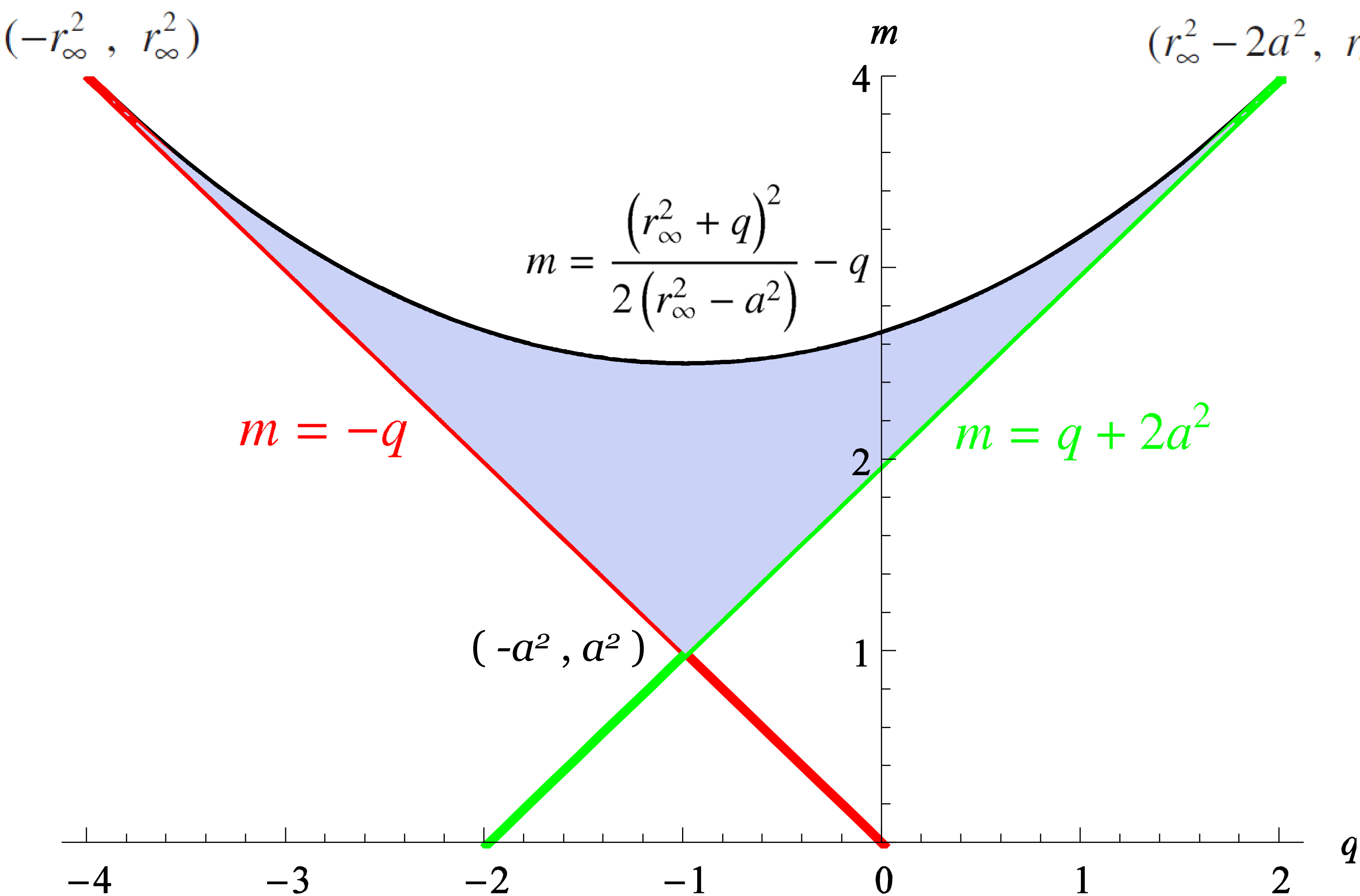
$$m > \frac{a^2 q^2 - r_\infty^6}{2a^2 r_\infty^2} - q.$$

Region of Parameters (m, q, a, r_∞)



Region of Parameters (m, q, a, r_∞)

$r_\infty = 2a$



Induced metric on black hole horizon ρ_H

$$ds^2 = -F(\rho)dt^2 + U(\rho)\left[\frac{d\rho^2}{V(\rho)} + \rho^2 d\Omega_{S^2}^2\right] + W(\rho)\left(d\psi + \cos\theta d\phi\right)^2 + 2K(\rho)dt\left(d\psi + \cos\theta d\phi\right)$$

- Two horizons : $\rho=\rho_\pm$

$$\rho_\pm = \frac{-2(m+q)a^2 - q^2 + \left(m \pm \sqrt{(-2a^2+m-q)(m+q)}\right)r_\infty^2}{4r_\infty^2\rho_0}$$

- Induced metric

$$ds^2\Big|_{t=const.,\ \rho=\rho_\pm} = \rho_\pm^2 U(\rho_\pm) d\Omega_{S^2}^2 + W(\rho_\pm)\left(d\psi + \cos\theta d\phi\right)^2$$

horizons are the squashed S^3 in the form of the Hopf bundle

Asymptotic structure of charged rotating squashed black hole

$$ds^2 = -F(\rho)dt^2 + U(\rho)\left[\frac{d\rho^2}{V(\rho)} + \rho^2 d\Omega_{S^2}^2\right] + W(\rho)\left(d\psi + \cos\theta d\phi\right)^2 + 2K(\rho)dt\left(d\psi + \cos\theta d\phi\right)$$

At the infinity, $\rho = \infty$,

$$ds^2 \simeq -dt^2 + d\rho^2 + \rho^2 d\Omega_{S^2}^2 + L^2\left(d\psi + \cos\theta d\phi\right)^2$$

A twisted S^1 fiber bundle over 4D Minkowski spacetime

余剰次元サイズ： $L^2 = \frac{r_\infty^6 - a^2(q^2 - 2(m+q)r_\infty^2)}{4r_\infty^4}$

“天体的ブラックホール” $\Leftrightarrow$ $L << \rho_{\mathrm{H}}$

Physical Quantities

$$ds^2 = -F(\rho)dt^2 + U(\rho)\left[\frac{d\rho^2}{V(\rho)} + \rho^2 d\Omega_{S^2}^2\right] + W(\rho)\left(d\psi + \cos\theta d\phi\right)^2 + 2K(\rho)dt\left(d\psi + \cos\theta d\phi\right)$$

Killing vector fields : $\partial/\partial t$, $\partial/\partial\varphi$, $\partial/\partial\psi$

- Komar mass :

$$M = \pi\frac{2r_\infty^6(mr_\infty^2 - q^2) - 2a^4(m+q)q^2 - a^2(q^4 - 4mq^2r_\infty^2 + (4m^2 + 4mq + 3q^2)r_\infty^4)}{2r_\infty^2(r_\infty^6 - a^2(q^2 - 2(m+q)r_\infty^2))\rho_0}L,$$

- Komar angular momenta :

$$J_\phi = 0,$$

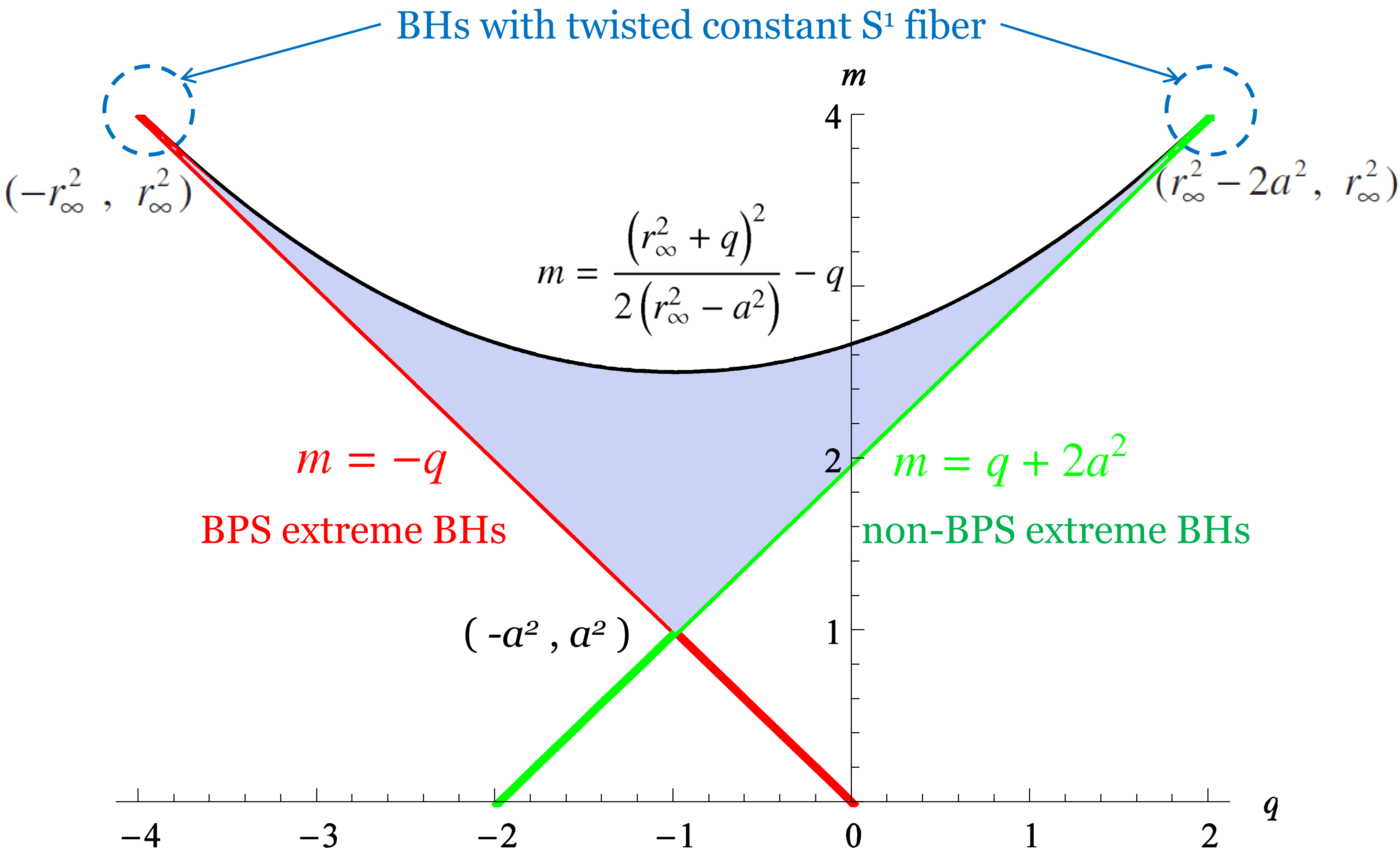
$$J_\psi = -\pi\frac{a(a^2q^3 + 3q^2r_\infty^4 - 2(2m+q)r_\infty^6)}{4r_\infty^4\sqrt{r_\infty^6 - a^2(q^2 - 2(m+q)r_\infty^2)}}L,$$

- Electric charge :

$$Q = -\frac{\sqrt{3}}{2}\pi q$$

Spacetime has only one angular momentum in extra direction

Various Limits



BHs with twisted constant S¹ fiber

$$(q, m) \longrightarrow (-r_\infty^2, r_\infty^2)$$

$$ds^2 = -\frac{4(\rho-\rho_+)(\rho-\rho_-)-a^2}{4\rho^2}dT^2 + \frac{d\rho^2}{(1-\frac{\rho_+}{\rho})(1-\frac{\rho_-}{\rho})} + \rho^2 d\Omega_{S^2}^2 + \frac{4\rho_+\rho_- - a^2}{4}\sigma_3^2 + a\frac{\sqrt{4\rho_+\rho_- - a^2}}{2\rho}dT\sigma_3,$$

$$(q, m) \longrightarrow (r_\infty^2 - 2a^2, r_\infty^2).$$

$$ds^2 = -\frac{16\rho_+^2\rho_-^2(\rho-\rho_+)(\rho-\rho_-) - a^2(a^2-6\rho_+\rho_-)^2}{16\rho_+^2\rho_-^2\rho^2}dT^2 + \frac{d\rho^2}{(1-\frac{\rho_+}{\rho})(1-\frac{\rho_-}{\rho})} + \rho^2 d\Omega_{S^2}^2 + \frac{(4\rho_+\rho_- - a^2)(2\rho_+\rho_- + a^2)^2}{16\rho_+^2\rho_-^2}\sigma_3^2 + a\frac{a^2-6\rho_+\rho_-}{2\rho_+\rho_-\rho}\sqrt{\frac{(4\rho_+\rho_- - a^2)(2\rho_+\rho_- + a^2)^2}{16\rho_+^2\rho_-^2}}dT\sigma_3,$$

Timelike Geodesics in Charged Rotating Squashed Kaluza-Klein Black Hole Spacetimes

Lagrangian and constants of motion for neutral test particles

- Metric

$$ds^2 = -F(\rho)dt^2 + U(\rho)\left[\frac{d\rho^2}{V(\rho)} + \rho^2 d\Omega_{S^2}^2\right] + W(\rho)\left(d\psi + \cos\theta d\phi\right)^2 + 2K(\rho)dt\left(d\psi + \cos\theta d\phi\right)$$

- Lagrangian : $2\mathcal{L} = -1$

$$\mathcal{L} = \frac{1}{2}\left[-F\dot{t}^2 + \frac{U}{V}\dot{\rho}^2 + U\rho^2\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right) + W\left(\dot{\psi} + \cos\theta\dot{\phi}\right)^2 + 2K\dot{t}\left(\dot{\psi} + \cos\theta\dot{\phi}\right)\right]$$

- Constants of motion :

$$E := F\dot{t} - K\left(\dot{\psi} + \cos\theta\dot{\phi}\right),$$

$$l := U\rho^2\sin^2\theta\dot{\phi} + \left[K\dot{t} + W\left(\dot{\psi} + \cos\theta\dot{\phi}\right)\right]\cos\theta,$$

$$p_\psi := K\dot{t} + W\left(\dot{\psi} + \cos\theta\dot{\phi}\right).$$

Timelike geodesics without extra dimensional direction

- Lagrangian : $2\mathcal{L} = -1$

$$\mathcal{L} = \frac{1}{2}\left[-F\dot{t}^2 + \frac{U}{V}\dot{\rho}^2 + U\rho^2\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right) + W\left(\dot{\psi} + \cos\theta\dot{\phi}\right)^2 + 2K\dot{t}\left(\dot{\psi} + \cos\theta\dot{\phi}\right)\right]$$

- Assumption : Particle has no momentum in extra direction

$$p_\psi = K\dot{t} + W\left(\dot{\psi} + \cos\theta\dot{\phi}\right) = 0$$

- Effective Lagrangian

$$\mathcal{L}_{\rm eff} = \frac{1}{2}\left[-\left(F + \frac{K^2}{W}\right)\dot{t}^2 + \frac{U}{V}\dot{\rho}^2 + U\rho^2\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right)\right] \qquad 2\mathcal{L}_{\rm eff} = -1$$

We can concentrate on orbits with $\theta=\pi/2$ on assumption of $p_\psi=0$

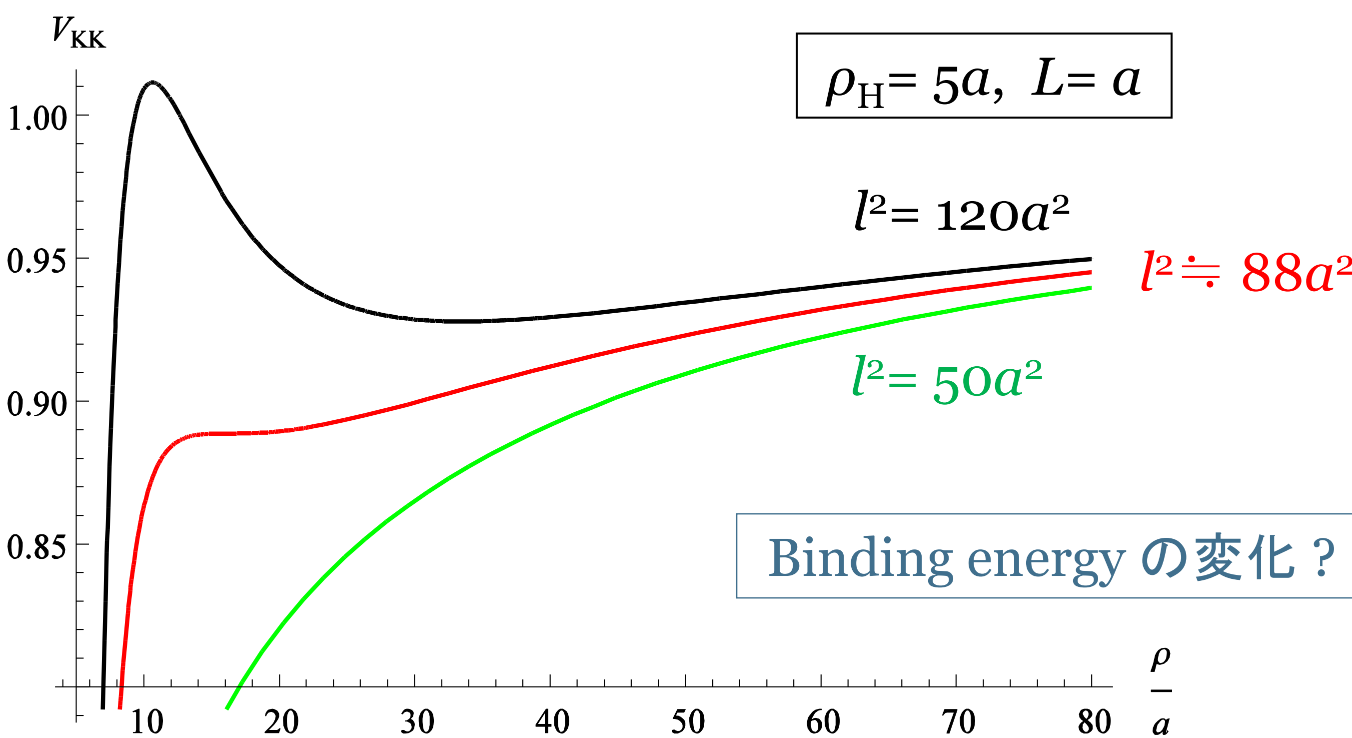
Timelike Geodesics

- Energy conservation equation

$$\left(F+\frac{K^2}{W}\right)\frac{U}{V}\left(\frac{d\rho}{d\tau}\right)^2+V_{\rm KK}(\rho)=E^2$$

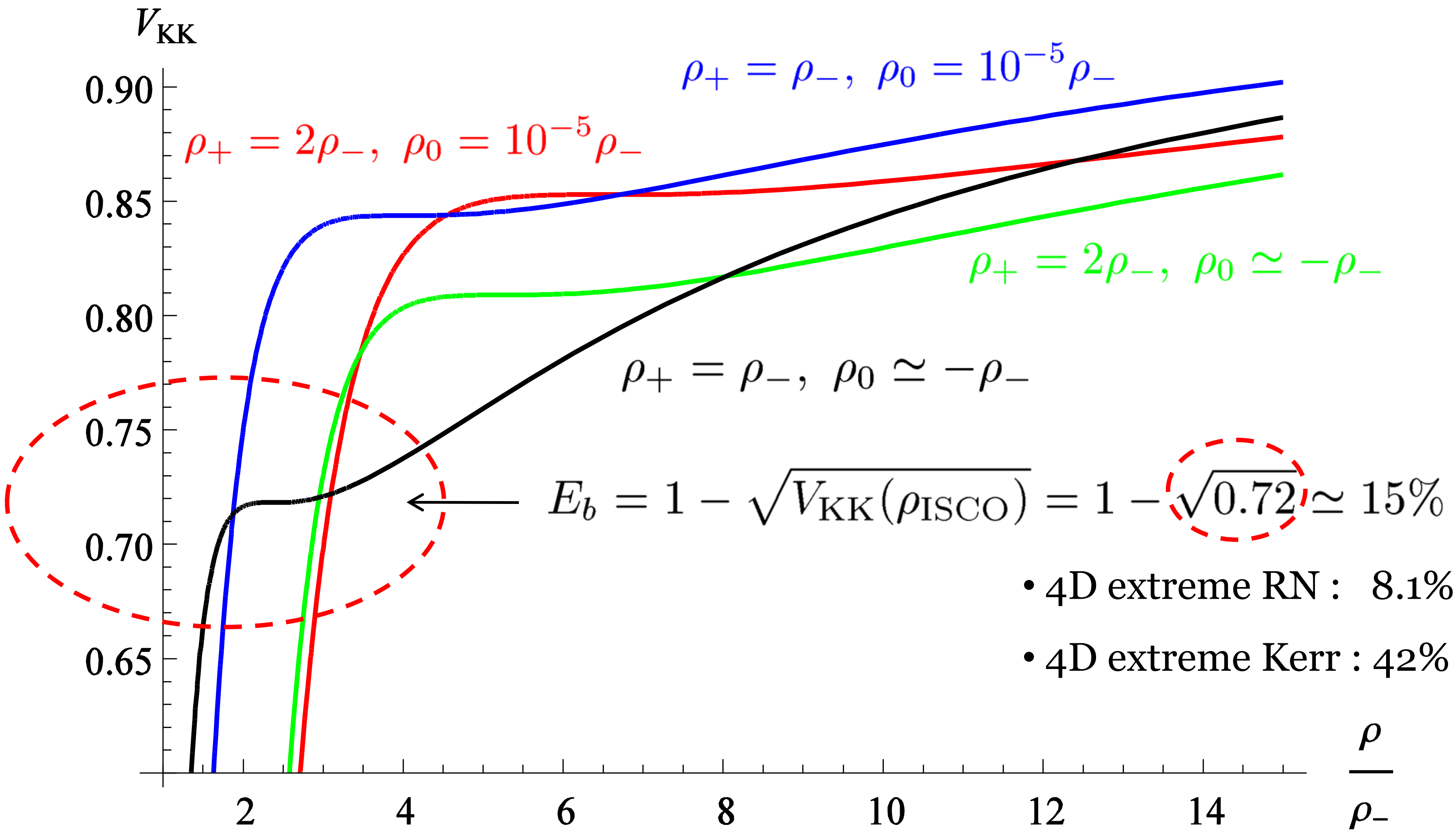
- Effective potential

$$V_{\rm KK}(\rho)=\frac{L^2\rho r_\infty^2\left(\rho^2r_\infty^4-2m\rho\left(\rho+\rho_0\right)r_\infty^2+\left(2(m+q)a^2+q^2\right)\left(\rho+\rho_0\right)^2\right)}{\rho_0^2\left(\rho^3r_\infty^6+2a^2(m+q)\rho\left(\rho+\rho_0\right)^2r_\infty^2-a^2q^2\left(\rho+\rho_0\right)^3\right)}\left(1+\frac{l^2}{\rho\left(\rho+\rho_0\right)}\right)$$



Binding energy の変化？

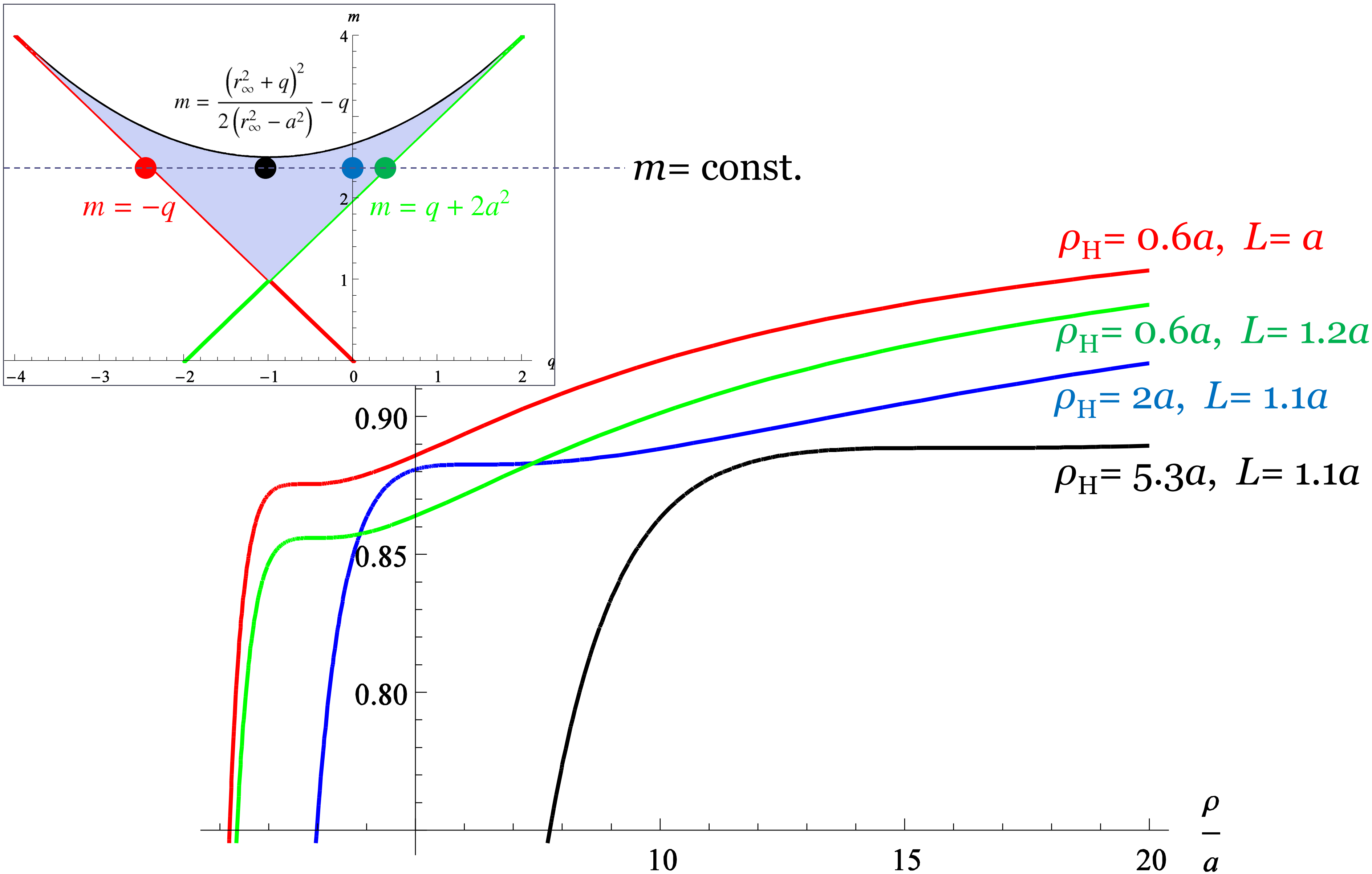
(復習) Innermost stable circular orbits around charged static squashed BHs



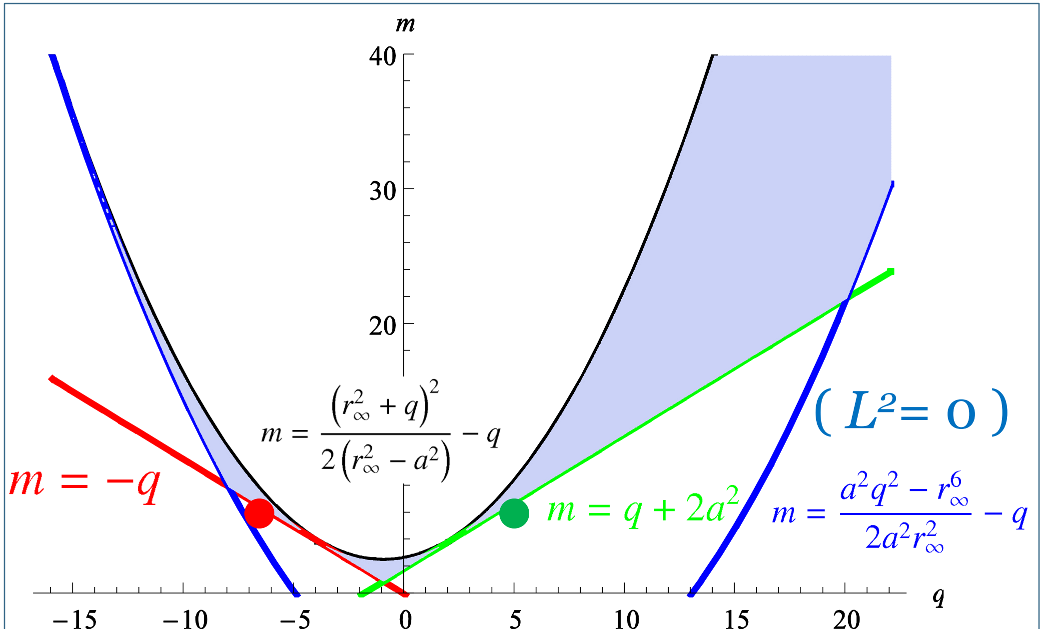
余剰次元サイズ： $r_\infty^2 = 4(\rho_+ + \rho_0)(\rho_- + \rho_0)$

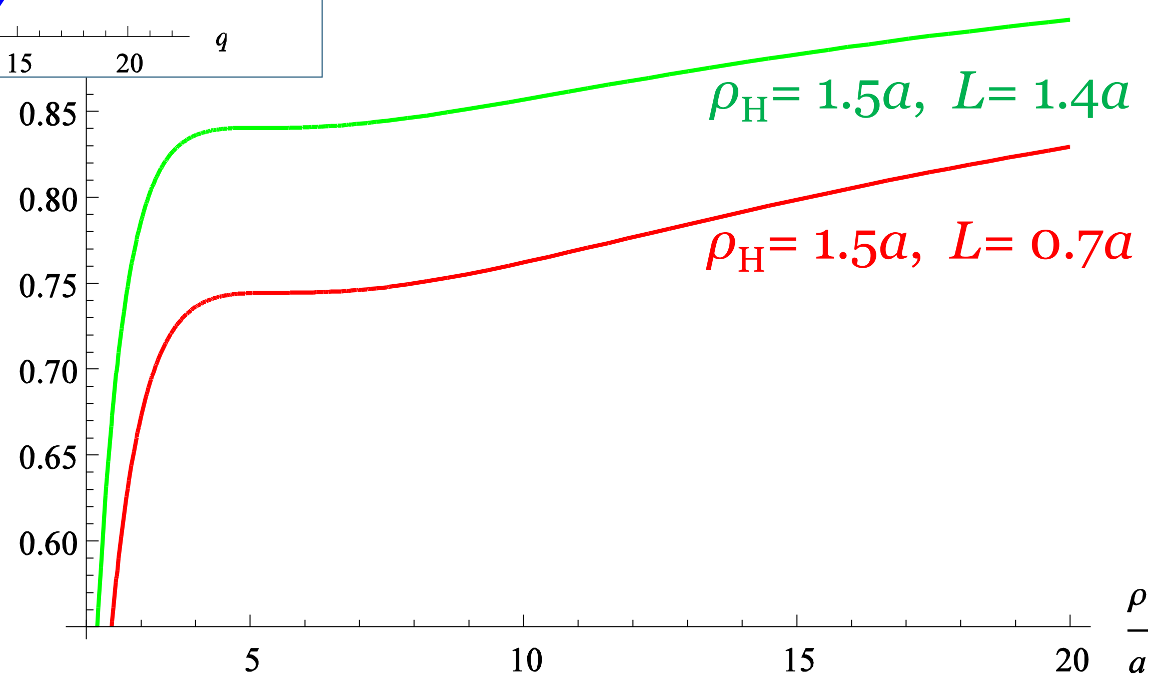
"天体的ブラックホール" 程、Binding Energy ⇒ 大

Innermost stable circular orbits around charged rotating squashed BHs

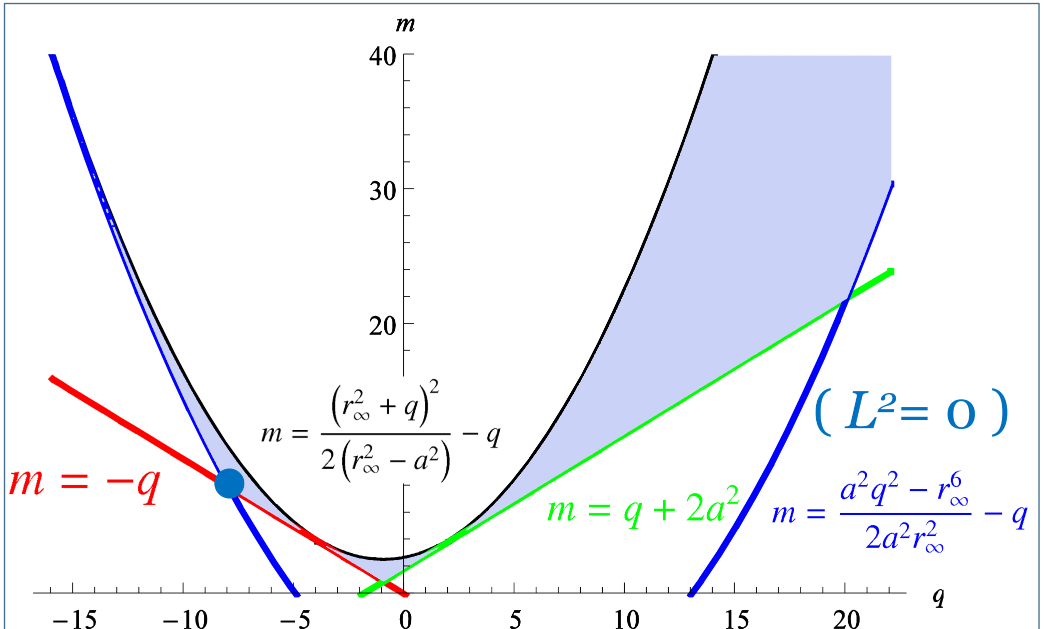


ISCOs around charged rotating squashed extreme BHs

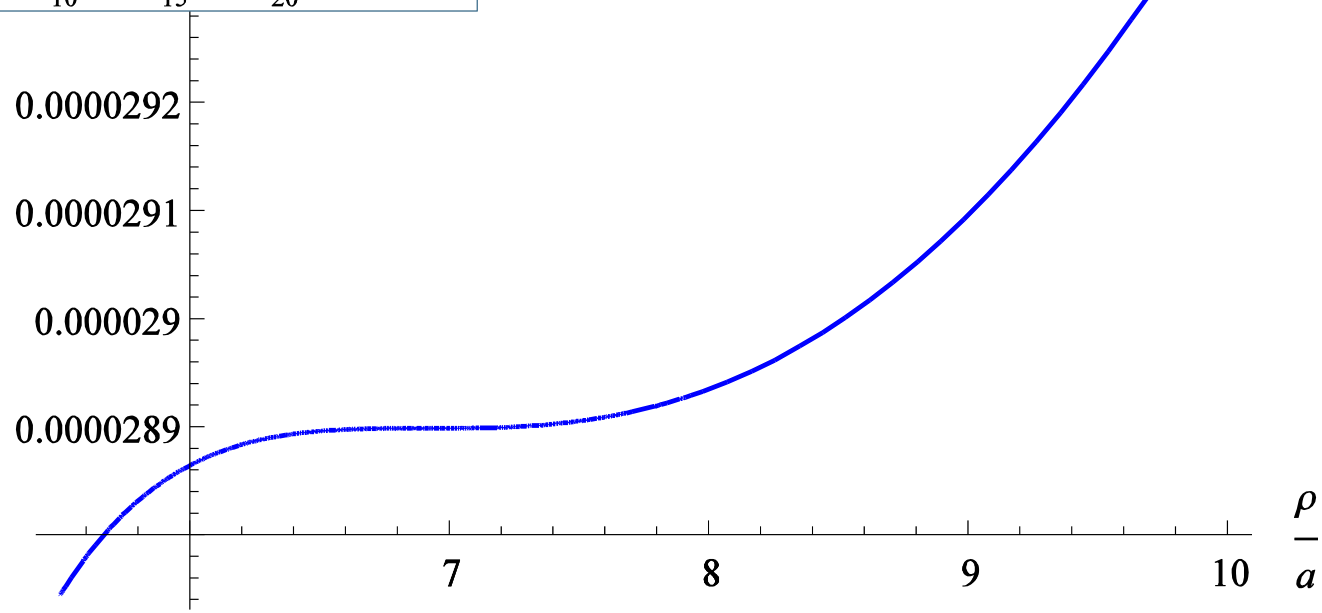




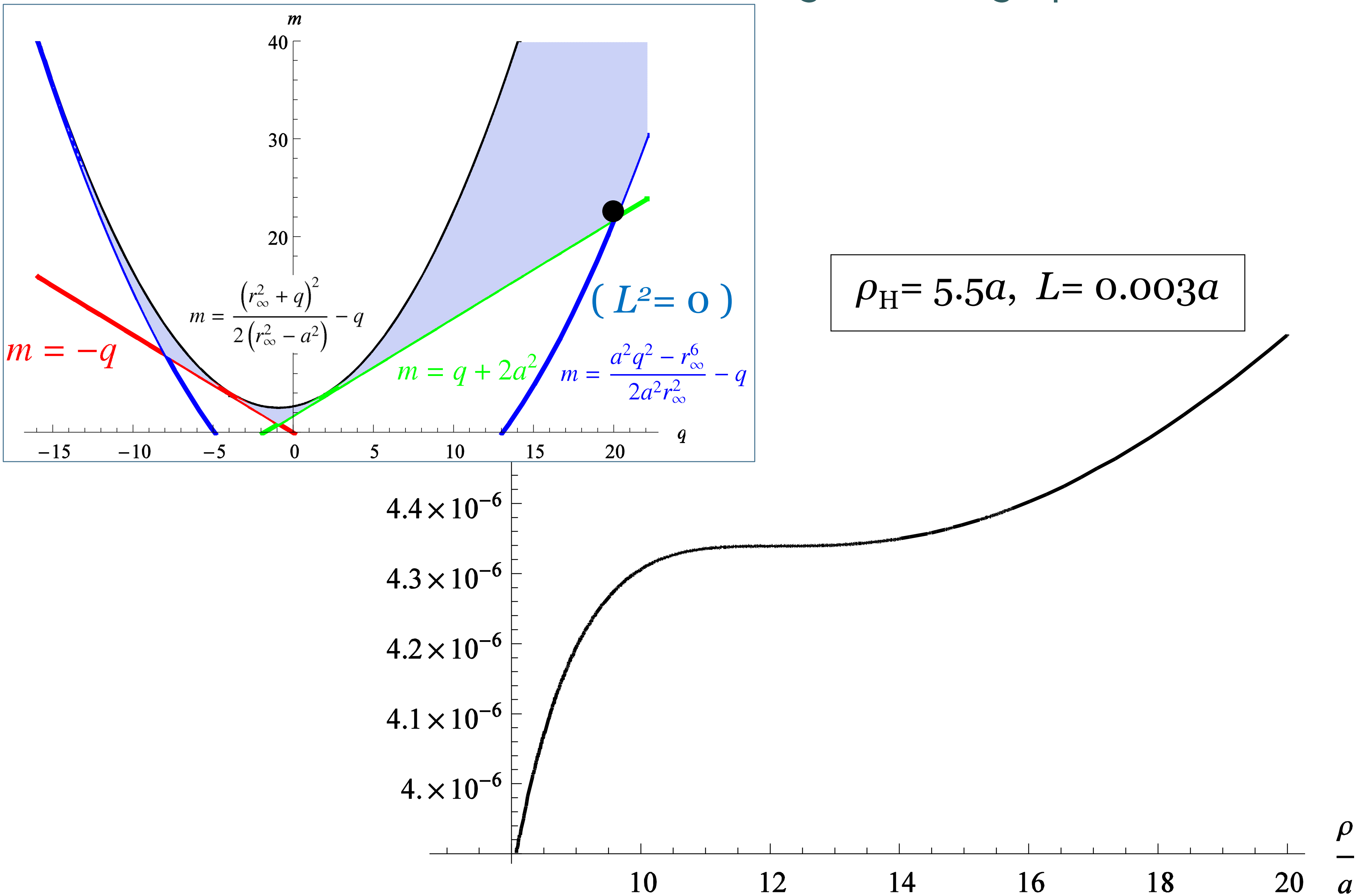
Innermost stable circular orbits around charged rotating squashed BHs



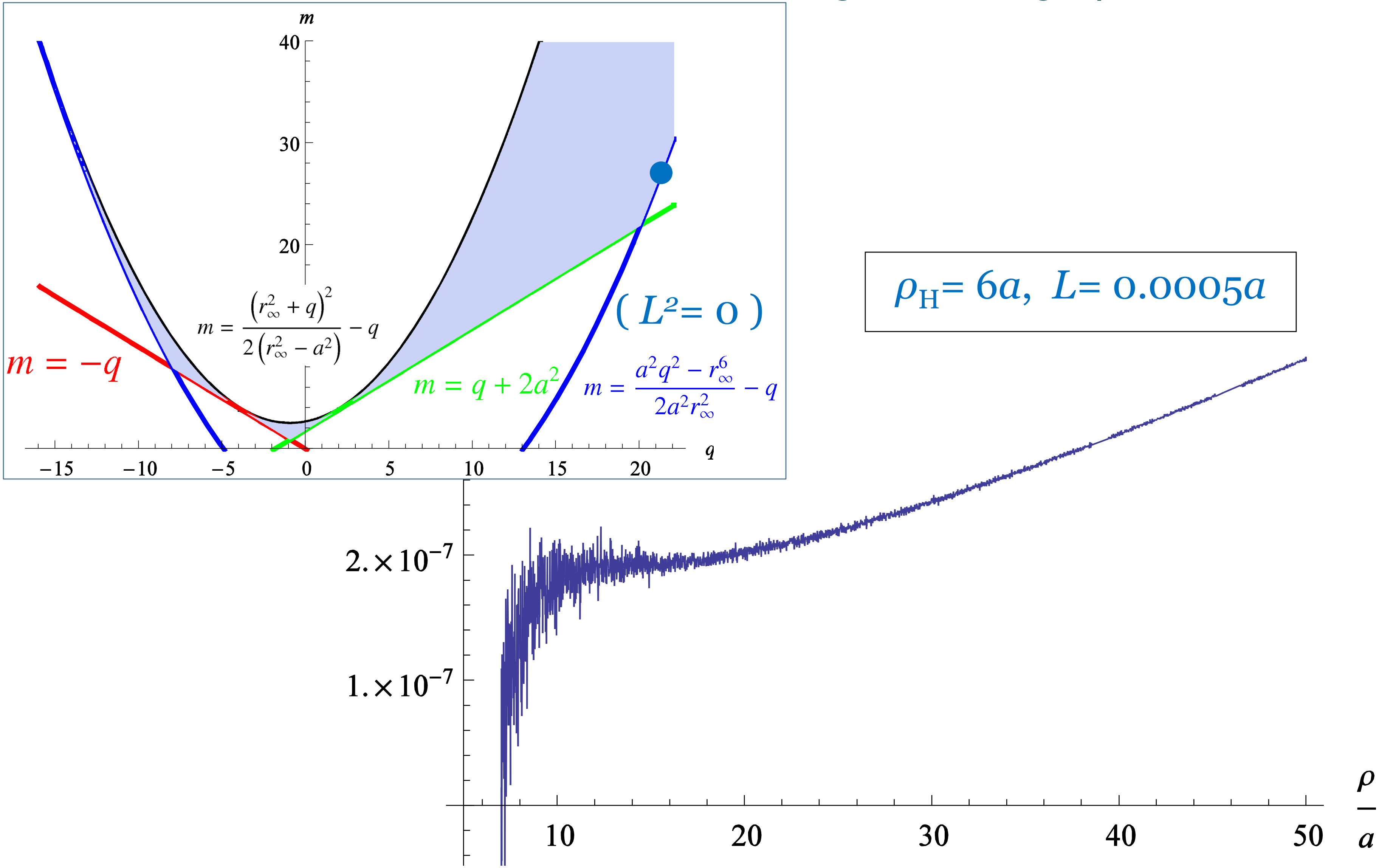
$\rho_H = 2a, \; L = 0.002a$



Innermost stable circular orbits around charged rotating squashed BHs



Innermost stable circular orbits around charged rotating squashed BHs



Binding energy $E_b \Rightarrow$ 100% for "astronomical" squashed BHs

Conclusion

We consider timelike geodesics of neutral test particles around 5D charged rotating squashed Kaluza-Klein black holes

- Types of geodesics are similar to those of 4D black holes
- Binding energy $E_b \Rightarrow$ 100% for "astronomical" squashed BHs

cf.

- Charged particles in 4D extreme RN : ~100% as $|q| \to$ large

  (Johnston & Ruffini, 1974)
- Neutral particles in superspinar of string theory : ~100%

  (Gimon & Horava, 2009)

Future works

- Classical tests in squashed Kaluza-Klein black hole spacetimes
  - Light deflection
  - Time delay
  - Perihelion shift

- D>5 squashed BPS Kaluza-Klein black holes
  - D=4, 5 : special cases ?

- 5D charged slowly rotating Kaluza-Klein dilaton black holes with arbitrary Chern-Simons coupling
  - Counterrotating horizons ?
  - Instabilities ?

Large Scale Extra Dimension in Brane world model

D次元時空（D ≧ 4）（余剰次元サイズ L）

$$G_D = L^{D-4} G_4$$

：D次元重力定数

$$E_{\mathrm{P,D}} = \left(\frac{\hbar^{D-3} c^{D+1}}{G_D} \right)^{1/(D-2)}$$

：D次元プランクエネルギー

cf. $E_{\mathrm{P,4}} = \sqrt{\hbar c^5 / G_4} \simeq 10^{19}\mathrm{GeV}$

- When $E_{\mathrm{P,D}}$ ≒ TeV ，D = 6

$$L = \hbar c \left(\frac{{E_{\mathrm{P,4}}}^2}{{E_{\mathrm{P,D}}}^{D-2}} \right)^{1/(D-4)} \simeq 0.1 \text{ mm}$$

ミニ・ブラックホールの形成条件

コンプトン波長 $r_{\mathrm{Comp}} \simeq \hbar / mc$

ブラックホール半径 r_{Sch}

$$r_{\mathrm{Sch}} \geq r_{\mathrm{Comp}}$$

［4次元］ $r_{\mathrm{Sch}} \simeq G_4 m / c^2$

Particle with $mc^2 \geq E_{\mathrm{P},4} = \sqrt{\dfrac{\hbar c^5}{G_4}} \simeq 10^{19}\ \mathrm{GeV}$ ≫ 1 GeV : 1 Proton

［D次元］ $r_{\mathrm{Sch}} \simeq \left(G_D m / c^2\right)^{1/(D-3)}$

Particle with $mc^2 \geq E_{\mathrm{P,D}} = \left(\dfrac{\hbar^{D-3} c^{D+1}}{G_D}\right)^{1/(D-2)}$

例．LHC 加速器内：$E_{\mathrm{P,D}}$ ≒ TeV

⇒ mc^2 ≧ TeV ≒ (proton mass)×10^3 ミニ・ブラックホール！

Three-sphere S^3

$$d\Omega_{S^3}^2 = d\chi^2 + \sin^2\chi\left(d\nu^2 + \sin^2\nu d\xi^2\right)$$

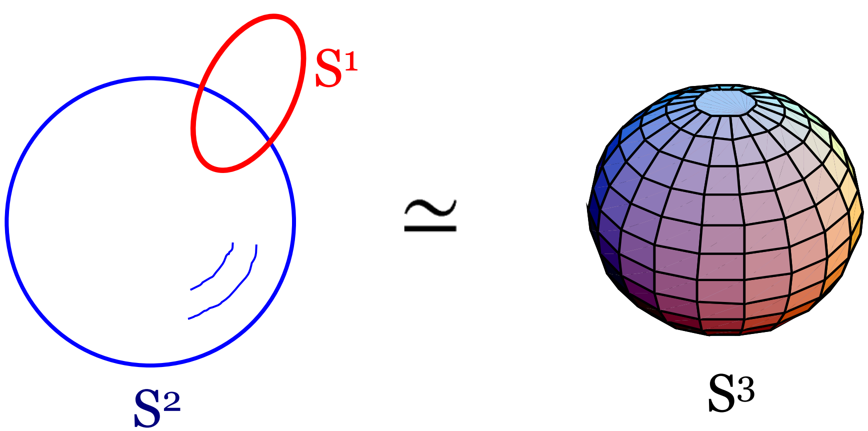
$$(\quad 0 \le \chi \le \pi, \quad 0 \le \nu \le \pi, \quad 0 \le \xi \le 2\pi \quad)$$

$$\Rightarrow \quad d\Omega_{S^3}^2 = \frac{1}{4}\left[d\Omega_{S^2}^2 + (d\psi + \cos\theta d\phi)^2\right]$$

(S^2 base) (twisted S^1 fiber)

$$\left(\quad d\Omega_{S^2}^2 = d\theta^2 + \sin^2\theta d\phi^2 \quad\right)$$

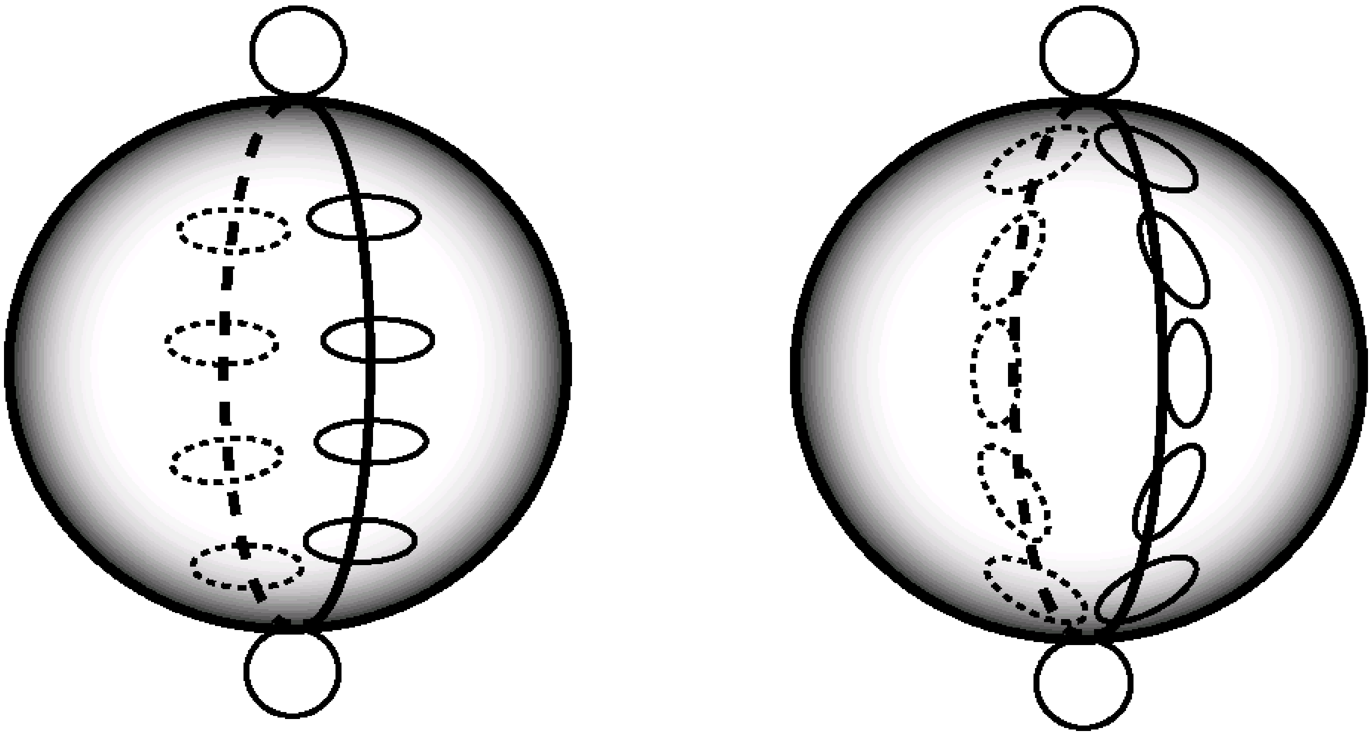
$$(\quad 0 \le \theta \le \pi, \quad 0 \le \phi \le 2\pi, \quad 0 \le \psi \le 4\pi \quad)$$



Three-sphere S^3

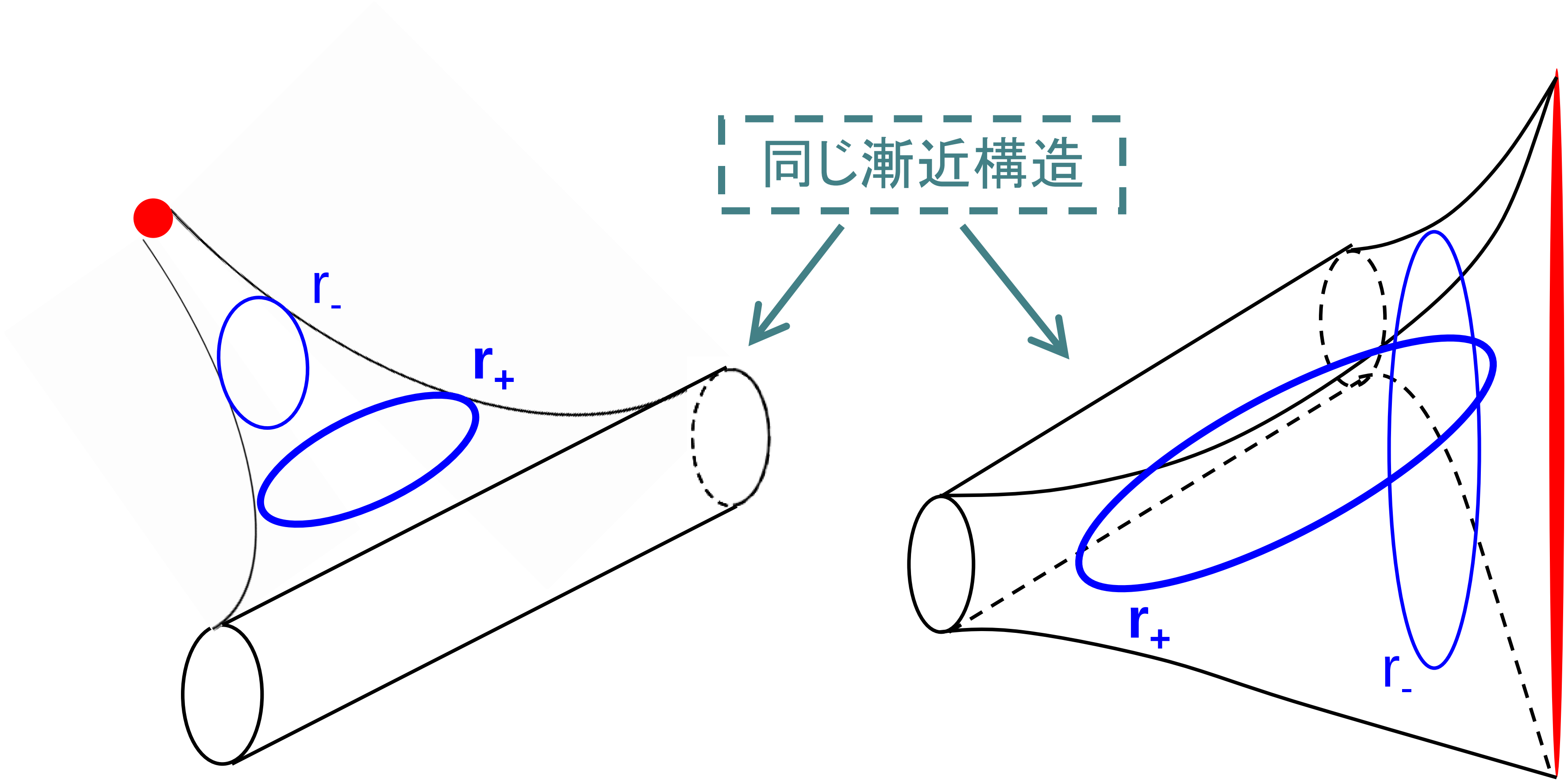
$$d\Omega^2_{S^3} = \frac{1}{4}\left[d\Omega^2_{S^2} + (d\psi + \cos\theta d\phi)^2\right]$$

(S^2 base) (twisted S^1 fiber)



$S^2 \times S^1$ S^3

Two types of Kaluza-Klein BHs



Point Singularity

Stretched Singularity