Y-system and perturbed W minimal model

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Y. Hatsuda, K.I. Yuji Satoh, JHEP 1302 (2013) 067, arXiv:1211.6225

Introduction

Thermodynamic Bethe Ansatz for homogeneous sine-Gordon models

- integrable perturbation of generalized parafermionic CFT
  multi-parameter deformation of CFT
- the null-polygonal Wilson loop in AdS$_n$ $(n = 3, 4)$
- (level-rank) duality among CFTs
  genelarized parafermions $\leftrightarrow$ W-minimal model

Homogeneous sine-Gordon model

[Fernandez Pouza-Gallas-Hollowood-Miramontes 9606032]

gauged WZW model based on the coset $G/H = SU(N)/U(1)^{N-1}$
deformed by weight zero adjoint operators

$$S_{HSG} = k\left(S_{gWZW} - \int d^2x \frac{\mu^2}{4\pi}\mathrm{tr}(\Lambda_+ g^\dagger \Lambda_- g)\right) \quad g \in G, \Lambda_\pm = i\lambda_\pm \cdot h$$

$$\lambda_\pm = \sum_{i=1}^{N-1} \tilde{\mu}_i e^{\pm\sigma_i}\omega_i, \quad \omega_i : \text{fundamental weight},$$

σ_i: resonance parameter (changes the ratio of left/right mass)
soliton mass ($k-1$ copies of A_{N-1} affine Toda theory)

$$\mu_i^a = \mu\tilde{\mu}_i \frac{\sin\frac{\pi a}{k}}{\sin\frac{\pi}{k}}, \quad a = 1, \cdots, k-1, i = 1, \cdots, N-1$$

potential term (weight zero fields in the adjoint representation)

$$\frac{\mu^2}{4\pi}\sum_{i,j=1}^{N-1} \tilde{\mu}_i^2 \tilde{\mu}_j^2 e^{2\sigma_i - 2\sigma_j}\mathrm{tr}(\omega_i \cdot hg^\dagger \omega_j \cdot hg)$$

In the UV limit, the HSG model is described by the conformal perturbation theory.

- CFT: generalized parafermion model $su(N)_k/u(1)^{N-1}$
  $su(2)_k/u(1)$: $\mathbf{Z}_k$ parafermion model ($k=2$ Ising model)
  Φ: primary field with weights (Λ,λ):

$$\Delta=\bar{\Delta}=\frac{\Lambda(\Lambda+2\rho)}{2(k+N)}-\frac{\lambda^2}{2k}$$

- perturbed CFT

$$S=S_{CFT}+\lambda\int d^2x\Phi(z,\bar{z})$$

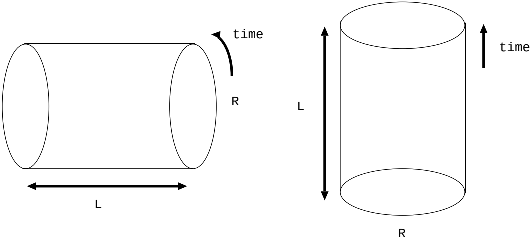
  Φ: $\Lambda=\omega_1+\omega_{N-1}$, $\lambda=0$ (weight zero in the adjoint rep.)

$$\Delta=\bar{\Delta}=\frac{N}{N+k}$$

Thermodynamic Bethe Ansatz

$1+1$ dim Euclidean QFT on a cylinder with periodic B.C.
partiton function

$$Z(R,L) = \mathrm{Tr} e^{-RH_L} = \mathrm{Tr} e^{-LH_R}$$



$$e^{-LRf(R)} \sim e^{-LE_0(R)} \quad (L\to\infty) \Longrightarrow E_0(R) = Rf(R)$$

$f(R)$: free energy per unit length at temperature $T=1/R$
$E_0(R)$: ground state energy of the Hamiltonian H_R defined on S^1

Free energy from TBA equations

particles (a,s) with masses $M_{a,s}=M_s\sin(\frac{\pi a}{k})/\sin(\frac{\pi}{k})$
S-matrix

$$S^{rs}_{ab}(\theta;\sigma_{rs})=\left[S^{\min}_{ab}(\theta)\right]^{\delta_{r,s}}\left[(\eta_{r,s})^{-ab}S^{F}_{ab}(\theta+\sigma_{rs})\right]^{-I_{r,s}}$$

$S^{\min}_{ab}$: the S-matrix of A_{k-1} affine Toda theory
TBA equations

$$\log Y_A(\theta)=-m_A\cosh\theta+\sum_B K_{AB}*\log(1+Y_B),$$

kernel

$$K_{AB}=\frac{1}{2\pi i}\frac{\partial}{\partial\theta}\log S_{AB}(\theta)$$

free energy

$$F=-\sum_A\int_{-\infty}^{+\infty}\frac{d\theta}{2\pi}|m_A|\cosh\theta\log(1+Y_A)$$

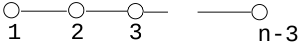
Y-system

TBA eqs $\Longleftrightarrow$ Y-system+ asymptotic conditions at $\theta \to \pm\infty$
$(Y_{a,s} \to -m_{a,s} \cosh\theta + const.)$

- $SU(N)_2/U(1)^{N-1}$ (AdS$_3$ minimal surface) $N = n - 2$

$$Y_s^+ Y_s^- = (1 + Y_{s+1})(1 + Y_{s-1}), \quad Y_s^\pm(\theta) = Y_s(\theta \pm i\frac{\pi}{4})$$

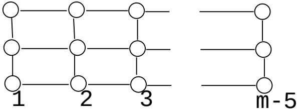
boundary cond. $Y_0 = Y_{n-2} = 0$



- $SU(N)_4/U(1)^{N-1}$ (AdS$_4$ minimal surface) $N = m - 5$

$$\frac{Y_{a,s}^- Y_{4-a,s}^+}{Y_{a+1,s} Y_{a-1,s}} = \frac{(1 + Y_{a,s+1})(1 + Y_{4-a,s-1})}{(1 + Y_{a+1,s})(1 + Y_{a-1,s})}, \quad Y_{1,s} = Y_{3,s}$$

The boundary conditions:
$Y_{a,0} = Y_{a,m-4} = 0 \ (a = 1, 2, 3)$
$Y_{0,s} = Y_{4,s} = \infty \ (s = 1, \cdots, m-5)$.



AdS minimal surface and CPT

- AdS_3 8-pt: [Alday-Maldacena] $c = \frac{1}{2}$ Ising model
- Y-system (AdS_5) [Alday-Maldacena-Sever-Vieira:1002.2459]
  Plücker relation for the Wronskians associated with the Hitchin eqs.
- TBA for $2n$-pt amplitudes in AdS_3=TBA for HSG
  $\hat{su}(n-2)_2/\hat{u}(1)^{n-3}$
- TBA for m-pt amplitudes in AdS_4= TBA for HSG
  $\hat{su}(m-4)_4/\hat{u}(1)^{m-5}$, central charge(UV limit)
  [Hatsuda-KI-Sakai-Satoh:1002.2491]
- $m = 6$ $\mathbf{Z}_4$ parafermion (AdS_5) [Hatsuda-I-Sakai-Satoh:1005.4487]
  $c = 1$ CFT
- $m = 7$ remainder function [Hatsuda-KI-Satoh:1211.6225]

ground state energy from perturbed CFT

Klassen-Melzer, NPB338 (1990) 485

$$S = S_{CFT} + \lambda \int d^2 w \Phi(w, \bar{w})$$

$\Phi(w, \bar{w})$ has conformal weight $(\Delta, \bar{\Delta})$
partition function $Z = \langle \exp \left(-\lambda \int d^2 w \Phi(w, \bar{w}) \right) \rangle_0$

$$F = \lim_{L \to \infty} \frac{R}{L} \log Z = RE(R) - \frac{1}{4}(mR)^2$$

$$E(R) = E_0 - R^2 \sum_{n=1}^{\infty} \frac{(-\lambda)^n}{n!} \left(\frac{2\pi}{R} \right)^{2+2n(\Delta-1)} \int d^2 z_2 \cdots d^2 z_n$$

$$\times \langle \Phi_0(0) \Phi(1) \epsilon(z_2) \cdots \Phi(z_n) \Phi_0(\infty) \rangle_{0,connected} \prod_i |z_i|^{2(\Delta-1)}$$

$E_0 \sim c_{eff} = c - 24\Delta$: effective central charge
Φ_0: the lowest dimensional oprator (appears in non-unitary CFT)

Free energy of HSG model

$SU(N)_k/U(1)^{N-1}$ CFT $+\Phi$ ($\Delta = \bar{\Delta} = \frac{N}{N+k}$)
UV limit ($l = ML \to 0$, $m_s = \tilde{M}_s M$)

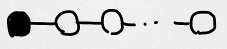
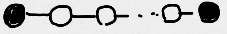
$$A_{\rm free} = -F = \frac{\pi}{6}c + f^{\rm bulk} + \sum_{p=2}^{\infty} f^{(p)} l^{2p(1-\Delta)}$$

c: central charge $= \frac{(k-1)N(N-1)}{N+k}$
$f^{\rm bulk}$: bulk term
$f^{(p)}$: connected part of p-point function of perturbation op.

- Numerical solution for the integral equation (instability[Castro-Alvaredo-Fring])
- multi-parameter deformation of CFT
- (unkown) mass coupling relation [Zamolodchikov, Fateev] $\lambda = -\kappa M^{2(1-\Delta)}$

TBA for HSG and W minimal models

$SU(N)_k/U(1)^{N-1}$ TBA $\Longleftarrow$ (extended) Dynkin diagrams $A_{N-1} \times A_{k-1}$
node $\Longleftrightarrow$ mass $m_{a,s}$
Introduce a single mass term.Ravanini-Tateo-Valleriani: 9207069, Ravanini
TBA eqs. $\Longrightarrow$ TBA eqs. for GKO coset $+\ \phi_{1,1,adj}$
For $SU(n-2)_2/U(1)^{n-3}$ ([Zamolodchikov, Itoyama-Moxay])

- RSOS limit $M_1 \neq 0, M_2 = \cdots = M_{n-3} = 0$ 
  unitary coset $\frac{SU(2)_1 \times SU(2)_{n-3}}{SU(2)_{n-2}}$ perturbed by $\phi_{1,1;adj}$
  minimal model $\mathcal{M}_{n-1,n} + \phi_{(1,3)}$
- $M_1 = M_{n-3} = M$, $M_2 = \cdots = M_{n-4} = 0$ $(n: odd)$
  
  $T_{(n-3)/2} = A_{n-3}/\mathbf{Z}_2$ tadpole diagram
  non-unitary coset $\frac{SU(2)_1 \times SU(2)_{n/2-3}}{SU(2)_{n/2-2}}$ perturbed by $\phi_{1,1;adj}$
  minimal model $\mathcal{M}_{n-2,n} + \phi_{(1,3)}$

$SU(N)_k/U(1)^{N-1}$ and GKO coset

- Both TBA equations coincide.
- level rank duality

$$SU(2)_k/U(1) = \frac{SU(k)_1 \times SU(k)_1}{SU(k)_2}$$

$$\frac{SU(N)_k}{U(1)^{N-1}} = \frac{(SU(k)_1)^N}{SU(k)_N}$$
$$= \frac{SU(k)_1 \times SU(k)_1}{SU(k)_2} \times \frac{SU(k)_1 \times SU(k)_2}{SU(k)_3} \times \cdots \frac{SU(k)_1 \times SU(k)_{N-1}}{SU(k)_N}$$

[Bagger-Nemeschansky]

CPT for $SU(n-2)_2/U(1)^{n-3}$ HSG model

Hatsuda-KI-Sakai-Satoh, 1102.2477

Leading contribution: 2-pt functions

$$f_n^{(2)} = \frac{\pi}{6} C_n^{(2)} \kappa_n^2 G^2(\tilde{M}_j)$$

$$C_n^{(2)} = 3(2\pi)^{\frac{2(n-4)}{n}} \gamma^2\left(\frac{n-2}{n}\right)\gamma\left(\frac{4-n}{n}\right),$$

$$\langle \Phi(z)\Phi(0)\rangle = \frac{G(\tilde{M}_j)^2}{|z|^{4\Delta}}, \quad G(\tilde{M}_j) = \sum_{i,j=1}^{n} \tilde{M}_i^{2/n} F_{ij} \tilde{M}_j^{2/n}$$

For $n=5$ (3-pt function $\langle \Phi_0 \Phi \Phi_0 \rangle$ Dotsenko-Fateev)

$$f_5^{(2)} = f_{(RSOS)_3}^{(2)} \left(\tilde{M}_1^{4/5} + \tilde{M}_2^{4/5} - B\tilde{M}_1^{2/5}\tilde{M}_2^{2/5}\right)^2$$

$$f_{(RSOS)_3}^{(2)} = \frac{\pi}{8\cdot 6^{2/5}}\gamma(-\frac{1}{5})\gamma(\frac{3}{5})\gamma(\frac{4}{5}) \quad B = 2 - \left(\frac{3}{\pi^2}\right)^{1/5}\gamma(\frac{1}{4})^{4/5}$$

CPT for $SU(n-4)_4/U(1)^{n-5}$ HSG

$SU(n-4)_4/U(1)^{n-5}$ coset CFT$+\Phi$ $(\Delta=\bar{\Delta}=\frac{n-4}{4})$
single mass perturbation

- $M_s = M\delta_{s,r}$ GKO coset

$$\frac{SU(4)_r \times SU(4)_{n-r-4}}{SU(4)_{n-4}} \quad + \quad \phi_{(1,1,adj)}$$

  leading contribution: 2-point function $\langle \Phi\Phi \rangle$

- $M_s = (\delta_{s,r} + \delta_{s,n-4-r})$ (n: odd) non-unitary GKO coset

$$\frac{SU(4)_r \times SU(4)_{n/2-r-4}}{SU(4)_{n/4-4}} \quad + \quad \phi_{(1,1,adj)}$$

  leading contribution 3-point function $\langle \Phi_0\Phi\Phi_0 \rangle$: Φ_0 vacuum operator

GKO coset and W-minimal model

fractional GKO [Frenkel-Kac-Wakimoto, Matheiu-Senechal-Walton]

$$\frac{SU(k)_1 \times SU(k)_m}{SU(k)_{1+m}} = WA^{(p,q)}_{k-1}, \quad m+k = \frac{o}{q-p}$$

weight vectors $(\mu_1, \mu_m, \mu_{m+1})$ ($\mu_1 + \mu_m - \mu_{m+1}$: root) $\Longrightarrow$

$$h_\Lambda = \frac{\Lambda^2 - \rho^2 (p-q)^2}{2pq}, \quad \pm\Lambda = q\mu_m - p\mu_{m+1} + (q-p)\rho$$

ρ: the Weyl vector

HSG with $r = 1$

$$\frac{SU(4)_1 \times SU(4)_{n-5}}{SU(4)_{n-4}} = WA_3^{(n-1,n)}$$

$$\frac{SU(4)_1 \times SU(4)_{n/2-5}}{SU(4)_{n/2-4}} = WA_3^{(n-2,n)}, \quad n\text{: odd}$$

Free energy for $SU(n-4)_4/U(1)^{n-5}$ HSG

$$A_{\rm free} = \frac{\pi}{6}c_n + f_n^{\rm bulk} + \sum_{p=2}^{\infty} f_n^{(p)} l^{8p/n}$$

$c_n = \frac{3(n-4)(n-5)}{n}$: central charge
$f_n^{\rm bulk} = K_{ss'} m_s \bar{m}_{s'}$: bulk term

$$f_n^{(2)} = \frac{\pi}{6}(\kappa_n G(\tilde{M}_s))^2 C_n^{(2)},$$

$$\langle \Phi(z)\Phi(0)\rangle = \frac{G^2(\tilde{M}_s)}{|z|^{4\Delta}}, \quad G(\tilde{M}_s) = \sum_{r,s=1}^{n} F_{rs}\tilde{M}_r^{4/n}\tilde{M}_s^{4/n}$$

$$C_n^{(2)} = 3(2\pi)^{2-16/n}\gamma^2(1-\frac{4}{n})\gamma(\frac{8}{n}-1)$$

$n=6$ [HISS] $\mu=1$

$$f_6^{(2)}=\frac{\pi}{6}\kappa_6^2G^2C_6^{(2)}=\frac{\pi}{2}\gamma^3(\frac{1}{3})\gamma(\frac{1}{6})\left[\frac{1}{2\sqrt{\pi}}\gamma(\frac{3}{4})\right]^{\frac{8}{3}}.$$

$n=7$ [HIK] 3-point function:[Fateev-Litvinov]

$$f_7^{(2)}=\frac{\pi}{6}(\kappa_7G(\tilde{M}_s))^2C_7^{(2)}$$

$$\begin{aligned}\kappa_7F_{11}&=\frac{2}{3\pi}\left[\gamma(\frac{2}{7})\gamma(\frac{4}{7})\right]^{1/2}\left[\frac{3}{4\sqrt{2}}\Gamma^2(\frac{3}{4})\right]^{8/7}.\\ \kappa_7G(\tilde{M}_1,\tilde{M}_2)&=\kappa_7F_{11}(\tilde{M}_1^{8/7}+\tilde{M}_2^{8/7}+B\tilde{M}_1^{4/7}\tilde{M}_2^{4/7})\\ B&=(\frac{2^{11}\pi^8}{3^2})^{\frac{1}{14}}\left[\gamma(\frac{4}{7})\gamma(\frac{6}{7})\gamma(\frac{1}{14})\right]^{1/2}\left[\frac{\Gamma(\frac{7}{8})}{\Gamma(\frac{3}{4})\Gamma(\frac{5}{8})}\right]^{\frac{8}{7}}-2.\end{aligned}$$

agree with the numerical results.

Outlook

- TBA equations for HSG $\Longrightarrow$ free energy $\sim$ minimal surface area $\Longrightarrow$ Gluon scattering amplitudes
- Free energy, g-function, T-functions
- BLZ
  ODE-IM correspondence (Dorey-Tateo)
- null-polygonal Wilson loops: strong coupling vs weak coupling
  strong coupling corrections, higher loop corrections
  Basso-Server-Vieira
- AdS_5
  Integrable system?
  HSG +twist operators (6-point amplitudes)