On Matrix Models of Chern-Simons-matter Theories

Takao Suyama (SNU)

Ref) 1304.????

Introduction

Chern-Simons-matter theories have been studied recently.

Introduction

Chern-Simons-matter theories have been studied recently.

- M2-branes and gravity duals [ABJM]

  $\exists$ interesting observables

  - Free energy $\Rightarrow$ # of d.o.f. [DMP]
  - Wilson loop $\Rightarrow$ minimal surface [Marino,Putrov]

Introduction

Chern-Simons-matter theories have been studied recently.

- M2-branes and gravity duals [ABJM]

  ∃ interesting observables

  - Free energy ⇨ # of d.o.f. [DMP]
  - Wilson loop ⇨ minimal surface [Marino,Putrov]

- Superconformal field theoies in 3-dim. [Gaiotto,Yin]

  - $N \geq 3$ SUSY ⇔ non-renormalization
  - Seiberg-like duality [Giveon,Kutasov]

Introduction

Chern-Simons-matter theories have been studied recently.

- M2-branes and gravity duals [ABJM]

  $\exists$ interesting observables

  - Free energy $\Rightarrow$ # of d.o.f. [DMP]
  - Wilson loop $\Rightarrow$ minimal surface [Marino,Putrov]

- Superconformal field theoies in 3-dim. [Gaiotto,Yin]

  - $N \geq 3$ SUSY $\Leftrightarrow$ non-renormalization
  - Seiberg-like duality [Giveon,Kutasov]

- Matrix models via localization

  A new family of non-trivial matrix models.

Introduction

Chern-Simons-matter theories have been studied recently.

- M2-branes and gravity duals [ABJM]

  ∃ interesting observables

  - Free energy ⇨ # of d.o.f. [DMP]
  - Wilson loop ⇨ minimal surface [Marino,Putrov]

- Superconformal field theoies in 3-dim. [Gaiotto,Yin]

  - $N \geq 3$ SUSY ⇔ non-renormalization
  - Seiberg-like duality [Giveon,Kutasov]

- Matrix models via localization

  A new family of non-trivial matrix models.

⇨ How to solve them, at least in the planar limit?

Systematic way to solve them?

A family of Chern-Simons-matter theories

A family $\boldsymbol{F}$ consists of N=3 CSM theories such that

- gauge group $G = \prod_{a=1}^{n_g} \mathrm{U}(N_a)_{k_a}$ (k_a: Chern-Simons level)
- matter hypermultiplets in $R \oplus \bar{R}$ compatible with planar limit.
  (I.e. matters with many indices are forbidden.)

A family of Chern-Simons-matter theories

A family $\boldsymbol{F}$ consists of N=3 CSM theories such that

- gauge group $G = \prod_{a=1}^{n_g} \mathrm{U}(N_a)_{k_a}$ (k_a : Chern-Simons level)
- matter hypermultiplets in $R \oplus \bar{R}$ compatible with planar limit.
  (I.e. matters with many indices are forbidden.)

Each theory in $\boldsymbol{F}$ is specified by G and $R \oplus \bar{R}$.

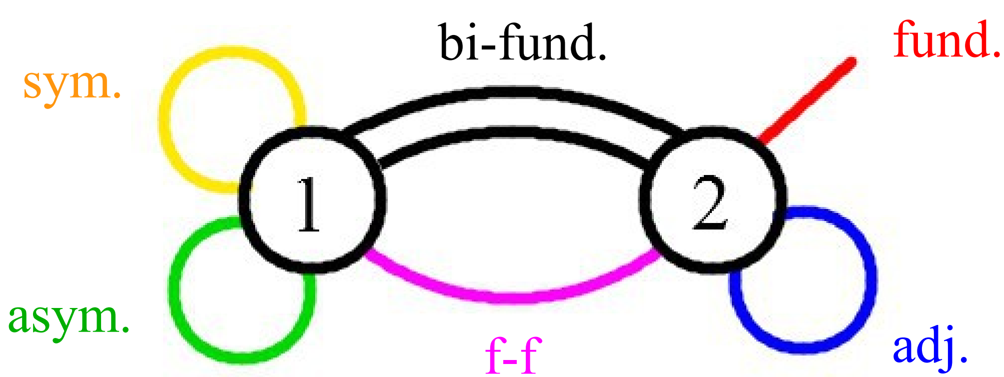
A family of Chern-Simons-matter theories

A family $\boldsymbol{F}$ consists of N=3 CSM theories such that

- gauge group $G = \prod_{a=1}^{n_g} \mathrm{U}(N_a)_{k_a}$ (k_a : Chern-Simons level)
- matter hypermultiplets in $R \oplus \bar{R}$ compatible with planar limit.
  (I.e. matters with many indices are forbidden.)

Each theory in $\boldsymbol{F}$ is specified by $\underline{G \text{ and } R \oplus \bar{R}}$.

$\Longleftrightarrow$ A diagram Γ with colored edges.



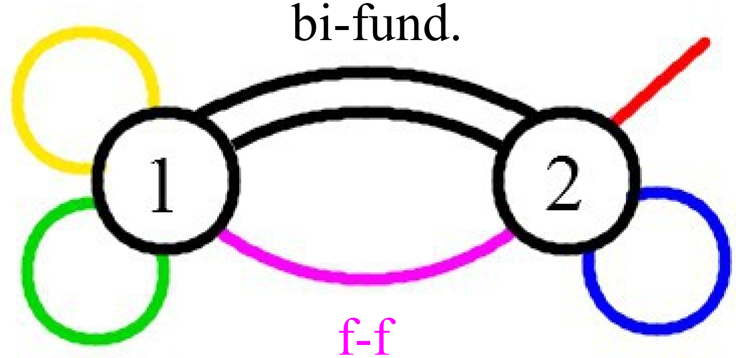
A family of Chern-Simons-matter theories

A family $\boldsymbol{F}$ consists of N=3 CSM theories such that

- gauge group $G = \prod_{a=1}^{n_g} \mathrm{U}(N_a)_{k_a}$ (k_a : Chern-Simons level)
- matter hypermultiplets in $R \oplus \bar{R}$ compatible with planar limit.
  (I.e. matters with many indices are forbidden.)

Each theory in $\boldsymbol{F}$ is specified by G and $R \oplus \bar{R}$.

$\Longleftrightarrow$ A diagram Γ with colored edges.



Note: Not all such diagrams define consistent QFT.

(E.g. ABJ theory with $|N_1 - N_2| > k$) [ABJ]

Saddle-point equations

The sphere partition functions are given as finite-dim integrals.

[Kapustin, Willett, Yaakov]

$$Z(\Gamma) \propto \int \prod du^a_{i_a} \exp\left(-S_{\text{tree}}[u] - S_v[u] - S_m[u]\right).$$

Saddle-point equations

The sphere partition functions are given as finite-dim integrals.

[Kapustin, Willett, Yaakov]

$$Z(\Gamma) \propto \int \prod du_{i_a}^{a} \exp\left(-S_{\text{tree}}[u]-S_{v}[u]-S_{m}[u]\right).$$

e.v. for nodes

vector

matters

Saddle-point equations

The sphere partition functions are given as finite-dim integrals.

[Kapustin, Willett, Yaakov]

$$Z(\Gamma) \propto \int \prod du^a_{i_a} \exp\left(-S_{\text{tree}}[u] - S_v[u] - S_m[u]\right).$$

e.v. for nodes vector matters

Consider the planar limit defined as

$$\underline{N_a, k_a, n^a_f \propto k, \quad k \to \infty}.$$

⇒ The exponent of the integrand scales as k^2.

A saddle-point dominates the integral.

Saddle-point equations

The sphere partition functions are given as finite-dim integrals.

[Kapustin, Willett, Yaakov]

$$Z(\Gamma) \propto \int \prod du^a_{i_a} \exp\left(-S_{\text{tree}}[u] - S_v[u] - S_m[u]\right).$$

e.v. for nodes   vector   matters

Consider the planar limit defined as

$$N_a, k_a, n^a_f \propto k, \quad k \to \infty.$$

⇨ The exponent of the integrand scales as k^2.

A saddle-point dominates the integral.

The saddle-point is determined by

$$\boxed{\frac{k_a}{2\pi} u^a_{i_a} = \sum_{j_a \neq i_a} \coth \frac{u^a_{i_a} - u^a_{j_a}}{2} - \frac{\partial S_m[u]}{\partial u^a_{i_a}}.}$$

Saddle-point equations

The sphere partition functions are given as finite-dim integrals.

[Kapustin, Willett, Yaakov]

$$Z(\Gamma) \propto \int \prod du^a_{i_a} \exp\left(-S_{\text{tree}}[u] - S_v[u] - S_m[u]\right).$$

e.v. for nodes   vector   matters

Consider the planar limit defined as

$$\underline{N_a, k_a, n^a_f \propto k, \quad k \to \infty}.$$

⇨ The exponent of the integrand scales as k^2.

A saddle-point dominates the integral.

The saddle-point is determined by

$$\boxed{\frac{k_a}{2\pi} u^a_{i_a} = \sum_{j_a \neq i_a} \coth \frac{u^a_{i_a} - u^a_{j_a}}{2} - \frac{\partial S_m[u]}{\partial u^a_{i_a}}.}$$

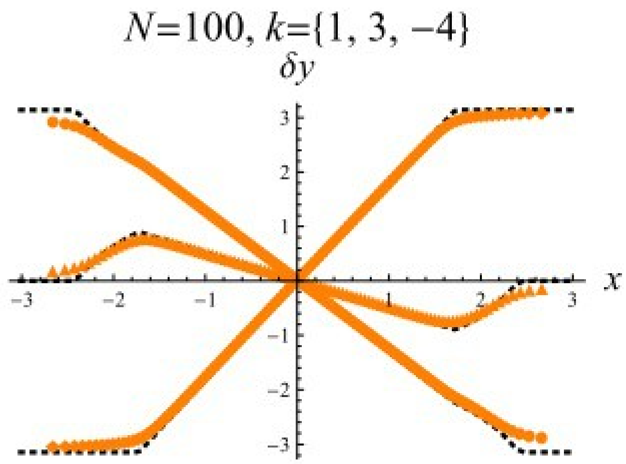
Note: k_a are assumed to be real positive.
Later, they will be analytically continued.

Planar relations

Assumption: E.v. distributions are symmetric under $u_{i_a}^a \to -u_{i_a}^a$.

(Based on numerical results in [Herzog,Klebanov,Pufu,Tesileanu])

⇨ Various theories share the same e.v. distributions.



Planar relations

Assumption: E.v. distributions are symmetric under $u^a_{i_a} \to -u^a_{i_a}$.

(Based on numerical results in [Herzog,Klebanov,Pufu,Tesileanu])

$\Rightarrow$ Various theories share the same e.v. distributions.

E.g. 1) bi-fund. vs fund.-fund.

$$\frac{\partial S_{\mathrm{bf}}}{\partial u^a_{i_a}} = -\frac{n^a_{\mathrm{bf}}}{2}\sum_{j_b}\tanh\frac{u^a_{i_a}-u^b_{j_b}}{2} \quad \Longleftrightarrow \quad \frac{\partial S_{\mathrm{ff}}}{\partial u^a_{i_a}} = -\frac{n^a_{\mathrm{ff}}}{2}\sum_{j_b}\tanh\frac{u^a_{i_a}+u^b_{j_b}}{2}$$

Planar relations

Assumption: E.v. distributions are symmetric under $u_{i_a}^a \to -u_{i_a}^a$.

(Based on numerical results in [Herzog,Klebanov,Pufu,Tesileanu])

⇨ Various theories share the same e.v. distributions.

E.g. 1) bi-fund. vs fund.-fund.

$$\frac{\partial S_{\text{bf}}}{\partial u_{i_a}^a} = -\frac{n_{\text{bf}}^a}{2}\sum_{j_b} \tanh\frac{u_{i_a}^a - u_{j_b}^b}{2} \quad \Leftrightarrow \quad \frac{\partial S_{\text{ff}}}{\partial u_{i_a}^a} = -\frac{n_{\text{ff}}^a}{2}\sum_{j_b} \tanh\frac{u_{i_a}^a + u_{j_b}^b}{2}$$

Planar relations

Assumption: E.v. distributions are symmetric under $u^a_{i_a} \to -u^a_{i_a}$.

(Based on numerical results in [Herzog,Klebanov,Pufu,Tesileanu])

$\Longrightarrow$ Various theories share the same e.v. distributions.

E.g. 1) bi-fund. vs fund.-fund.

$$\frac{\partial S_{\rm bf}}{\partial u^a_{i_a}} = -\frac{n^a_{\rm bf}}{2}\sum_{j_b}\tanh\frac{u^a_{i_a}-u^b_{j_b}}{2} \quad \Longleftrightarrow \quad \frac{\partial S_{\rm ff}}{\partial u^a_{i_a}} = -\frac{n^a_{\rm ff}}{2}\sum_{j_b}\tanh\frac{u^a_{i_a}+u^b_{j_b}}{2}$$

E.g. 2) sym. asym. vs adj.

$$\frac{\partial S}{\partial u^a_{i_a}} = -\frac{n^a}{2}\sum_{j_a}\tanh\frac{u^a_{i_a}+u^a_{j_a}}{2} \mp \frac{n^a}{2}\tanh u^a_{i_a} \qquad \text{for} \begin{cases}\text{sym.}\\ \text{asym.}\end{cases}$$

$$\Longleftrightarrow \quad \frac{\partial S_{\rm adj}}{\partial u^a_{i_a}} = -n^a_{\rm adj}\sum_{j_a}\tanh\frac{u^a_{i_a}-u^a_{j_a}}{2}$$

Planar relations

Assumption: E.v. distributions are symmetric under $u_{i_a}^a \to -u_{i_a}^a$.

(Based on numerical results in [Herzog,Klebanov,Pufu,Tesileanu])

$\Longrightarrow$ Various theories share the same e.v. distributions.

E.g. 1) bi-fund. vs fund.-fund.

$$\frac{\partial S_{\mathrm{bf}}}{\partial u_{i_a}^a} = -\frac{n_{\mathrm{bf}}^a}{2} \sum_{j_b} \tanh \frac{u_{i_a}^a - u_{j_b}^b}{2} \quad \Longleftrightarrow \quad \frac{\partial S_{\mathrm{ff}}}{\partial u_{i_a}^a} = -\frac{n_{\mathrm{ff}}^a}{2} \sum_{j_b} \tanh \frac{u_{i_a}^a + u_{j_b}^b}{2}$$

E.g. 2) sym. asym. vs adj.

$$\frac{\partial S}{\partial u_{i_a}^a} = -\frac{n^a}{2} \sum_{j_a} \tanh \frac{u_{i_a}^a + u_{j_a}^a}{2} \mp \frac{n^a}{2} \tanh u_{i_a}^a \qquad \text{for} \begin{cases} \text{sym.} \\ \text{asym.} \end{cases}$$

$$\Longleftrightarrow \quad \frac{\partial S_{\mathrm{adj}}}{\partial u_{i_a}^a} = -n_{\mathrm{adj}}^a \sum_{j_a} \tanh \frac{u_{i_a}^a - u_{j_a}^a}{2}$$

Planar relations

Assumption: E.v. distributions are symmetric under $u_{i_a}^a \to -u_{i_a}^a$.

(Based on numerical results in [Herzog,Klebanov,Pufu,Tesileanu])

$\Longrightarrow$ Various theories share the same e.v. distributions.

E.g. 1) bi-fund. vs fund.-fund.

$$\frac{\partial S_{\mathrm{bf}}}{\partial u_{i_a}^a} = -\frac{n_{\mathrm{bf}}^a}{2}\sum_{j_b}\tanh\frac{u_{i_a}^a - u_{j_b}^b}{2} \quad \Longleftrightarrow \quad \frac{\partial S_{\mathrm{ff}}}{\partial u_{i_a}^a} = -\frac{n_{\mathrm{ff}}^a}{2}\sum_{j_b}\tanh\frac{u_{i_a}^a + u_{j_b}^b}{2}$$

E.g. 2) sym. asym. vs adj.

$$\frac{\partial S}{\partial u_{i_a}^a} = -\frac{n^a}{2}\sum_{j_a}\tanh\frac{u_{i_a}^a + u_{j_a}^a}{2} \mp \frac{n^a}{2}\tanh u_{i_a}^a \quad \text{for} \begin{cases}\text{sym.}\\ \text{asym.}\end{cases}$$

$$\Longleftrightarrow \quad \frac{\partial S_{\mathrm{adj}}}{\partial u_{i_a}^a} = -n_{\mathrm{adj}}^a\sum_{j_a}\tanh\frac{u_{i_a}^a - u_{j_a}^a}{2}$$

negligible in the planar limit.

$(\, n^a = O(1)\,)$

Planar relations

Assumption: E.v. distributions are symmetric under $u^a_{i_a} \to -u^a_{i_a}$.

(Based on numerical results in [Herzog,Klebanov,Pufu,Tesileanu])

$\Longrightarrow$ Various theories share the same e.v. distributions.

E.g. 1) bi-fund. vs fund.-fund.

$$\frac{\partial S_{\rm bf}}{\partial u^a_{i_a}} = -\frac{n^a_{\rm bf}}{2}\sum_{j_b}\tanh\frac{u^a_{i_a} - u^b_{j_b}}{2} \quad \Longleftrightarrow \quad \frac{\partial S_{\rm ff}}{\partial u^a_{i_a}} = -\frac{n^a_{\rm ff}}{2}\sum_{j_b}\tanh\frac{u^a_{i_a} + u^b_{j_b}}{2}$$

E.g. 2) sym. asym. vs adj.

$$\frac{\partial S}{\partial u^a_{i_a}} = -\frac{n^a}{2}\sum_{j_a}\tanh\frac{u^a_{i_a}+u^a_{j_a}}{2} \mp \frac{n^a}{2}\tanh u^a_{i_a} \qquad \text{for} \begin{cases}\text{sym.}\\ \text{asym.}\end{cases}$$

$$\Longleftrightarrow \quad \frac{\partial S_{\rm adj}}{\partial u^a_{i_a}} = -n^a_{\rm adj}\sum_{j_a}\tanh\frac{u^a_{i_a} - u^a_{j_a}}{2}$$

negligible in the planar limit.

$(\, n^a = O(1)\,)$

$\Longrightarrow$ fund-fund, sym and asym do not need to be considered.

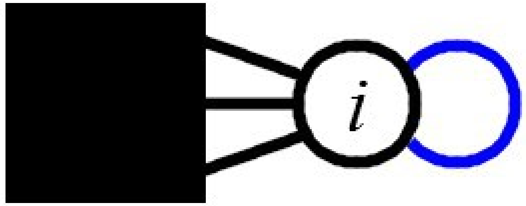
E.g. 3) adj. vs bi-fund.

adj. can be replaced with bi-fund. by duplicating diagram.

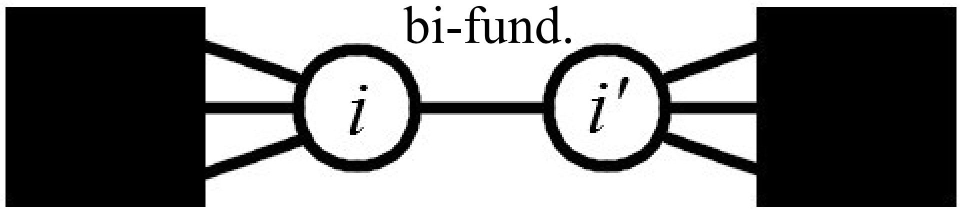
(constraint: e.v. distributions of the left are the same with the right.)

$$\frac{\partial S_{\text{adj}}}{\partial u^a_{i_a}} = -n^a_{\text{adj}} \sum_{j_a} \tanh \frac{u^a_{i_a} - u^a_{j_a}}{2}, \qquad \frac{\partial S_{\text{bf}}}{\partial u^a_{i_a}} = -\frac{n^a_{\text{bf}}}{2} \sum_{j_b} \tanh \frac{u^a_{i_a} - u^b_{j_b}}{2}.$$

E.g. 3) adj. vs bi-fund.



$\Longleftrightarrow$



adj. can be replaced with bi-fund. by duplicating diagram.

(constraint: e.v. distributions of the left are the same with the right.)

$\Longrightarrow$ All theories considered can be solved if any theory Γ consisting of only fund. and bifund. can be solved.

E.g. 3) adj. vs bi-fund.

adj. can be replaced with bi-fund. by duplicating diagram.

(constraint: e.v. distributions of the left are the same with the right.)

⇨ All theories considered can be solved if any theory Γ consisting of only fund. and bifund. can be solved.

In terms of new variables

$$z_{i_a}^a := \exp\left(u_{i_a}^a\right),$$

the saddle-point equations to be solved are

$$\boxed{\frac{k_a}{2\pi}\log z_{i_a}^a + \frac{n^a}{2}\frac{z_{i_a}^a - 1}{z_{i_a}^a + 1} = \sum_{j_a \neq i_a} \frac{z_{i_a}^a + z_{j_a}^a}{z_{i_a}^a - z_{j_a}^a} - \sum_b \frac{n^{ab}}{2}\sum_{j_b}\frac{z_{i_a}^a - z_{j_b}^b}{z_{i_a}^a + z_{j_b}^b}.}$$

E.g. 3) adj. vs bi-fund.

adj. can be replaced with bi-fund. by duplicating diagram.

(constraint: e.v. distributions of the left are the same with the right.)

⇨ All theories considered can be solved if any theory Γ consisting of only fund. and bifund. can be solved.

In terms of new variables

$$z^a_{i_a} := \exp\left(u^a_{i_a}\right),$$

the saddle-point equations to be solved are

$$\boxed{\frac{k_a}{2\pi}\log z^a_{i_a} + \frac{n^a}{2}\frac{z^a_{i_a}-1}{z^a_{i_a}+1} = \sum_{j_a\neq i_a}\frac{z^a_{i_a}+z^a_{j_a}}{z^a_{i_a}-z^a_{j_a}} - \sum_b \frac{n^{ab}}{2}\sum_{j_b}\frac{z^a_{i_a}-z^b_{j_b}}{z^a_{i_a}+z^b_{j_b}}.}$$

different signs

Resolvent equations

In terms of resolvents defined as

$$v^a(z) = \frac{t^a}{N_a} \sum_{i_a=1}^{N_a} \frac{z + z^a_{i_a}}{z - z^a_{i_a}}, \qquad t^a := \frac{2\pi N_a}{k},$$

Resolvent equations

In terms of resolvents defined as

$$v^a(z) = \frac{t^a}{N_a}\sum_{i_a=1}^{N_a}\frac{z+z^a_{i_a}}{z-z^a_{i_a}}, \qquad t^a := \frac{2\pi N_a}{k},$$

The saddle-point equations can be written as

$$2\kappa^a \log y_a + \frac{n^a}{k}\frac{y_a-1}{y_a+1} = v^a(y_a^+) + v^a(y_a^-) - \sum_b n^{ab}\tilde{v}^b(y_a),$$

$$2\kappa^b \log(-\tilde{y}_b) + \frac{n^b}{k}\frac{-\tilde{y}_b-1}{-\tilde{y}_b+1} = \tilde{v}^b(\tilde{y}_b^+) + \tilde{v}^b(\tilde{y}_b^-) - \sum_a n^{ba} v^a(y_b),$$

$$(y^\pm := y \pm i0)$$

where

$$\kappa^a := \frac{k_a}{k}, \qquad \tilde{v}^a(z) := v^a(-z), \qquad \tilde{y}_a := -y_{a.}$$

Resolvent equations

In terms of resolvents defined as

$$v^a(z) = \frac{t^a}{N_a}\sum_{i_a=1}^{N_a}\frac{z+z^a_{i_a}}{z-z^a_{i_a}}, \qquad t^a := \frac{2\pi N_a}{k},$$

The saddle-point equations can be written as

$$2\kappa^a \log y_a + \frac{n^a}{k}\frac{y_a-1}{y_a+1} = v^a(y_a^+) + v^a(y_a^-) - \sum_b n^{ab}\tilde{v}^b(y_a),$$

$$2\kappa^b \log(-\tilde{y}_b) + \frac{n^b}{k}\frac{-\tilde{y}_b-1}{-\tilde{y}_b+1} = \tilde{v}^b(\tilde{y}_b^+) + \tilde{v}^b(\tilde{y}_b^-) - \sum_a n^{ba} v^a(y_b),$$

$$(y^\pm := y \pm i0)$$

where

$$\kappa^a := \frac{k_a}{k}, \qquad \tilde{v}^a(z) := v^a(-z), \qquad \tilde{y}_a := -y_a.$$

If the graph Γ is bipartite, then one can choose

white nodes $\Leftrightarrow$ $v^a(z)$, black nodes $\Leftrightarrow$ $\tilde{v}^a(z)$.

$\Rightarrow$ No redundant equations.

Resolvent equations

In terms of resolvents defined as

$$v^a(z) = \frac{t^a}{N_a}\sum_{i_a=1}^{N_a}\frac{z+z^a_{i_a}}{z-z^a_{i_a}}, \qquad t^a := \frac{2\pi N_a}{k},$$

The saddle-point equations can be written as

$$2\kappa^a \log y_a + \frac{n^a}{k}\frac{y_a-1}{y_a+1} = v^a(y_a^+) + v^a(y_a^-) - \sum_b n^{ab}\tilde{v}^b(y_a),$$

$$2\kappa^b \log(-\tilde{y}_b) + \frac{n^b}{k}\frac{-\tilde{y}_b-1}{-\tilde{y}_b+1} = \tilde{v}^b(\tilde{y}_b^+) + \tilde{v}^b(\tilde{y}_b^-) - \sum_a n^{ba} v^a(y_b),$$

$$(y^\pm := y \pm i0)$$

where

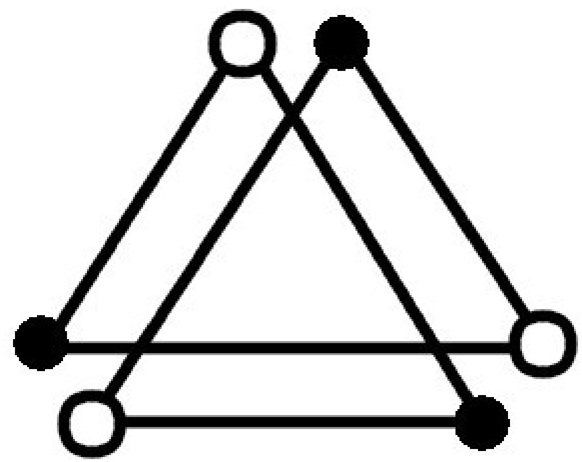
$$\kappa^a := \frac{k_a}{k}, \qquad \tilde{v}^a(z) := v^a(-z), \qquad \tilde{y}_a := -y_a.$$

If the graph Γ is bipartite, then one can choose

white nodes $\Longleftrightarrow$ $v^a(z)$, black nodes $\Longleftrightarrow$ $\tilde{v}^a(z)$.

$\Longrightarrow$ No redundant equations.

Note: If Γ is not bipartite, one may replace it with its bipartite double-cover.



Bipartite double cover of a traingle.

Homogeneous equations

The resolvent equations can be written simply

$$f^{a}(y_{a}) = v^{a}(y_{a}^{+}) + v^{a}(y_{a}^{-}) - \sum_{b} n^{ab} v^{b}(y_{a}).$$

Homogeneous equations

The resolvent equations can be written simply

$$f^a(y_a) = v^a(y_a^+) + v^a(y_a^-) - \sum_b n^{ab} v^b(y_a).$$

Let $v^a(z)$ be written as

$$v^a(z) = r^a(z) + \omega^a(z), \quad r^a(z): \text{rational functions}$$

such that

$$f^a(z) = 2r^a(z) - \sum_b n^{ab} r^b(z).$$

Homogeneous equations

The resolvent equations can be written simply

$$f^a(y_a) = v^a(y_a^+) + v^a(y_a^-) - \sum_b n^{ab} v^b(y_a).$$

Let $v^a(z)$ be written as

$$v^a(z) = r^a(z) + \omega^a(z), \qquad r^a(z) : \text{rational functions}$$

such that

$$f^a(z) = 2r^a(z) - \sum_b n^{ab} r^b(z).$$

$\Rightarrow$ $\omega^a(z)$ satisfy simpler equations

$$\boxed{\omega^a(y_a^+) + \omega^a(y_a^-) - \sum_b n^{ab} \omega^b(y_a) = 0.}$$

Homogeneous equations

The resolvent equations can be written simply

$$f^a(y_a) = v^a(y_a^+) + v^a(y_a^-) - \sum_b n^{ab} v^b(y_a).$$

Let $v^a(z)$ be written as

$$v^a(z) = r^a(z) + \omega^a(z), \quad r^a(z): \text{rational functions}$$

such that

$$f^a(z) = 2r^a(z) - \sum_b n^{ab} r^b(z).$$

$\Longrightarrow$ $\omega^a(z)$ satisfy simpler equations

$$\boxed{\omega^a(y_a^+) + \omega^a(y_a^-) - \sum_b n^{ab} \omega^b(y_a) = 0.}$$

Note: $f^a(z)$ contain log. $\Longrightarrow$ No such $r^a(z)$! Consider again.

Homogeneous equations

The resolvent equations can be written simply

$$f^a(y_a, \xi) = v^a(y_a^+, \xi) + v^a(y_a^-, \xi) - \sum_b n^{ab} v^b(y_a, \xi),$$

where

$$f^a(z) = \int_C d\xi \, f^a(z, \xi), \qquad v^a(z) = \int_C d\xi \, v^a(z, \xi).$$

Homogeneous equations

The resolvent equations can be written simply

$$f^a(y_a,\xi) = v^a(y_a^+,\xi) + v^a(y_a^-,\xi) - \sum_b n^{ab} v^b(y_a,\xi),$$

where

$$f^a(z) = \int_C d\xi \, f^a(z,\xi), \qquad v^a(z) = \int_C d\xi \, v^a(z,\xi).$$

rational

Homogeneous equations

The resolvent equations can be written simply

$$f^a(y_a,\xi)=v^a(y_a^+,\xi)+v^a(y_a^-,\xi)-\sum_b n^{ab}v^b(y_a,\xi),$$

where

$$f^a(z)=\int_C d\xi\, f^a(z,\xi),\qquad v^a(z)=\int_C d\xi\, v^a(z,\xi).$$

rational

Let $v^a(z,\xi)$ be written as

$$v^a(z,\xi)=r^a(z,\xi)+\omega^a(z,\xi),\quad r^a(z,\xi)\text{: rational functions}$$

such that

$$f^a(z,\xi)=2r^a(z,\xi)-\sum_b n^{ab}r^b(z,\xi).$$

Homogeneous equations

The resolvent equations can be written simply

$$f^a(y_a,\xi) = v^a(y_a^+,\xi)+v^a(y_a^-,\xi)-\sum_b n^{ab}v^b(y_a,\xi),$$

where

$$f^a(z) = \int_C d\xi\, f^a(z,\xi), \qquad v^a(z) = \int_C d\xi\, v^a(z,\xi).$$

rational

Let $v^a(z,\xi)$ be written as

$$v^a(z,\xi) = r^a(z,\xi)+\omega^a(z,\xi), \quad r^a(z,\xi): \text{rational functions}$$

such that

$$f^a(z,\xi) = 2r^a(z,\xi)-\sum_b n^{ab}r^b(z,\xi).$$

$\Longrightarrow$ $\omega^a(z,\xi)$ satisfy simpler equations

$$\boxed{\omega^a(y_a^+,\xi)+\omega^a(y_a^-,\xi)-\sum_b n^{ab}\omega^b(y_a,\xi) = 0.}$$

Homogeneous equations

The resolvent equations can be written simply

$$f^a(y_a,\xi) = v^a(y_a^+,\xi)+v^a(y_a^-,\xi)-\sum_b n^{ab} v^b(y_a,\xi),$$

where

$$f^a(z) = \int_C d\xi\, f^a(z,\xi), \qquad v^a(z) = \int_C d\xi\, v^a(z,\xi).$$

rational

Let $v^a(z,\xi)$ be written as

$$v^a(z,\xi) = r^a(z,\xi)+\omega^a(z,\xi), \quad r^a(z,\xi): \text{rational functions}$$

such that

$$f^a(z,\xi) = 2r^a(z,\xi)-\sum_b n^{ab} r^b(z,\xi).$$

⇨ $\omega^a(z,\xi)$ satisfy simpler equations

$$\boxed{\omega^a(y_a^+,\xi)+\omega^a(y_a^-,\xi)-\sum_b n^{ab}\omega^b(y_a,\xi) = 0.}$$

Note: The reduction is possible iff

$$\underline{\det[2\delta^{ab}-n^{ab}] \neq 0}.$$

⇨ Not applicable to ABJM (and other CSM with gravity duals...)

Riemann-Hilbert problem

Rewrite the homogeneous equations as (ξ -dependence ignored)

$$\boxed{\omega(y_a^+) = \omega(y_a^-)\, M_a .}$$

Here

$$\omega(z) = (\omega^1(z), \cdots, \omega^{n_g}(z)), \qquad M_a = \begin{bmatrix} 1 & & & n^{a1} & & & \\ & \ddots & & \vdots & & & \\ & & 1 & n^{a,a-1} & & & \\ & & & -1 & & & \\ & & & n^{a,a+1} & 1 & & \\ & & & \vdots & & \ddots & \\ & & & n^{a,n_g} & & & 1 \end{bmatrix}.$$

Note: This implies

$$\omega^a(y_b^+) = \omega^a(y_b^-) \quad (a \neq b), \qquad (M_a)^2 = I .$$

Riemann-Hilbert problem

Rewrite the homogeneous equations as (ξ-dependence ignored)

$$\boxed{\omega\left(y_a^{+}\right) = \omega\left(y_a^{-}\right) M_a .}$$

Note: This implies

$$\omega^a\left(y_b^{+}\right) = \omega^a\left(y_b^{-}\right) \quad \left(a \neq b\right), \qquad \left(M_a\right)^2 = I .$$

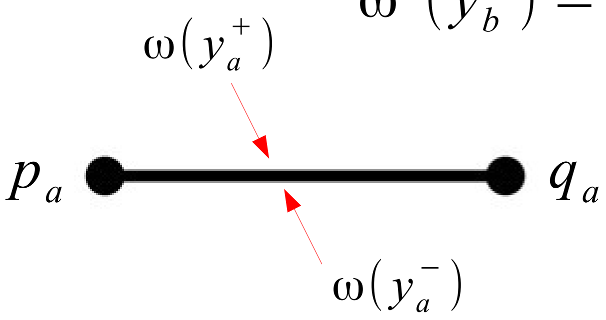
Riemann-Hilbert problem

Rewrite the homogeneous equations as (ξ -dependence ignored)

$$\boxed{\omega(y_a^+) = \omega(y_a^-)\, M_a .}$$

Note: This implies

$$\omega^a(y_b^+) = \omega^a(y_b^-) \quad (a \neq b), \qquad (M_a)^2 = I .$$



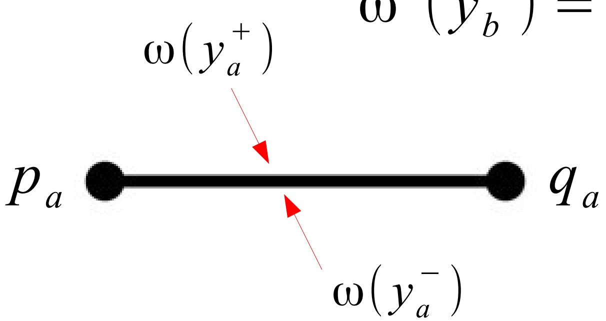
Riemann-Hilbert problem

Rewrite the homogeneous equations as (ξ-dependence ignored)

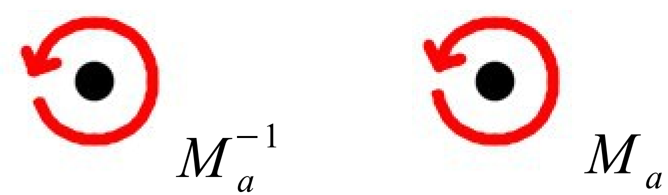
$$\boxed{\omega(y_a^+) = \omega(y_a^-)\, M_a .}$$

Note: This implies

$$\omega^a(y_b^+) = \omega^a(y_b^-) \quad (a \neq b), \qquad (M_a)^2 = I .$$



$\Longleftrightarrow$



specified monodromy
at specified points.

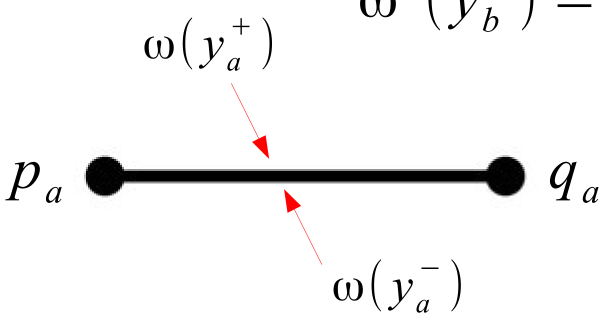
Riemann-Hilbert problem

Rewrite the homogeneous equations as (ξ-dependence ignored)

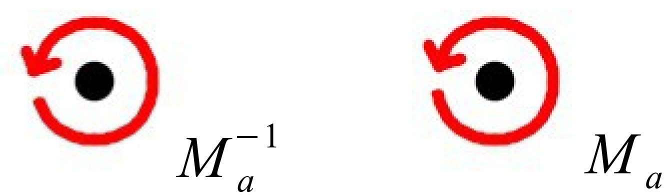
$$\boxed{\omega(y_a^+) = \omega(y_a^-)\, M_a .}$$

Note: This implies

$$\omega^a(y_b^+) = \omega^a(y_b^-) \quad (a \neq b), \qquad (M_a)^2 = I .$$



$\Longleftrightarrow$



$\exists$ a Fuchsian system $\overset{\text{R-H}}{\Longleftrightarrow}$ specified monodromy at specified points.

$$\frac{dy}{dz} = \sum_{a=1}^{n_g}\left[\frac{A_a}{z-p_a} + \frac{B_a}{z-q_a}\right] y .$$

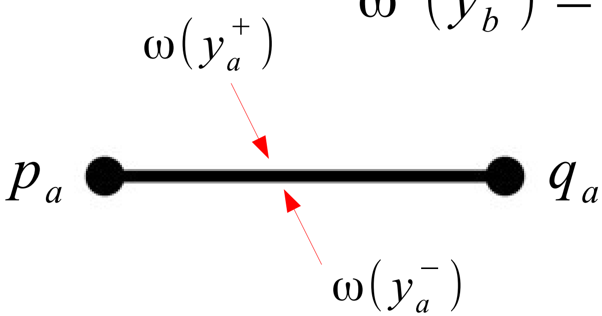
Riemann-Hilbert problem

Rewrite the homogeneous equations as (ξ-dependence ignored)

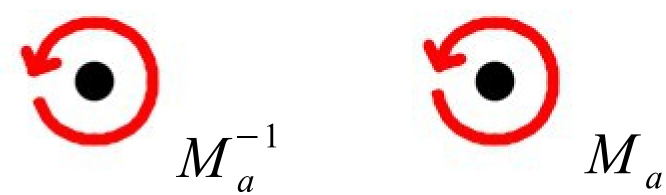
$$\boxed{\omega(y_a^+) = \omega(y_a^-)\,M_a .}$$

Note: This implies

$$\omega^a(y_b^+) = \omega^a(y_b^-) \quad (a \neq b), \qquad (M_a)^2 = I .$$



$\Longleftrightarrow$



R-H

$\exists$ a Fuchsian system $\Longleftrightarrow$ specified monodromy at specified points.

$$\frac{dy}{dz} = \sum_{a=1}^{n_g}\left[\frac{A_a}{z-p_a} + \frac{B_a}{z-q_a}\right] y .$$

A set of independent solutions form a matrix $Y(z)$.

$\Longrightarrow$ $$\boxed{\omega(z) = r(z)\,Y(z).}$$

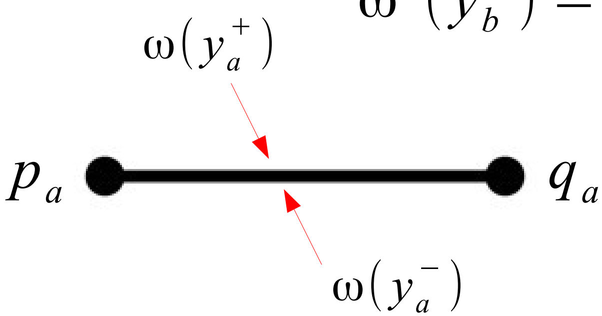
Riemann-Hilbert problem

Rewrite the homogeneous equations as (ξ-dependence ignored)

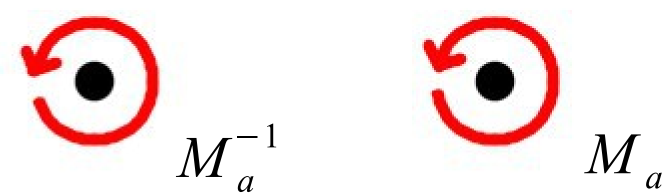
$$\boxed{\omega(y_a^+) = \omega(y_a^-)\, M_a .}$$

Note: This implies

$$\omega^a(y_b^+) = \omega^a(y_b^-) \quad (a \neq b), \qquad (M_a)^2 = I .$$



$\Longleftrightarrow$



$\exists$ a Fuchsian system $\overset{\text{R-H}}{\Longleftrightarrow}$ specified monodromy at specified points.

$$\frac{dy}{dz} = \sum_{a=1}^{n_g} \left[\frac{A_a}{z-p_a} + \frac{B_a}{z-q_a} \right] y .$$

A set of independent solutions form a matrix $Y(z)$.

$$\Longrightarrow \qquad \boxed{\omega(z) = r(z)\, Y(z).}$$

Note: $r(z)$ is determined by poles of $\omega(z,\xi)$ at $z = \pm\xi$.

Th. Irreducible monodromy $\Longrightarrow$ $\exists$ Fuchsian system. [Bolibruch][Kostov]

Th. Irreducible monodromy $\Longrightarrow$ $\exists$ Fuchsian system. [Bolibruch][Kostov]

In our case, $\{M_a\}$ is irreducible iff $\det[2\delta^{ab}-n^{ab}]\neq 0$.

$\Longrightarrow$ $\exists$ Fuchsian system $\Longrightarrow$ $\exists$ $Y_0(z)$ s.t. $Y_0(z)\to Y_0(z)M_a$.

Th. Irreducible monodromy $\Longrightarrow$ $\exists$ Fuchsian system. [Bolibruch][Kostov]

In our case, $\{M_a\}$ is irreducible iff $\det[2\delta^{ab}-n^{ab}]\neq 0$.

$\Longrightarrow$ $\exists$ Fuchsian system $\Longrightarrow$ $\exists$ $Y_0(z)$ s.t. $Y_0(z)\to Y_0(z)M_a$.

$Y_0(z)$ has the following local form

$$Y_0(z)=U(z)(z-p_a)^{\Lambda_a}(z-p_a)^{E_a}S_a, \qquad M_a=S_a^{-1}e^{2\pi i E_a}S_a.$$

Th. Irreducible monodromy $\Longrightarrow$ $\exists$ Fuchsian system. [Bolibruch][Kostov]

In our case, $\{M_a\}$ is irreducible iff $\det[2\delta^{ab}-n^{ab}]\neq 0$.

$\Longrightarrow$ $\exists$ Fuchsian system $\Longrightarrow$ $\exists$ $Y_0(z)$ s.t. $Y_0(z)\to Y_0(z)M_a$.

$Y_0(z)$ has the following local form

Jordan form

$$Y_0(z)=U(z)(z-p_a)^{\Lambda_a}(z-p_a)^{E_a}S_a, \qquad M_a=S_a^{-1}e^{2\pi i E_a}S_a.$$

Th. Irreducible monodromy $\Longrightarrow$ $\exists$ Fuchsian system. [Bolibruch][Kostov]

In our case, $\{M_a\}$ is irreducible iff $\det[2\delta^{ab} - n^{ab}] \neq 0$.

$\Longrightarrow$ $\exists$ Fuchsian system $\Longrightarrow$ $\exists$ $Y_0(z)$ s.t. $Y_0(z) \to Y_0(z) M_a$.

$Y_0(z)$ has the following local form

Jordan form

$$Y_0(z) = U(z)(z-p_a)^{\Lambda_a}(z-p_a)^{E_a} S_a, \qquad M_a = S_a^{-1} e^{2\pi i E_a} S_a.$$

$0 \leq \mathrm{Re}\,(\text{e.v.}\, E_a) < 1$

Th. Irreducible monodromy $\Longrightarrow$ $\exists$ Fuchsian system. [Bolibruch][Kostov]

In our case, $\{M_a\}$ is irreducible iff $\det[2\delta^{ab}-n^{ab}]\neq 0$.

$\Longrightarrow$ $\exists$ Fuchsian system $\Longrightarrow$ $\exists\ Y_0(z)$ s.t. $Y_0(z)\to Y_0(z)M_a$.

$Y_0(z)$ has the following local form

Jordan form

$$Y_0(z)=U(z)(z-p_a)^{\Lambda_a}(z-p_a)^{E_a}S_a,\qquad M_a=S_a^{-1}e^{2\pi i E_a}S_a.$$

$\exp[E_a\log(z-p_a)]$ $\qquad 0\leq \mathrm{Re}\,(\text{e.v.}\,E_a)<1$

Th. Irreducible monodromy $\Longrightarrow$ $\exists$ Fuchsian system. [Bolibruch][Kostov]

In our case, $\{M_a\}$ is irreducible iff $\det[2\delta^{ab} - n^{ab}] \neq 0$.

$\Longrightarrow$ $\exists$ Fuchsian system $\Longrightarrow$ $\exists$ $Y_0(z)$ s.t. $Y_0(z) \to Y_0(z) M_a$.

$Y_0(z)$ has the following local form

diagonal, integer

Jordan form

$$\underline{Y_0(z) = U(z)(z-p_a)^{\Lambda_a}(z-p_a)^{E_a} S_a}, \qquad M_a = S_a^{-1} e^{2\pi i E_a} S_a.$$

$\exp[E_a \log(z-p_a)]$ $\qquad 0 \leq \mathrm{Re}\,(\text{e.v.}\, E_a) < 1$

Th. Irreducible monodromy $\Longrightarrow$ $\exists$ Fuchsian system. [Bolibruch][Kostov]

In our case, $\{M_a\}$ is irreducible iff $\det[2\delta^{ab} - n^{ab}] \neq 0$.

$\Longrightarrow$ $\exists$ Fuchsian system $\Longrightarrow$ $\exists$ $Y_0(z)$ s.t. $Y_0(z) \to Y_0(z) M_a$.

$Y_0(z)$ has the following local form

diagonal, integer

Jordan form

$$\underbrace{Y_0(z) = U(z)(z-p_a)^{\Lambda_a}(z-p_a)^{E_a} S_a}, \qquad M_a = S_a^{-1} e^{2\pi i E_a} S_a .$$

holomorphic

$\exp[E_a \log(z-p_a)]$

$0 \leq \mathrm{Re}\,(\text{e.v.}\, E_a) < 1$

Th. Irreducible monodromy $\Longrightarrow$ $\exists$ Fuchsian system. [Bolibruch][Kostov]

In our case, $\{M_a\}$ is irreducible iff $\det[2\delta^{ab} - n^{ab}] \neq 0$.

$\Longrightarrow$ $\exists$ Fuchsian system $\Longrightarrow$ $\exists$ $Y_0(z)$ s.t. $Y_0(z) \to Y_0(z) M_a$.

$Y_0(z)$ has the following local form

diagonal, integer

Jordan form

$$\underbrace{Y_0(z) = U(z)(z-p_a)^{\Lambda_a}(z-p_a)^{E_a} S_a}, \qquad M_a = S_a^{-1} e^{2\pi i E_a} S_a.$$

holomorphic $\qquad \exp[E_a \log(z-p_a)]$ $\qquad 0 \leq \mathrm{Re}\,(\text{e.v. } E_a) < 1$

$\exists \Gamma(z)$ meromorphic on $\boldsymbol{P}^1$, holomorphic on $\boldsymbol{P}^1 \backslash \{p_a, q_a\}$, s.t.

$$Y(z) := \Gamma(z) Y_0(z) = U(z)(z-p_a)^{E_a} S_a.$$

(finite at singularity)

Th. Irreducible monodromy $\Longrightarrow$ $\exists$ Fuchsian system. [Bolibruch][Kostov]

In our case, $\{M_a\}$ is irreducible iff $\det[2\delta^{ab}-n^{ab}]\neq 0$.

$\Longrightarrow$ $\exists$ Fuchsian system $\Longrightarrow$ $\exists$ $Y_0(z)$ s.t. $Y_0(z)\to Y_0(z)M_a$.

$Y_0(z)$ has the following local form

diagonal, integer

Jordan form

$$\underline{Y_0(z) = U(z)(z-p_a)^{\Lambda_a}(z-p_a)^{E_a}S_a}, \qquad M_a = S_a^{-1}e^{2\pi i E_a}S_a.$$

holomorphic

$\exp[E_a\log(z-p_a)]$

$0\leq \mathrm{Re}\,(\text{e.v.}\,E_a) < 1$

$\exists\Gamma(z)$ meromorphic on $\boldsymbol{P}^1$, holomorphic on $\boldsymbol{P}^1\backslash\{p_a, q_a\}$, s.t.

$$Y(z) := \Gamma(z)Y_0(z) = U(z)(z-p_a)^{E_a}S_a.$$

(finite at singularity)

Near infinity,

holomorphic at infinity

$$Y(z) = z\,V(z).$$

Th. Irreducible monodromy $\Longrightarrow$ $\exists$ Fuchsian system. [Bolibruch][Kostov]

In our case, $\{M_a\}$ is irreducible iff $\det[2\delta^{ab} - n^{ab}] \neq 0$.

$\Longrightarrow$ $\exists$ Fuchsian system $\Longrightarrow$ $\exists\ Y_0(z)$ s.t. $Y_0(z) \to Y_0(z) M_a$.

$Y_0(z)$ has the following local form

diagonal, integer

Jordan form

$$\underline{Y_0(z) = U(z)(z-p_a)^{\Lambda_a}(z-p_a)^{E_a} S_a}, \qquad M_a = S_a^{-1} e^{2\pi i E_a} S_a .$$

holomorphic $\qquad \exp[E_a \log(z-p_a)]$ $\qquad 0 \leq \mathrm{Re}\,(\text{e.v.}\, E_a) < 1$

$\exists \Gamma(z)$ meromorphic on $\boldsymbol{P}^1$, holomorphic on $\boldsymbol{P}^1 \setminus \{p_a, q_a\}$, s.t.

$$Y(z) := \Gamma(z) Y_0(z) = U(z)(z-p_a)^{E_a} S_a .$$

(finite at singularity)

Near infinity,

holomorphic at infinity

$$Y(z) = z\, V(z).$$

Note: For a normalization $Y(0) = I$, one finds

$$\underline{Y(z^{-1}) = z^{-1} Y(z)}.$$

Resolvents

Assume that $\omega(z,\xi)$ is a solution of $\omega(y_a^+) = \omega(y_a^-) M_a$.

$\Rightarrow$ $\tilde{r}(z,\xi) := \omega(z,\xi) Y(z)^{-1}$ has no monodronomy on $\boldsymbol{P}^1$, i.e. rational.

Resolvents

Assume that $\omega(z,\xi)$ is a solution of $\omega(y_a^+) = \omega(y_a^-)M_a$.

⇨ $\tilde{r}(z,\xi) := \omega(z,\xi)Y(z)^{-1}$ has no monodronomy on $\boldsymbol{P}^1$, i.e. rational.

Recall

$$v^a(z) = \frac{t^a}{N_a}\sum_{i_a=1}^{N_a}\frac{z-z_{i_a}^a}{z+z_{i_a}^a} = \int_C d\xi\, v^a(z,\xi), \quad v^a(z,\xi) = r^a(z,\xi)+\omega^a(z,\xi).$$

Resolvents

Assume that $\omega(z,\xi)$ is a solution of $\omega(y_a^+) = \omega(y_a^-) M_a$.

⇨ $\tilde{r}(z,\xi) := \omega(z,\xi) Y(z)^{-1}$ has no monodronomy on $\boldsymbol{P}^1$, i.e. rational.

Recall

$$v^a(z) = \frac{t^a}{N_a} \sum_{i_a=1}^{N_a} \frac{z - z_{i_a}^a}{z + z_{i_a}^a} = \int_C d\xi\, v^a(z,\xi), \quad v^a(z,\xi) = r^a(z,\xi) + \omega^a(z,\xi).$$

holomorphic on $\boldsymbol{P}^1 \backslash [p_a, q_a]$
finite at branch points.

Resolvents

Assume that $\omega(z,\xi)$ is a solution of $\omega(y_a^+) = \omega(y_a^-) M_a$.

$\Rightarrow$ $\tilde{r}(z,\xi) := \omega(z,\xi) Y(z)^{-1}$ has no monodronomy on $\boldsymbol{P}^1$, i.e. rational.

Recall

poles at $z = \pm\xi$.

$$v^a(z) = \frac{t^a}{N_a} \sum_{i_a=1}^{N_a} \frac{z - z_{i_a}^a}{z + z_{i_a}^a} = \int_C d\xi \, v^a(z,\xi), \quad v^a(z,\xi) = r^a(z,\xi) + \omega^a(z,\xi).$$

holomorphic on $\boldsymbol{P}^1 \setminus [p_a, q_a]$
finite at branch points.

Resolvents

Assume that $\omega(z,\xi)$ is a solution of $\omega(y_a^+) = \omega(y_a^-) M_a$.

$\Rightarrow$ $\tilde{r}(z,\xi) := \omega(z,\xi) Y(z)^{-1}$ has no monodronomy on $\boldsymbol{P}^1$, i.e. rational.

Recall

poles at $z = \pm\xi$.

$$v^a(z) = \frac{t^a}{N_a} \sum_{i_a=1}^{N_a} \frac{z - z_{i_a}^a}{z + z_{i_a}^a} = \int_C d\xi \, v^a(z,\xi), \quad v^a(z,\xi) = r^a(z,\xi) + \omega^a(z,\xi).$$

holomorphic on $\boldsymbol{P}^1 \backslash [p_a, q_a]$
finite at branch points.

$$\Rightarrow \qquad \tilde{r}(z,\xi) = \frac{a(\xi) z + b(\xi)}{z^2 - \xi^2}.$$

Resolvents

Assume that $\omega(z,\xi)$ is a solution of $\omega(y_a^+) = \omega(y_a^-) M_a$.

$\Rightarrow$ $\tilde{r}(z,\xi) := \omega(z,\xi) Y(z)^{-1}$ has no monodronomy on $\boldsymbol{P}^1$, i.e. rational.

Recall

poles at $z = \pm\xi$.

$$v^a(z) = \frac{t^a}{N_a} \sum_{i_a=1}^{N_a} \frac{z - z_{i_a}^a}{z + z_{i_a}^a} = \int_C d\xi \, v^a(z,\xi), \quad v^a(z,\xi) = r^a(z,\xi) + \omega^a(z,\xi).$$

holomorphic on $\boldsymbol{P}^1 \setminus [p_a, q_a]$
finite at branch points.

$$\Rightarrow \quad \tilde{r}(z,\xi) = \frac{a(\xi) z + b(\xi)}{z^2 - \xi^2}.$$

Let

$$\rho_\pm := \mathrm{Res}_{\pm\xi} \, \omega(z,\xi).$$

$$\Rightarrow \quad \boxed{\omega(z,\xi) = \left[\frac{1}{z-\xi} \rho_+ Y(\xi)^{-1} + \frac{1}{z+\xi} \rho_- Y(-\xi)^{-1} \right] Y(z).}$$

Resolvents

Assume that $\omega(z,\xi)$ is a solution of $\omega(y_a^+) = \omega(y_a^-) M_a$.

$\Rightarrow$ $\tilde{r}(z,\xi) := \omega(z,\xi) Y(z)^{-1}$ has no monodronomy on $\boldsymbol{P}^1$, i.e. rational.

Recall

poles at $z = \pm\xi$.

$$v^a(z) = \frac{t^a}{N_a} \sum_{i_a=1}^{N_a} \frac{z - z_{i_a}^a}{z + z_{i_a}^a} = \int_C d\xi\, v^a(z,\xi), \qquad v^a(z,\xi) = r^a(z,\xi) + \omega^a(z,\xi).$$

holomorphic on $\boldsymbol{P}^1 \backslash [p_a, q_a]$
finite at branch points.

$$\Rightarrow \qquad \tilde{r}(z,\xi) = \frac{a(\xi) z + b(\xi)}{z^2 - \xi^2}.$$

Let

$$\rho_{\pm} := \mathrm{Res}_{\pm\xi}\, \omega(z,\xi).$$

$$\Rightarrow \qquad \boxed{\omega(z,\xi) = \left[\frac{1}{z-\xi} \rho_+ Y(\xi)^{-1} + \frac{1}{z+\xi} \rho_- Y(-\xi)^{-1}\right] Y(z).}$$

Note: This solution satisfies

$$\omega(z^{-1},\xi) = -\xi^{-2} \omega(z,\xi^{-1}) \quad \Rightarrow \quad \underline{v(z^{-1}) = -v(z)}.$$

't Hooft couplings

The 't Hooft couplings can be written as

$$\boxed{t = \int_C d\xi \left[r(0,\xi) + \frac{1}{\xi}\left(-\rho_+ Y(\xi)^{-1} + \rho_- Y(-\xi)^{-1} \right) \right].}$$

The behavior of t w.r.t. parameters is complicated.

't Hooft couplings

The 't Hooft couplings can be written as

$$\boxed{t = \int_C d\xi \left[r(0,\xi) + \frac{1}{\xi}\left(-\rho_+ Y(\xi)^{-1} + \rho_- Y(-\xi)^{-1} \right) \right].}$$

The behavior of t w.r.t. parameters is complicated.

⇨ The divergent behavior can be investigated.

't Hooft couplings

The 't Hooft couplings can be written as

$$\boxed{t = \int_C d\xi \left[r(0,\xi) + \frac{1}{\xi}\left(-\rho_+ Y(\xi)^{-1} + \rho_- Y(-\xi)^{-1} \right) \right].}$$

The behavior of t w.r.t. parameters is complicated.

⇒ The divergent behavior can be investigated.

⇒ appears when one of singularities of $Y(\xi)$ approach the endpoints of C.

't Hooft couplings

The 't Hooft couplings can be written as

$$\boxed{t = \int_C d\xi \left[r(0,\xi) + \frac{1}{\xi}\left(-\rho_+ Y(\xi)^{-1} + \rho_- Y(-\xi)^{-1} \right) \right].}$$

The behavior of t w.r.t. parameters is complicated.

⇨ The divergent behavior can be investigated.

⇨ appears when one of singularities of $Y(\xi)$ approach the endpoints of C.

If p_a is small, $Y(\xi)$ behaves near the origin as

$$Y(\xi)^{-1} \sim \sqrt{\frac{-p_a}{\xi - p_a}}\, I.$$

't Hooft couplings

The 't Hooft couplings can be written as

$$\boxed{t = \int_C d\xi \left[r(0,\xi) + \frac{1}{\xi}\left(-\rho_+ Y(\xi)^{-1} + \rho_- Y(-\xi)^{-1} \right) \right].}$$

The behavior of t w.r.t. parameters is complicated.

⇨ The divergent behavior can be investigated.

⇨ appears when one of singularities of $Y(\xi)$ approach the endpoints of C.

If p_a is small, $Y(\xi)$ behaves near the origin as

$$Y(\xi)^{-1} \sim \sqrt{\frac{-p_a}{\xi - p_a}} I.$$

This implies

$$t \sim -\underbrace{(\rho_+ - \rho_-)}_{\text{real}} \log p_a, \qquad (p_a \to 0).$$

't Hooft couplings

The 't Hooft couplings can be written as

$$\boxed{t = \int_C d\xi \left[r(0,\xi) + \frac{1}{\xi}\left(-\rho_+ Y(\xi)^{-1} + \rho_- Y(-\xi)^{-1}\right)\right].}$$

The behavior of t w.r.t. parameters is complicated.

⇨ The divergent behavior can be investigated.

⇨ appears when one of singularities of $Y(\xi)$ approach the endpoints of C.

If p_a is small, $Y(\xi)$ behaves near the origin as

$$Y(\xi)^{-1} \sim \sqrt{\frac{-p_a}{\xi - p_a}}\, I.$$

This implies

$$t \sim -\underbrace{(\rho_+ - \rho_-)}_{\text{real}} \log p_a, \qquad (p_a \to 0).$$

⇨ Im(t) is finite, irrelevant for CSM.
To obtain a large Im(t), $\arg(p_a)$ must be large.

't Hooft couplings

The 't Hooft couplings can be written as

$$\boxed{t = \int_C d\xi \left[r(0,\xi) + \frac{1}{\xi}\left(-\rho_+ Y(\xi)^{-1} + \rho_- Y(-\xi)^{-1} \right) \right].}$$

The behavior of t w.r.t. parameters is complicated.

⇨ The divergent behavior can be investigated.

⇨ appears when one of singularities of $Y(\xi)$ approach the endpoints of C.

If p_a is small, $Y(\xi)$ behaves near the origin as

$$Y(\xi)^{-1} \sim \sqrt{\frac{-p_a}{\xi - p_a}}\, I.$$

This implies

$$t \sim -\underbrace{(\rho_+ - \rho_-)}_{\text{real}} \log p_a, \qquad (p_a \to 0).$$

⇨ Im(t) is finite, irrelevant for CSM.

To obtain a large Im(t), $\arg(p_a)$ must be large.

A finite e.v. distribution. Finite Wilson loops.

Isomonodromic deformations

So far, we obtained the solution for given monodromies.

The solution should be continuous w.r.t. positions $\{p_a, q_a\}$.

Isomonodromic deformations

So far, we obtained the solution for given monodromies.

The solution should be continuous w.r.t. positions $\{p_a, q_a\}$.

$\Longrightarrow$ For a given M_a , one has to find a continuous family of Fuchsian systems.

$\Longleftrightarrow$ Isomonodromic deformation

Isomonodromic deformations

So far, we obtained the solution for given monodromies.

The solution should be continuous w.r.t. positions $\{p_a, q_a\}$.

$\Longrightarrow$ For a given M_a , one has to find a continuous family of Fuchsian systems.

$\Longleftrightarrow$ Isomonodromic deformation

The continuous family

$$\frac{dy}{dz} = \sum_i \frac{A_i(a)}{z-a_i} y$$

should have matrices $A_i(a)$ satisfying

$$dA_i(a) = \sum_{j\neq i} \frac{[A_i(a), A_j(a)]}{a_i - a_j} d(a_i - a_j) \quad \text{: Schlesinger eqs.}$$

Isomonodromic deformations

So far, we obtained the solution for given monodromies.

The solution should be continuous w.r.t. positions $\{p_a, q_a\}$.

$\Rightarrow$ For a given M_a, one has to find a continuous family of Fuchsian systems.

$\Leftrightarrow$ Isomonodromic deformation

The continuous family

$$\frac{dy}{dz} = \sum_i \frac{A_i(a)}{z - a_i} y$$

should have matrices $A_i(a)$ satisfying

$$dA_i(a) = \sum_{j \neq i} \frac{[A_i(a), A_j(a)]}{a_i - a_j} d(a_i - a_j) \quad : \text{Schlesinger eqs.}$$

It was shown [Miwa] that the solution has movable singularities.

Isomonodromic deformations

So far, we obtained the solution for given monodromies.

The solution should be continuous w.r.t. positions $\{p_a, q_a\}$.

$\Rightarrow$ For a given M_a, one has to find a continuous family of Fuchsian systems.

$\Leftrightarrow$ Isomonodromic deformation

The continuous family

$$\frac{dy}{dz} = \sum_i \frac{A_i(a)}{z - a_i} y$$

should have matrices $A_i(a)$ satisfying

$$dA_i(a) = \sum_{j \neq i} \frac{[A_i(a), A_j(a)]}{a_i - a_j} d(a_i - a_j) \quad \text{: Schlesinger eqs.}$$

It was shown [Miwa] that the solution has movable singularities.

$\Leftrightarrow$ No solution of RH problem for particular positions.

$\Leftrightarrow$ Extra divergences in the 't Hooft couplings.

Isomonodromic deformations

So far, we obtained the solution for given monodromies.

The solution should be continuous w.r.t. positions $\{p_a, q_a\}$.

⇨ For a given M_a, one has to find a continuous family of Fuchsian systems.

⇔ Isomonodromic deformation

The continuous family

$$\frac{dy}{dz} = \sum_i \frac{A_i(a)}{z-a_i} y$$

should have matrices $A_i(a)$ satisfying

$$dA_i(a) = \sum_{j\neq i} \frac{[A_i(a), A_j(a)]}{a_i - a_j} d(a_i - a_j) \quad \text{: Schlesinger eqs.}$$

It was shown [Miwa] that the solution has movable singularities.

⇔ No solution of RH problem for particular positions.

⇔ Extra divergences in the 't Hooft couplings.

This can be explicitly observed for the case $n_g = 2$.

Summary

- A large family of Chern-Simons-matter theories are studied.
- Matrix models associated to most of CSM are solvend in the planar limit.
- Wilson loops for those theories behave differently from e.g. ABJM.
- Interesting relation to Fuchsian system and related mathematics.

Open issues

- How to solve the exceptional theories?
- How to characterize the exceptional theories in terms of QFT?
- How to determine the planar free energy?
- Higher genus?
- etc.